

UNIFORMLY STABLE MIXED *HP*-FINITE ELEMENTS ON MULTILEVEL ADAPTIVE GRIDS WITH HANGING NODES

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Abstract. We consider a family of quadrilateral or hexahedral mixed *hp*-finite elements for an incompressible flow problem with Q_r -elements for the velocity and discontinuous P_{r-1} -elements for the pressure where the order r can vary from element to element between 2 and an arbitrary bound. For multilevel adaptive grids with hanging nodes and a sufficiently small mesh size, we prove the inf-sup condition uniformly with respect to the mesh size and the polynomial degree.

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INTRODUCTION

For the numerical solution of incompressible flow problems, the mixed *hp*-finite element method is a well established efficient technique (see [22] and the references therein). It is well known that the inf-sup condition has to be satisfied in order to guarantee stability and accuracy of the discrete solution [9]. The inf-sup condition has been proven for many pairs of finite element spaces for the *h*-version (see *e.g.* [5,8,9,11,16,20,23]) as well as for the *p*- and *hp*-version (see *e.g.* [1,4,10,21,24,25]). In the above-mentioned papers on the *h*- and *hp*-version, it has been shown that the inf-sup constant is independent of the mesh size h . However, in the literature on the *h*-version (see *e.g.* [8,11,16,20,23]), the dependence of the inf-sup constant on the polynomial degree has not been investigated.

For the *hp*-version, the inf-sup condition was proved in [24] for different families of finite element pairs, in particular for the pairs $(Q_r, \mathbb{P}_{r-1}^{\text{disc}})$ and $(Q_{r+1}, Q_{r-1}^{\text{disc}})$ where “disc” means that the space consists of discontinuous functions which are elementwise defined by polynomials. It was proved that the inf-sup constant decreases as $Cr^{-(d-1)/2}$ where d denotes the dimension of the domain and C a constant independent of the mesh size h and the polynomial degree r . In [21,25], for the two and three dimensional case, respectively, the inf-sup condition was proved for the family of finite element pairs $(Q_r, Q_{r-2}^{\text{disc}})$ on so-called anisotropically and geometrically refined meshes with hanging nodes. It has been shown that the inf-sup constant can be estimated from below by $Cr^{-1/2}$ in the 2D-case and $Cr^{-3/2}$ in the 3D-case where the constant C is independent of the mesh size h , the polynomial degree r and the element aspect ratio. In [21,24,25], the cells of the mesh are assumed to be affine equivalent to a reference element. Therefore, the analysis covers only the case where the quadrilateral or

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hexahedral elements are parallelograms or parallelepipeds, respectively, which seems to be too restrictive from the practical point of view.

For two dimensional non-affine equivalent quadrilateral meshes with hanging nodes, Ainsworth and Coggins [1] proposed a family of finite element pairs $(\tilde{\mathbb{Q}}_r, \mathbb{Q}'_{r-1})$ which is inf-sup stable uniformly with respect to the mesh size h and the polynomial degree r . Here \mathbb{Q}'_{r-1} denotes a continuous pressure space constructed from the discontinuous space $\mathbb{P}_r^{\text{disc}}$ by some additional \mathbb{Q}_{r-1} -functions. The velocity space $\tilde{\mathbb{Q}}_r$ denotes a modification of the usual space \mathbb{Q}_r where the basis functions associated with nodes at the element boundary are defined by means of the usual reference transformation but the functions corresponding to interior element nodes are based on the *Piola transformation* (see [10] for a similar construction). This causes some extra difficulties in the implementation, in particular for the derivatives of the velocity functions.

In this paper, we prove, for the d -dimensional case, $d \in \{2, 3\}$, a uniform inf-sup condition for the family of finite element pairs $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{\text{disc}})$ on so called *multilevel adaptive grids* of quadrilateral or hexahedral elements with hanging nodes. By $\mathbb{P}_{r-1}^{\text{disc}}$ we denote the space of mapped discontinuous functions which are defined on the reference element by polynomials of total degree not greater than $r - 1$. Note that this mapped space may have non-optimal approximation properties on general quadrilateral or hexahedral grids [2,5]. However, one can show that it has optimal approximation properties on multilevel adaptive grids [17,19]. Multilevel adaptive grids are created from a regular coarse grid by means of recursive refinement of existing elements into 2^d many son-elements combined with the introduction of hanging nodes. From the practical point of view, such grids occur very naturally in applications with adaptive refinement based on *a posteriori* error estimators (see e.g. [13,14]). Typical grids are depicted in Figure 1. Note that we do not assume that the elements are affine equivalent.

The new contribution of this paper compared to the results of [16,20,24] on the element pair $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{\text{disc}})$ is that we prove the inf-sup constant to be independent of the polynomial degree r if the mesh size h is sufficiently small. We use the result of Bernardi and Maday [4] that, for the situation of just one element $K = (-1, 1)^d$, the inf-sup constant of this element pair is independent of the polynomial degree r . The main idea is to use a new *modified Piola transformation* to transform the inf-sup condition from the reference element to the original element. This proves the local inf-sup condition. The proof of the global inf-sup condition is based on the technique of Boland and Nicolaides [6] and an result in [16] for low order subspaces. Note that we assume shape-regular grids with possibly hanging nodes and that we do not consider anisotropic mesh refinements.

As a model for an incompressible flow problem we consider the Stokes equations with homogeneous Dirichlet conditions: *Find a velocity u and a pressure p such that*

$$\begin{aligned} -\nu \Delta u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned} \tag{0.1}$$

Here, Ω denotes a bounded domain in \mathbb{R}^d , $d \in \{2, 3\}$, with a polygonal or polyhedral boundary Γ , $\nu > 0$ the kinematic viscosity and f a given body force. Then, a pair (X^N, M^N) of conforming finite element spaces for the approximation of the velocity $u \in (H_0^1(\Omega))^d$ and the pressure $p \in L_0^2(\Omega)$ is said to fulfill the *discrete inf-sup condition* if the following estimate holds:

$$\exists \beta > 0 : \quad \inf_{q^N \in M^N \setminus \{0\}} \sup_{v^N \in X^N \setminus \{0\}} \frac{(\operatorname{div} v^N, q^N)}{|v^N|_{1,\Omega} \|q^N\|_{0,\Omega}} \geq \beta. \tag{0.2}$$

For the stability and accuracy of the solution $(u^N, p^N) \in X^N \times M^N$ of the discrete Stokes problem, it is desirable that the inf-sup constant β in (0.2) is independent of the mesh size and the polynomial degree. Note that the same requirement arises for solving the incompressible Navier-Stokes equations.

The outline of this paper is as follows. In Section 1, we introduce the notations, the concept of multilevel adaptive grids and the used finite element spaces. In Section 2.1, we present the global and local inf-sup conditions that are used for the technique of Boland and Nicolaides. The modified Piola transformation is

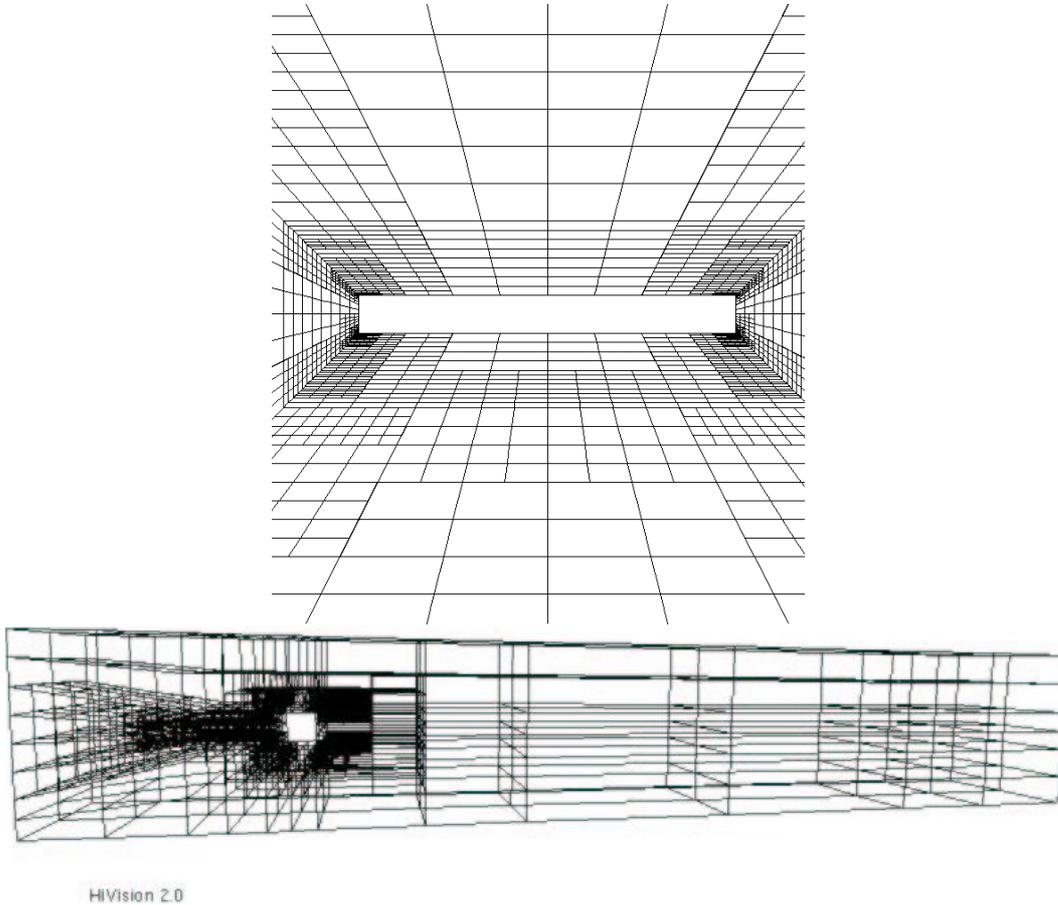


FIGURE 1. Typical local refined grids with hanging nodes; (top) two-dimensional grid for the computation of an exterior flow problem [7]; (bottom) three-dimensional grid for a channel flow problem [12].

introduced and studied in Section 2.2. In Section 2.3, we prove the local inf-sup condition. Finally, we summarize our main result in Section 2.4.

1. PRELIMINARIES AND NOTATIONS

1.1. General notations

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with a polygonal or polyhedral boundary. For a set $G \subset \Omega$, let $|\cdot|_{m,p,G}$ and $\|\cdot\|_{m,p,G}$ denote the usual semi-norm and norm in the Sobolev spaces $W^{m,p}(G)$ and $(W^{m,p}(G))^d$, respectively. For the Hilbert spaces $H^m(G)$ and $(H^m(G))^d$, we omit the index p and denote the semi-norm and norm by $|\cdot|_{m,G}$ and $\|\cdot\|_{m,G}$. The scalar product in $L^2(G)$ and $(L^2(G))^d$ will be denoted by $(\cdot, \cdot)_G$. Let $L_0^2(G) \subset L^2(G)$ be the subspace of functions with an integral mean value of zero. By $\mathbb{P}_m(G)$ we denote the space of all polynomials on the domain $G \subset \mathbb{R}^d$ with total degree less than or equal to m and by $\mathbb{Q}_m(G)$ the space of those polynomials where the maximum power in each coordinate is less than or equal to m . For $p \in [1, \infty]$, both the usual p -norm of a vector in \mathbb{R}^d and the induced p -norm of a $d \times d$ -matrix are denoted by $\|\cdot\|_p$. For a set $G \subset \mathbb{R}^d$, we denote by $\text{int}(G)$ and \overline{G} the interior and closure of G , respectively. Throughout this paper, C will denote a generic constant which may have different values at different places. All these constants

occurring inside of any estimate will be – unless it is explicitly stated – independent of the local and global mesh parameters h_K and h and the polynomial degree vector \underline{r} which will be defined below.

1.2. Multilevel adaptive grids

Let the bounded domain $\Omega \subset \mathbb{R}^d$ be partitioned by a *grid* \mathcal{T}_h consisting of elements $K \in \mathcal{T}_h$ which are assumed to be open quadrilaterals in the 2D-case and open hexahedrons in the 3D-case such that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} \overline{K}$. For an element $K \in \mathcal{T}_h$, we denote by h_K the diameter of the element K . The *mesh size* h of \mathcal{T}_h is given by $h := \max_{K \in \mathcal{T}_h} h_K$. We denote by $F_K : \widehat{K} \rightarrow K$ the mapping between the reference element $\widehat{K} := (-1, +1)^d$ and the original element K . In the sequel of this paper, the mapping F_K is assumed to be multi-linear, *i.e.* $F_K \in (\mathbb{Q}_1(\widehat{K}))^d$. Therefore, the two dimensional faces of 3D hexahedral elements $K \in \mathcal{T}_h$ can be curved in general.

For our analysis, we assume that the mesh family $\{\mathcal{T}_h\}$ is *shape-regular* in the sense presented in [20], see also [16,19]. Let us recall the definition. By a Taylor expansion of $F_K(\hat{x})$ at the point $\hat{x} = 0$ we get

$$F_K(\hat{x}) = F_K(0) + B_K \hat{x} + G_K(\hat{x}), \tag{1.1}$$

with $B_K := DF_K(0)$ and $G_K(\hat{x}) := F_K(\hat{x}) - F_K(0) - B_K \hat{x}$. We denote by $\widehat{S} \subset \widehat{K}$ the d -simplex with the vertices $0, e_1, \dots, e_d$, where $e_i \in \mathbb{R}^d$ denotes the i -th unit vector. By S_K we denote the d -simplex which is the image of \widehat{S} under the affine mapping $\hat{x} \mapsto B_K \hat{x} + F_K(0)$. We assume that the simplices S_K satisfy the usual shape regularity condition

$$\frac{h_{S_K}}{\rho_{S_K}} \leq C_s \quad \forall K \in \mathcal{T}_h, \tag{1.2}$$

where h_{S_K} denotes the diameter of S_K and ρ_{S_K} the diameter of the largest ball that can be inscribed into S_K . For an element $K \in \mathcal{T}_h$ and $p \in [1, \infty]$, we define the quantity $\gamma_p(K)$ by

$$\gamma_p(K) := \sup_{\hat{x} \in \widehat{K}} \|B_K^{-1} DF_K(\hat{x}) - I\|_p. \tag{1.3}$$

$\gamma_p(K)$ is a measure of the deviation of K from a parallelogram or a parallelepiped, respectively. Note that $\gamma_p(K) = 0$ is equivalent to the fact that the mapping F_K is affine.

Definition 1. A family $\{\mathcal{T}_h\}$ of meshes consisting of quadrilateral or hexahedral elements is called **shape-regular** if there exist a fixed $p \in [1, \infty]$ and h -independent constants C_s and γ_0 such that (1.2) and the estimates

$$\gamma_p(K) \leq \gamma_0 < 1 \quad \forall K \in \mathcal{T}_h \tag{1.4}$$

are satisfied.

It has been proven in [20] that for a family $\{\mathcal{T}_h\}$ of meshes, which is shape-regular in the sense of Definition 1, the reference mappings $F_K : \widehat{K} \rightarrow K$ are bijective for all elements $K \in \mathcal{T}_h$. Furthermore, one can find sufficient conditions in [20], Section 2, that guarantee the fulfillment of the assumption (1.4). These conditions can be easily checked in practical computations. For example, in the 2D-case, the condition $\gamma_1(K) < 1$ is equivalent to the fact that the quadrilateral element K is convex. Our assumption that the meshes are shape-regular in the sense of Definition 1 guarantees that the reference mapping $F_K : \widehat{K} \rightarrow K$ satisfies for all $K \in \mathcal{T}_h$ the following properties (for the proof, see [15]):

$$\|DF_K(\hat{x})\|_\infty \leq Ch_K \quad \forall \hat{x} \in \widehat{K}, \tag{1.5}$$

$$Ch_K^d \leq |\det(DF_K(\hat{x}))| \leq C'h_K^d \quad \forall \hat{x} \in \widehat{K}. \tag{1.6}$$

In this paper, we study the case of hanging nodes, *i.e.*, the usual assumption of a *regular* grid \mathcal{T}_h has to be weakened. In the following, we will describe the type of grids that are treated in this paper. We assume that

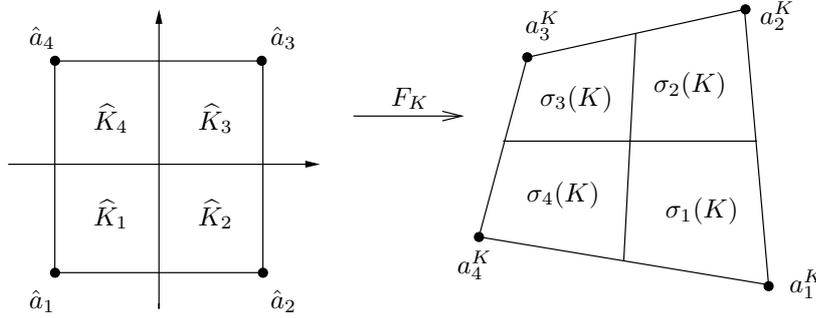


FIGURE 2. Refinement of element K into son-elements $\sigma_i(K)$, $i = 1, \dots, 2^d$.

\mathcal{T}_h is a *multilevel adaptive grid* generated by a refinement process in the following way. We start with an initial partition \mathcal{T}^0 of the domain Ω into elements $K \in \mathcal{T}^0$ of grid-level 0, *i.e.* $\Omega = \text{int}(\bigcup_{K \in \mathcal{T}^0} \overline{K})$. The grid \mathcal{T}^0 is assumed to be *regular* in the usual sense, *i.e.*, for any two different elements $K_1, K_2 \in \mathcal{T}^0$ the intersection $\overline{K_1} \cap \overline{K_2}$ is either empty or a common $(d - m)$ -dimensional face of K_1 and K_2 where $m \in \{1, \dots, d\}$. Now, starting with the elements $K \in \mathcal{T}^0$, an existing element K can be refined, *i.e.*, it can be split into 2^d many new elements called *son-elements* which are denoted by $\sigma_i(K)$, $i = 1, \dots, 2^d$, see Figure 2. These son-elements are defined as follows. Let \widehat{K}_i , $i \in \{1, \dots, 2^d\}$, be the son-elements of the reference element $\widehat{K} = (-1, 1)^d$ which are congruent d -cubes with $|\widehat{K}_i| = 1$. Then, the i -th son-element $\sigma_i(K)$ of the original element K is defined as $\sigma_i(K) := F_K(\widehat{K}_i)$ for all $i = 1, \dots, 2^d$. Note that in the 3D-case, even if an element K has only planar faces, it can happen that the son-elements $\sigma_i(K)$ have curved faces in the interior of K (see [18], Rem. 3.3, for an example). For a new element $K' = \sigma_i(K)$, we will say that K is the *father-element* of K' and we will write $K = \mathcal{F}(K')$. If an element K is refined then it is replaced in the partition of the domain Ω by the set of its son-elements $\sigma_i(K)$, $i = 1, \dots, 2^d$. The new elements can be refined repeatedly until the final partition \mathcal{T}_h of Ω is created. Examples of such grids used in practical computations can be found in [13,14].

Definition 2. For an element $K \in \mathcal{T}_h$, generated from the initial grid \mathcal{T}^0 by the refinement process described above, we define the **refinement level** $\ell(K)$ as $\ell(K) := 0$ if $K \in \mathcal{T}^0$ and $\ell(K) := m \geq 1$ if there exists a chain of m father-elements K_i , $i = m - 1, \dots, 0$, starting from $K_m := K$ and defined by $K_{i-1} := \mathcal{F}(K_i)$ for $i = m, \dots, 1$, such that $K_0 \in \mathcal{T}^0$. The grid \mathcal{T}_h , generated by the refinement process defined above, is called **1-irregular** if the condition

$$|\ell(K) - \ell(K')| \leq 1 \tag{1.7}$$

holds for any pair of different adjacent elements $K, K' \in \mathcal{T}_h$ where $\partial K \cap \partial K'$ is a one- or two-dimensional manifold.

Note that, in the 3D-case, the condition (1.7) has to be satisfied if the elements K and K' have a common edge or a common face. The *refinement level* $\ell(K)$ is equal to the number of refinement steps that are needed to generate the element K from an element of the coarsest grid \mathcal{T}^0 .

Our concept of 1-irregular grids is a special case of the concept of *K-meshes* introduced by Babuška and Miller [3]. In many practical computations, 1-irregular grids are used. They are sufficient to describe complicated structures in a reasonable way, see *e.g.* [13,14]. An example that shows the capabilities of 1-irregular quadrilateral grids to resolve an inner fine scale structure is presented in Figure 3.

An advantage of multilevel adaptive grids is that the reference mapping F_K behaves – up to a power of the mesh size h – like an affine mapping. This is one of the key points of the analysis in this paper. We have the following result.

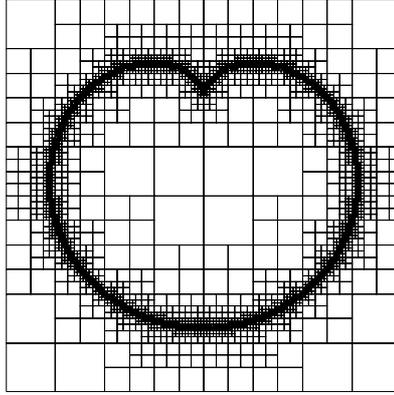


FIGURE 3. Resolution of an inner fine scale structure by a multilevel adaptive 1-irregular grid.

Lemma 3. *Let $\{\mathcal{T}_h\}$ be a family of multilevel adaptive grids generated from a shape-regular coarse grid \mathcal{T}^0 as described above. Then, there exist constants $C_1(m, \mathcal{T}^0)$ and $C_2(\mathcal{T}^0)$ such that the following estimates are satisfied for all $K \in \mathcal{T}_h$*

$$|F_K|_{m, \infty, \widehat{K}} \leq C_1(m, \mathcal{T}^0) h_K^m \quad \forall m \in \{0, 1, 2, 3\}, \quad (1.8)$$

$$\|J_K(\widehat{x}) - B_K\|_p \leq C_2(\mathcal{T}^0) h_K^2 \quad \forall p \in [1, \infty], \quad \forall \widehat{x} \in \widehat{K}, \quad (1.9)$$

where $J_K(\widehat{x}) := DF_K(\widehat{x})$ and $B_K := DF_K(0)$.

Proof. See Lemma 5 in [19]. □

1.3. Finite element spaces

Let \underline{r} denote the degree vector $\underline{r} := \{r_K : K \in \mathcal{T}_h\}$ which contains the polynomial degrees r_K of the velocity approximation on each element K . In this paper, we assume that

$$r_K \geq 2 \quad \forall K \in \mathcal{T}_h. \quad (1.10)$$

The finite element space of the velocity $X^N \subset (H_0^1(\Omega))^d$ associated with the grid \mathcal{T}_h and the degree vector \underline{r} is defined as $X^N := (S^N)^d$ with the scalar finite element space

$$S^N := S(\Omega; \underline{r}, \mathcal{T}_h) := \{\phi \in H_0^1(\Omega) : \phi|_K \circ F_K \in \mathbb{Q}_{r_K}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}. \quad (1.11)$$

The finite element space $M^N \subset L_0^2(\Omega)$ for the approximation of the pressure is chosen as the following mapped space

$$M^N := M(\Omega; \underline{r}, \mathcal{T}_h) := \{q \in L_0^2(\Omega) : q|_K \circ F_K \in \mathbb{P}_{r_K-1}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}. \quad (1.12)$$

Then, the discrete Stokes problem reads: Find $(u^N, p^N) \in X^N \times M^N$ such that

$$\begin{aligned} \nu(\nabla u^N, \nabla v^N)_\Omega - (p^N, \nabla \cdot v^N)_\Omega &= (f, v^N)_\Omega & \forall v^N \in X^N \\ (q^N, \nabla \cdot u^N)_\Omega &= 0 & \forall q^N \in M^N. \end{aligned} \quad (1.13)$$

For the subsequent analysis we need also the finite element space for the velocity with the polynomial order of 2, i.e. $X_h^2 := (S_h^2)^d$ based on the scalar finite element space

$$S_h^2 := \{\phi \in H_0^1(\Omega) : \phi|_K \circ F_K \in \mathbb{Q}_2(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}, \quad (1.14)$$

as well as the finite element space of the piecewise constant pressure functions

$$M_h^0 := \{q \in L_0^2(\Omega) : q|_K \in \mathbb{P}_0(K) \quad \forall K \in \mathcal{T}_h\}. \tag{1.15}$$

Some attention is required to ensure interelement continuity in (1.11) and (1.14) in the case of hanging nodes or if r_K is different on two adjacent cells of an inner element face. We refer to [22] for a detailed treatment of this issue. Note that, due to the assumption (1.10), it holds $X_h^2 \subset X^N$.

2. PROVING THE INF-SUP CONDITION

2.1. Global and local inf-sup conditions

Our approach to prove the inf-sup condition (0.2) is based on the technique proposed by Boland and Nicolaides [6] (see also [11], Sect. II.1.4). For the partitioning of the domain Ω into non-overlapping subdomains, we take the elements $K \in \mathcal{T}_h$. According to the finite element spaces $X^N = (S^N)^d$ and M^N defined in (1.11) and (1.12), respectively, we define for each element $K \in \mathcal{T}_h$ the following local spaces

$$X^N(K) := \left\{ v \in (H_0^1(K))^d : v \circ F_K \in (\mathbb{Q}_{r_K}(\widehat{K}))^d \right\}, \tag{2.1}$$

$$M^N(K) := \left\{ q \in L_0^2(K) : q \circ F_K \in \mathbb{P}_{r_K-1}(\widehat{K}) \right\}. \tag{2.2}$$

We say that the finite element spaces X^N and M^N fulfill the *local inf-sup condition* uniformly with respect to the mesh size h and the polynomial degree vector \underline{r} if there exists a constant $\beta_0 > 0$, such that it holds

$$\inf_{q^N \in M^N(K) \setminus \{0\}} \sup_{v^N \in X^N(K) \setminus \{0\}} \frac{(\nabla \cdot v^N, q^N)_K}{|v^N|_{1,K} \|q^N\|_{0,K}} \geq \beta_0, \quad \forall K \in \mathcal{T}_h. \tag{2.3}$$

Now, in order to prove the global inf-sup condition for our finite element spaces $X^N = (S^N)^d$ and M^N we only have to verify the local inf-sup condition (2.3) and the following global inf-sup condition for a suitable pair of low order subspaces (\overline{X}^N, M_h^0) .

Lemma 4. *Let the local inf-sup condition (2.3) be fulfilled uniformly with respect to the mesh size h and the degree vector \underline{r} . Furthermore, let there exist a subspace $\overline{X}^N \subset X^N$ such that the pair of finite element spaces (\overline{X}^N, M_h^0) fulfills the global inf-sup condition*

$$\inf_{q_h \in M_h^0 \setminus \{0\}} \sup_{v_h \in \overline{X}^N \setminus \{0\}} \frac{(\nabla \cdot v_h, q_h)_\Omega}{|v_h|_{1,\Omega} \|q_h\|_{0,\Omega}} \geq \bar{\beta}, \tag{2.4}$$

with a constant $\bar{\beta} > 0$ independent of h and \underline{r} . Then, the pair (X^N, M^N) satisfies the global inf-sup condition (0.2) with a constant $\beta > 0$ which depends on $\bar{\beta}$ and β_0 but not on h and \underline{r} .

Proof. See Theorem II.1.12 in [11]. □

The global inf-sup condition (2.4) of the pair of spaces (\overline{X}^N, M_h^0) can be proved for the second order subspace $\overline{X}^N = X_h^2 \subset X^N$.

Lemma 5. *Let \mathcal{T}_h be a multilevel adaptive grid, which is assumed to be 1-irregular and shape-regular. Then, the pair of finite element spaces (\overline{X}^N, M_h^0) with $\overline{X}^N := X_h^2$ satisfies the global inf-sup condition (2.4) with a constant $\bar{\beta} > 0$ independent of h and \underline{r} .*

Proof. See [16], Lemma 9. □

2.2. Modified Piola transformation

The Piola transformation is a well known tool to establish the local inf-sup condition [1,10]. For an element $K \in \mathcal{T}_h$ and a given function $\hat{v} \in (H^1(\hat{K}))^d$, the Piola transformation $\mathcal{P}_K \hat{v} \in (H^1(K))^d$ is defined by

$$(\mathcal{P}_K \hat{v})(x) := \frac{1}{\det(J_K(\hat{x}))} J_K(\hat{x}) \hat{v}(\hat{x}) \quad \forall x \in K, \tag{2.5}$$

where $J_K(\hat{x}) := DF_K(\hat{x})$ and $\hat{x} := F_K^{-1}(x)$. Due to the property (1.6) of grids which are shape-regular in the sense of Definition 1, the expression in (2.5) is well defined. The Piola transformation has the following essential property.

Lemma 6. *For a given element $K \in \mathcal{T}_h$ and functions $\hat{v} \in (H_0^1(\hat{K}))^d$ and $\hat{q} \in H^1(\hat{K})$, let $v \in (H_0^1(K))^d$ and $q \in H^1(K)$, be the associated functions $v := \mathcal{P}_K \hat{v}$ and $q := \hat{q} \circ F_K^{-1}$, respectively. Then, it holds*

$$(\operatorname{div} v, q)_K = (\operatorname{div} \hat{v}, \hat{q})_{\hat{K}}. \tag{2.6}$$

Proof. We easily obtain (2.6) if we apply partial integration and transform the integral over K into an integral over \hat{K} , see [10]. □

For our local velocity space $X^N(K)$ defined in (2.1), a problem in the proof of the local inf-sup condition is the fact that for a given function $\hat{v} \in (\mathbb{Q}_{r_K}(\hat{K}) \cap H_0^1(\hat{K}))^d$ the transformed function $\mathcal{P}_K \hat{v}$ in general is not contained in $X^N(K)$. Due to this fact, the local velocity space in [1,10] has been defined as the image of the space $(\mathbb{Q}_{r_K}(\hat{K}) \cap H_0^1(\hat{K}))^d$ under the operator \mathcal{P}_K . However, we will not use this modification of the internal finite element basis functions since it complicates the implementation, in particular for the derivatives of the velocity.

The idea to overcome this problem is to use a *modified Piola transformation* which is close to the original one. For an element $K \in \mathcal{T}_h$ and a given function $\hat{v} \in (H^1(\hat{K}))^d$, we define the modified Piola transformation $\tilde{\mathcal{P}}_K \hat{v} \in (H^1(K))^d$ by

$$(\tilde{\mathcal{P}}_K \hat{v})(x) := \frac{1}{\det(B_K)} B_K \hat{v}(F_K^{-1}(x)) \quad \forall x \in K, \tag{2.7}$$

where $B_K := DF_K(0)$. Then, one easily verifies that $\tilde{\mathcal{P}}_K \hat{v} \in X^N(K)$ for all functions $\hat{v} \in (\mathbb{Q}_{r_K}(\hat{K}) \cap H_0^1(\hat{K}))^d$. Moreover, we can prove the following results.

Lemma 7. *Let $\{\mathcal{T}_h\}$ be a family of shape-regular and 1-irregular multilevel adaptive grids generated from a coarse grid \mathcal{T}^0 . Then, for all functions $\hat{v} \in (H_0^1(\hat{K}))^d$ and elements $K \in \mathcal{T}_h$, the following estimates are satisfied*

$$|\tilde{\mathcal{P}}_K \hat{v}|_{1,K} \leq C_3 h_K^{-d/2} |\hat{v}|_{1,\hat{K}}, \tag{2.8}$$

$$|\hat{v}|_{1,\hat{K}} \leq C h_K^{d/2} |\tilde{\mathcal{P}}_K \hat{v}|_{1,K}, \tag{2.9}$$

$$|\mathcal{P}_K \hat{v} - \tilde{\mathcal{P}}_K \hat{v}|_{1,K} \leq C_4(\mathcal{T}^0) h_K |\tilde{\mathcal{P}}_K \hat{v}|_{1,K}, \tag{2.10}$$

where the constants are independent of \hat{v} and the local mesh size h_K .

Proof. From the definition (2.7) of the modified Piola transformation and the estimates (1.6) and (1.5) it follows

$$|\tilde{\mathcal{P}}_K \hat{v}|_{1,K} \leq C h_K^{1-d} |\hat{v} \circ F_K^{-1}|_{1,K}.$$

Furthermore, we have

$$|\hat{v} \circ F_K^{-1}|_{1,K} \leq C |\det(DF_K)|_{0,\infty,\hat{K}}^{1/2} |F_K^{-1}|_{1,\infty,K} |\hat{v}|_{1,\hat{K}} \leq C h_K^{d/2-1} |\hat{v}|_{1,\hat{K}}, \tag{2.11}$$

which yields the estimate (2.8). From

$$\widehat{v}(\widehat{x}) = \det(B_K) B_K^{-1} (\widetilde{\mathcal{P}}_K \widehat{v})(F_K(\widehat{x}))$$

we get by means of (1.6) and (1.5) the estimates

$$|\widehat{v}|_{1,\widehat{K}} \leq Ch_K^{d-1} |(\widetilde{\mathcal{P}}_K \widehat{v}) \circ F_K|_{1,\widehat{K}} \leq Ch_K^{d-1} |\det(DF_K^{-1})|_{0,\infty,K}^{1/2} |F_K|_{1,\infty,\widehat{K}} |\widetilde{\mathcal{P}}_K \widehat{v}|_{1,K},$$

which imply (2.9).

To prove (2.10) we start with the splitting

$$\mathcal{P}_K \widehat{v} - \widetilde{\mathcal{P}}_K \widehat{v} = \underbrace{\left\{ \frac{1}{\det(J_K)} - \frac{1}{\det(B_K)} \right\}}_{=:T_1} B_K (\widehat{v} \circ F_K^{-1}) + \underbrace{\frac{1}{\det(J_K)} \{J_K - B_K\}}_{=:T_2} (\widehat{v} \circ F_K^{-1}).$$

Let us define the function $g_K : \widehat{K} \rightarrow \mathbb{R}$ as

$$g_K := \frac{1}{\det(J_K)} - \frac{1}{\det(B_K)}.$$

Using (1.5) with $\widehat{x} = 0$ and (2.11), we get

$$|T_1|_{1,K} \leq C \|g_K\|_{0,\infty,\widehat{K}} h_K^{d/2} |\widehat{v}|_{1,\widehat{K}} + C |g_K|_{1,\infty,\widehat{K}} |F_K^{-1}|_{1,\infty,K} h_K^{d/2+1} \|\widehat{v}\|_{0,\widehat{K}}.$$

If we apply Friedrichs inequality and the estimate $|F_K^{-1}|_{1,\infty,K} \leq Ch_K^{-1}$, which follows from (1.5) and (1.6), we conclude

$$|T_1|_{1,K} \leq C \left\{ \|g_K\|_{0,\infty,\widehat{K}} + |g_K|_{1,\infty,\widehat{K}} \right\} h_K^{d/2} |\widehat{v}|_{1,\widehat{K}}. \quad (2.12)$$

For the function g_K we have

$$\begin{aligned} |g_K|_{1,\infty,\widehat{K}} &= \left| \frac{1}{\det(J_K)} \right|_{1,\infty,\widehat{K}} \leq C \left| \frac{1}{(\det(J_K))^2} \right|_{0,\infty,\widehat{K}} |\det(J_K)|_{1,\infty,\widehat{K}} \\ &\leq Ch_K^{-2d} |F_K|_{2,\infty,\widehat{K}} |F_K|_{1,\infty,\widehat{K}}^{d-1}. \end{aligned}$$

Using the estimate (1.8) for the multilevel adaptive grid \mathcal{T}_h , we obtain

$$|g_K|_{1,\infty,\widehat{K}} \leq C(\mathcal{T}^0) h_K^{1-d}. \quad (2.13)$$

From $g_K(0) = 0$ we get

$$\|g_K\|_{0,\infty,\widehat{K}} \leq \text{diam}(\widehat{K}) |g_K|_{1,\infty,\widehat{K}},$$

which yields with (2.13)

$$\|g_K\|_{0,\infty,\widehat{K}} \leq C(\mathcal{T}^0) h_K^{1-d}. \quad (2.14)$$

If we combine (2.12) with (2.14) and (2.13) we get the estimate

$$|T_1|_{1,K} \leq C(\mathcal{T}^0) h_K^{1-d/2} |\widehat{v}|_{1,\widehat{K}}. \quad (2.15)$$

For the estimation of the term T_2 , we introduce the functions $\phi_K : \widehat{K} \rightarrow \mathbb{R}$ and $G_K : \widehat{K} \rightarrow \mathbb{R}^{d \times d}$ defined as

$$\phi_K(\widehat{x}) := \frac{1}{\det(J_K(\widehat{x}))}, \quad G_K(\widehat{x}) := J_K(\widehat{x}) - B_K.$$

Now, T_2 can be estimated as follows

$$\begin{aligned}
 |T_2|_{1,K} &\leq Ch_K^{d/2} \left\{ |\phi_K|_{1,\infty,\hat{K}} \|G_K\|_{0,\infty,\hat{K}} \|\widehat{v}\|_{0,\hat{K}} + \|\phi_K\|_{0,\infty,\hat{K}} |G_K|_{1,\infty,\hat{K}} \|\widehat{v}\|_{0,\hat{K}} \right. \\
 &\quad \left. + \|\phi_K\|_{0,\infty,\hat{K}} \|G_K\|_{0,\infty,\hat{K}} |\widehat{v}|_{1,\hat{K}} \right\} |F_K^{-1}|_{1,\infty,K},
 \end{aligned}
 \tag{2.16}$$

where the norm and semi norm of G_K are defined by

$$\|G_K\|_{0,\infty,\hat{K}} := \sup_{\widehat{x} \in \hat{K}} \|G_K(\widehat{x})\|_\infty, \quad |G_K|_{1,\infty,\hat{K}} := \max_{j=1,\dots,d} \sup_{\widehat{x} \in \hat{K}} \left\| \frac{\partial G_K}{\partial \widehat{x}_j}(\widehat{x}) \right\|_\infty.$$

For the function ϕ_K , we have by (1.6) and (2.13)

$$\|\phi_K\|_{0,\infty,\hat{K}} \leq Ch_K^{-d}, \quad |\phi_K|_{1,\infty,\hat{K}} = |g_K|_{1,\infty,\hat{K}} \leq C(\mathcal{T}^0)h_K^{1-d}.$$

Using (1.9) and (1.8), the matrix function G_K can be estimated as follows

$$\|G_K\|_{0,\infty,\hat{K}} \leq C_2(\mathcal{T}^0)h_K^2, \quad |G_K|_{1,\infty,\hat{K}} \leq C|F_K|_{2,\infty,\hat{K}} \leq C(\mathcal{T}^0)h_K^2.$$

If we apply Friedrichs inequality and the estimate $|F_K^{-1}|_{1,\infty,K} \leq Ch_K^{-1}$, we obtain from (2.16)

$$\begin{aligned}
 |T_2|_{1,K} &\leq C(\mathcal{T}^0)h_K^{d/2} \left\{ h_K^{1-d} h_K^2 + h_K^{-d} h_K^2 + h_K^{-d} h_K^2 \right\} h_K^{-1} |\widehat{v}|_{1,\hat{K}}, \\
 &\leq C(\mathcal{T}^0)h_K^{1-d/2} |\widehat{v}|_{1,\hat{K}}.
 \end{aligned}
 \tag{2.17}$$

Now, the assertion (2.10) follows from (2.15), (2.17) and (2.9). □

2.3. Proof of the local inf-sup condition

For our pair of finite element spaces, the following local inf-sup condition is known.

Lemma 8. *Let \mathcal{T}_h be a shape-regular grid. Then, for each element $K \in \mathcal{T}_h$, the pair $(X^N(K), M^N(K))$ of the mapped local finite element spaces defined in (2.1) and (2.2), satisfies the local inf-sup condition (2.3) with a constant $\beta_0 = \lambda(r_K) > 0$ which depends on the local degree r_K but which is independent of K and the mesh size h .*

Proof. See the proof of Theorem 8 in [20]. □

This result implies a local inf-sup condition (2.3) with a constant

$$\beta_0 = \beta_0^*(r^*) := \min_{r=1,\dots,r^*} \lambda(r) \quad \text{where} \quad r^* := \max_{K \in \mathcal{T}_h} r_K.
 \tag{2.18}$$

However, the dependence of $\beta_0^*(r^*)$ on the maximum polynomial degree r^* has not been investigated in [20]. For a grid \mathcal{T}_h consisting of affine equivalent elements, the estimate $\beta_0^*(r^*) \geq C(r^*)^{-(d-1)/2}$ has been shown in [24]. In the following, we will prove that, for a sufficiently small mesh size, the local inf-sup condition (2.3) holds with a constant β_0 which is independent of the mesh size h and the polynomial degree vector \underline{r} .

A main ingredient of our analysis is the following result of Bernardi and Maday which we will apply on the reference element.

Lemma 9. For $\widehat{K} = (-1, 1)^d$ and an arbitrary integer $r \geq 2$, let $\widehat{X}_r := (\mathbb{Q}_r(\widehat{K}) \cap H_0^1(\widehat{K}))^d$ and $\widehat{M}_r := \mathbb{P}_{r-1}(\widehat{K}) \cap L_0^2(\widehat{K})$. Then, it holds

$$\forall \widehat{q} \in \widehat{M}_r \quad \exists \widehat{v} \in \widehat{X}_r : \quad (\widehat{\operatorname{div}} \widehat{v}, \widehat{q})_{\widehat{K}} \geq \widehat{\beta}_0 |\widehat{v}|_{1, \widehat{K}} \|\widehat{q}\|_{0, \widehat{K}}, \quad (2.19)$$

with a constant $\widehat{\beta}_0 > 0$ which is independent of the polynomial degree r .

Proof. See [4]. □

Now, using the original and modified Piola transformation, the estimate (2.19) can be transformed to the pair of the local finite element spaces $(X^N(K), M^N(K))$ defined in (2.1) and (2.2).

Lemma 10. Let $\{\mathcal{T}_h\}$ be a family of shape-regular and 1-irregular multilevel adaptive grids generated from a coarse grid \mathcal{T}^0 . Then, there exists a sufficiently small mesh size $h_0 = h_0(\mathcal{T}^0)$ such that for all grids \mathcal{T}_h with $h \leq h_0$ and all $K \in \mathcal{T}_h$ it holds

$$\forall q \in M^N(K) \quad \exists v \in X^N(K) : \quad (\operatorname{div} v, q)_K \geq \beta_0 |v|_{1, K} \|q\|_{0, K}, \quad (2.20)$$

with a constant $\beta_0 > 0$ which is independent of the mesh size and the polynomial degree vector \underline{r} . Therefore, the local inf-sup condition (2.3) is satisfied with this constant β_0 .

Proof. For a given $q \in M^N(K)$, let $\widehat{q} := q \circ F_K \in \widehat{M}_r$. Due to Lemma 9, there exists a $\widehat{v} \in \widehat{X}_r$ satisfying (2.19). Choose $v := h_K^d \mathcal{P}_K \widehat{v}$. Then, it holds

$$(\operatorname{div} v, q)_K = h_K^d (\widehat{\operatorname{div}} \widehat{v}, \widehat{q})_{\widehat{K}} \geq \widehat{\beta}_0 h_K^d |\widehat{v}|_{1, \widehat{K}} \|\widehat{q}\|_{0, \widehat{K}}.$$

Using a standard argument for the estimation between the norms of \widehat{q} and q we obtain

$$(\operatorname{div} v, q)_K \geq C_5 h_K^{d/2} |\widehat{v}|_{1, \widehat{K}} \|q\|_{0, K}, \quad (2.21)$$

with a constant C_5 which is independent of the mesh size and the degree vector \underline{r} . Since v in general is not contained in $X^N(K)$, we define the function $\widetilde{v} := h_K^d \widetilde{\mathcal{P}}_K \widehat{v}$ which satisfies $\widetilde{v} \in X^N(K)$. From Lemma 7 we get

$$|\widehat{v}|_{1, \widehat{K}} \geq C_3^{-1} h_K^{-d/2} |\widetilde{v}|_{1, K}, \quad (2.22)$$

$$|(\operatorname{div}(\widetilde{v} - v), q)_K| \leq C_4(\mathcal{T}^0) \sqrt{d} h_K |\widetilde{v}|_{1, K} \|q\|_{0, K}. \quad (2.23)$$

Using (2.21)-(2.23), this implies

$$\begin{aligned} (\operatorname{div} \widetilde{v}, q)_K &= (\operatorname{div} v, q)_K + (\operatorname{div}(\widetilde{v} - v), q)_K \\ &\geq C_5 h_K^{d/2} |\widehat{v}|_{1, \widehat{K}} \|q\|_{0, K} - C_4(\mathcal{T}^0) \sqrt{d} h_K |\widetilde{v}|_{1, K} \|q\|_{0, K} \\ &\geq \left(\frac{C_5}{C_3} - C_4(\mathcal{T}^0) \sqrt{d} h_K \right) |\widetilde{v}|_{1, K} \|q\|_{0, K} \\ &\geq \beta_0 |\widetilde{v}|_{1, K} \|q\|_{0, K} \end{aligned}$$

for all $K \in \mathcal{T}_h$ and a grid \mathcal{T}_h with $h \leq h_0 := C_5 / (2C_3 C_4(\mathcal{T}^0) \sqrt{d})$ where $\beta_0 = C_5 / (2C_3)$. Therefore, (2.20) is proved since $\widetilde{v} \in X^N(K)$. □

2.4. Summary

Now, we can state the main result on the discrete inf-sup condition (0.2) for our finite element spaces.

Theorem 11. *Let $\{\mathcal{T}_h\}$ be a family of shape-regular and 1-irregular multilevel adaptive grids generated from a coarse grid \mathcal{T}^0 . Then, there exists a sufficiently small mesh size $h_0 = h_0(\mathcal{T}^0)$ such that the following result holds. For the pair of finite element spaces (X^N, M^N) , defined in (1.11) and (1.12) with $X^N = (S^N)^d$ and the degree vector \underline{r} , the global inf-sup condition (0.2) is fulfilled with a constant $\beta > 0$ that satisfies the following estimates*

$$\beta \geq \begin{cases} \beta^*(r^*), & \forall \mathcal{T}_h, \underline{r} \text{ with } \max_{K \in \mathcal{T}_h} r_K \leq r^*, \\ \beta_1, & \forall \mathcal{T}_h, \underline{r} \text{ with } h \leq h_0, \end{cases} \quad (2.24)$$

where β_1 is a positive constant which is independent of the mesh size h and the degree vector \underline{r} .

Proof. Due to Lemma 4, it is sufficient to verify the global and local inf-sup condition. The global inf-sup condition is guaranteed by Lemma 5. Then, the first estimate in (2.24) is a consequence of Lemma 8 and (2.18). The second estimate in (2.24) follows from Lemma 10. \square

By Theorem 11, the uniform inf-sup stability of our hp -finite element pair on multilevel adaptive grids is guaranteed under the following hp -regime. For coarser grids \mathcal{T}_h with $h > h_0$, we do not allow the polynomial degree to be beyond a moderate degree r^* , i.e. $\max_{K \in \mathcal{T}_h} r_K \leq r^*$, but for finer grids \mathcal{T}_h with $h \leq h_0$ the polynomial degree can be chosen arbitrarily.

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