RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

P. CHÉVRIER
H. GALLEY

A Van Leer finite volume scheme for the Euler equations on unstructured meshes

RAIRO – Modélisation mathématique et analyse numérique, tome 27, nº 2 (1993), p. 183-201.
<http://www.numdam.org/item?id=M2AN_1993__27_2_183_0>

© AFCET, 1993, tous droits réservés.

L'accès aux archives de la revue « RAIRO – Modélisation mathématique et analyse numérique » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 27, n° 2, 1993, p. 183 à 201)

A VAN LEER FINITE VOLUME SCHEME FOR THE EULER EQUATIONS ON UNSTRUCTURED MESHES (*)

by P. CHÉVRIER (1) and H. GALLEY (1)

Communicated by R. TEMAM

Abstract. — *The feasibility of the Finite Volume method using a Van Leer scheme on irregular meshes made of quadrilaterals or triangles is shown for the Euler 2D equations. The results are compared with those of a first order scheme.*

Résumé. — *On propose une méthode de volumes finis de type Van Leer (du second ordre en espace sur des maillages réguliers) pour résoudre les équations d'Euler sur des maillages non structurés. Dans le cas du tube à choc, les résultats obtenus sont comparés avec ceux d'un schéma du premier ordre en espace. L'efficacité du schéma sur des maillages non structurés est mise en évidence sur un cas vraiment bidimensionnel.*

1. INTRODUCTION

We aim at solving numerically the two-dimensional equations of gas dynamics (Euler equations). A Finite Volume method, using a Van Leer-like scheme ([Van Leer], [Vila]) is described here. This method is both accurate and less burdensome when compared to a Finite Element method, and also applies to irregular or unstructured meshes.

Unstructured meshes are common practice for industrial cases, especially with complex domain shapes involving the use of an automatic mesh

Both authors are research engineers at the Merlin Gerin Research Center. During this work, completed in 1989, P. Chévrier was also a member of the laboratory LMC (team EDP) at Grenoble (France), where he was completing his doctoral thesis, involving numerical analysis of phenomena in a circuit breaker, under the direction of P. Baras.

(*) Received 18 November 1991, revised 12 June 1992.

(1) Centre de Recherches A2, Merlin Gerin, 38050 Grenoble Cedex, France.

generator. But mesh regularity has an important influence on the accuracy of the numerical schemes for the two-dimensional problems. The use of second order schemes in space (the second order is only obtained on regular meshes) is then desirable in order to limit numerical damping.

Finite Volume schemes which have been proposed by Leveque and Jouve-Le Floch (see [Leveque], [Jouve]) are based on the same ideas. However, the crucial step of the scheme (i.e. slopes computation), is notably different.

2. CONSERVATIVE FORM OF THE EULER EQUATIONS

The Euler equations model the dynamics of unviscous gases that do not conduct heat. We note

ρ = density, or volumic mass, of the gas.
u = x coordinate of the gas velocity.
v = y coordinate of the gas velocity.
e = inner energy of the gas.
T = temperature of the gas.
E = $\rho\left((u^2+v^2)/2+e\right)$ total energy of the gas.
P = pressure of the gas, verifying the state equation $P=f(\rho,T)$.

The system of conservation laws (mass, momentum, energy) is :

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

$$\frac{\partial \rho u}{\partial t}+\frac{\partial(\rho u^2+P)}{\partial x}+\frac{\partial \rho u v}{\partial y}=0$$

$$\frac{\partial \rho v}{\partial t}+\frac{\partial \rho u v}{\partial x}+\frac{\partial(\rho v^2+P)}{\partial y}=0$$

$$\frac{\partial E}{\partial t}+\frac{\partial((E+P)u)}{\partial x}+\frac{\partial((E+P)v)}{\partial y}=0\,.$$

This system can be written using the following form, said conservative form :

$$\frac{\partial(F_0(U))}{\partial t}+\frac{\partial(F_1(U))}{\partial x}+\frac{\partial(F_2(U))}{\partial y}=0$$

with

$$\begin{aligned} U &= (\rho, u, v, P)^t \\ F_0(U) &= (\rho, \rho u, \rho v, E)^t \\ F_1(U) &= (\rho u, P+\rho u^2, \rho u v, (P+E)\,u)^t \\ F_2(U) &= (\rho v, \rho u v, P+\rho v^2, (P+E)\,v)^t \end{aligned}$$

$F_0(U)$ represents the conservative variables of the two-dimensional system.

3. TWO-DIMENSIONAL VAN LEER SCHEME

Let us consider the two-dimensional Euler system of equations :

$$\frac{\partial(F_0(U))}{\partial t} + \frac{\partial(F_1(U))}{\partial x} + \frac{\partial(F_2(U))}{\partial y} = 0 .$$

Given a meshing of space with quadrilateral or triangular elements, and given the value of time t, we look for an approximate solution which is linear on each element and discontinuous from one element to another. This approximate solution function is determined by its mean value on each element (i.e., value at the center of the element) and by two slopes Px and Py, constant on each element.

We recall that the two-dimensional Godunov scheme looks for an approximate solution constant on each element, and discontinuous between two elements. Mean values (on each element) and fluxes are obtained as these of the Van Leer scheme (see [Godunov]). The only difference is the absence of slopes. Godunov scheme is a first order scheme on regular meshes of rectangles (*fig.* 1).

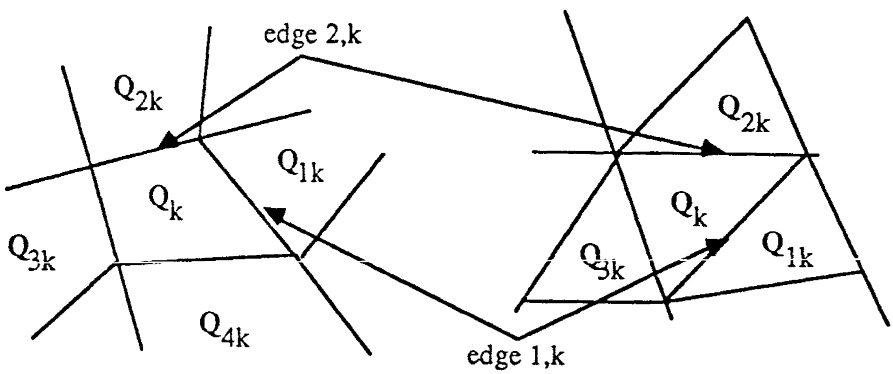


Figure 1.

The approximate solution U_h has on each element Q_k the following form : if $(x, y) \in Q_k$, then

$$U_h(x, y) = U_h(x_k, y_k) + (x - x_k) Px + (y - y_k) Py$$

where (x_k, y_k) are the coordinates of the center of the element Q_k.

On regular meshes of rectangles, second order accuracy in space is obtained if the slopes are well approximated. Each element has 12 degrees of freedom :

$U_h(x_k, y_k, t = t_n)$ which will be noted U_k^n (4 degrees of freedom)

Px_k^n (4 degrees of freedom, there is indeed one different slope for each unknown).

Py_k^n (idem).

We note :

S_k : surface of the element Q_k

$l_{i,k}$: length of the edge i, k

$U_{i,k,-}$: « left value » of the approximate solution at the center of the edge i, k

$U_{i,k,+}$: « right value » of the approximate solution at the center of the edge i, k

where (i, k) denotes the edge i of the element Q_k, and (ik) the element, other than Q_k, facing the edge (i, k). The « left » and « right » positions are defined for each edge by arbitrarily orientating the edge.

4. NUMERICAL SCHEME

Given the value of the approximate solution, linear on each element, at the time $t = t_n$, one wants to compute the value of the solution at $t = t_{n+1}$. This is done in two steps :

1) calculation of the mean values U_k^{n+1} on each element

2) computation of the slopes Px and Py on each element.

4.1. Calculation of the mean values U_k^{n+1} on each element

The system of equations is integrated on each Finite Volume $Q_k \times [t_n, t_{n+1}]$

$$\int_K \int_{t_n}^{t_{n+1}} \left[\frac{\partial F_0(U)}{\partial t} + \frac{\partial F_1(U)}{\partial x} + \frac{\partial F_2(U)}{\partial y} \right] dt\, d\omega = 0$$

which leads, after an Euler explicit discretization of the F_0 term (time term) and an integration by parts of the space term averaged in time, to the formula :

$$\frac{F_0(U_k^{n+1}) - F_0(U_k^n)}{t_{n+1} - t_n} = \frac{-1}{S_k} \left[\sum_{i=1}^{m} l_{i,k} \{ F_1(U_{i,k}) \cdot n_{xi,k} + F_2(U_{i,k}) \cdot n_{yi,k} \} \right]$$

with the notations of the following figure : (*fig.* 2).

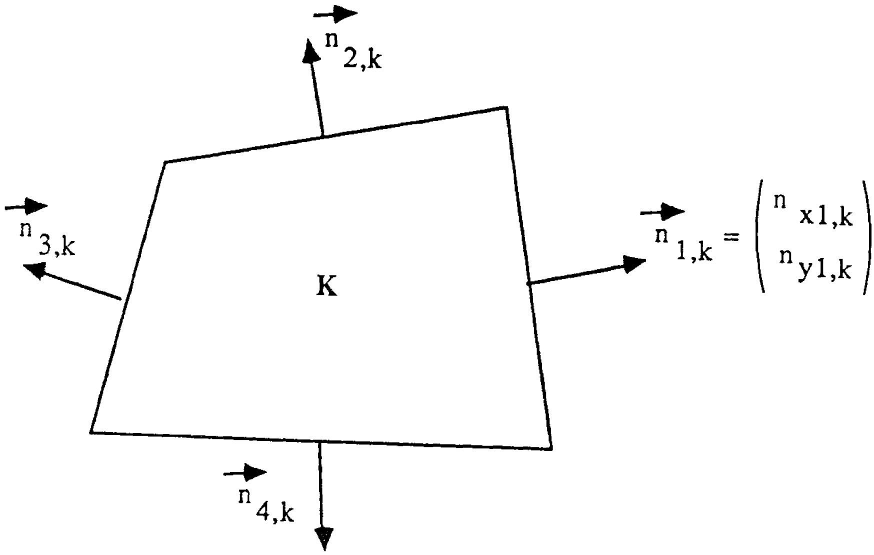


Figure 2.

This formula can be used to compute U^{n+1} from the value of U^n once one has solved the Riemann problem (see [Roe], [Collela]) on the m edges of each element ($m = 3$ or 4) in order to reckon the terms $F_j(U_{i,k})$. These Riemann problems are reduced, by local projection, to quasi-one-dimensional problems of Riemann on each edge (i, k) : find U such as :

$$\frac{\partial(F_0(U))}{\partial t} + \frac{\partial(F_1(U))}{\partial x} + \frac{\partial(F_2(U))}{\partial y} = 0$$

with

$$U(x, y, t_n) = \begin{cases} U^n_{i,k,-} & \text{if } (x, y) \text{ is on the right side of the edge } i, k \\ U^n_{i,k,+} & \text{if } (x, y) \text{ is on the left side of the edge } i, k\,. \end{cases}$$

4.2. Computation of the Slopes Px^{n+1} and Py^{n+1} ([Gallouët])

There are two steps for computing the slopes :

1. prediction : an approximate value of the slopes of the highest possible order is calculated
2. correction : to prevent scheme's instabilities, the values of the slopes previously computed are limited.

4.2.1. *Prediction step*

Let us consider an element Q_k. This element has m bordering elements Q_{ik}, $i : 1, m$. Given the $m + 1$ mean values $U^{n+1} k$, $U^{n+1} 1 k$, $U^{n+1} 2 k$, ..., $U^{n+1} mk$, we want to estimate the slopes on the element Q_k in a coherent way. With a given neighbour Q_{ik} we would like the slopes Px_k^{n+1} and Py_k^{n+1} to follow as closely as possible :

$$\Delta(\mathbf{U}, \mathbf{k}, \mathbf{i}, \mathbf{n+1}) = U_{ik}^{n+1} - U_k^{n+1} + Px_k^{n+1}(x - x_k) + Py_k^{n+1}(y - y_k) \approx \mathbf{0}$$

with (x, y) denoting the center of the element Q_{ik}, and (x_k, y_k) the center of the element Q_k. The expression $\Delta(U, k, i, n+1)$ represents the difference between the effective value of U_h at the center of the element Q_{ik} and the value obtained at the center of the element Q_{ik} by extending the function U_h defined on the element Q_k. So we choose to determine Px_k^{n+1} and Py_k^{n+1}, constant on each element Q_k, by minimizing the sum of squares of the differences Δ between the element Q_k and its neighbours, i.e. :

$$\sum_{i=1, m} [\Delta(U, k, i, n+1)]^2 .$$

The slopes Px_k^{n+1} and Py_k^{n+1} minimizing this expression are solution of the linear system 2×2 :

$$\frac{\partial \left\{ \sum_{i=1}^{m} [\Delta(U, k, i, n+1)]^2 \right\}}{\partial P_x} = 0 \quad \text{and} \quad \frac{\partial \left\{ \sum_{i=1}^{m} [\Delta(U, k, i, n+1)]^2 \right\}}{\partial P_y} = 0 .$$

4.2.2. *Correction step*

The correction step is the crux of the implementation of the two-dimensional Van Leer scheme. The slopes are corrected on each element Q_k, to make sure that the « inner value » (i.e., calculated with the mean value and the slopes of the element Q_k) of U_h at the center of every edge (i, k) belongs to the interval $[U_k^{n+1}, U_{ik}^{n+1}]$.

We use the following algorithm, said « coupled limitation », for each element Q_k, with (x_{ik}, y_{ik}) denoting the coordinates of the center of the element (ik) :

for $i = 1$ to m **do**

if the inner value on the edge (i, k) does not belong to $[U_k^{n+1}, U_{ik}^{n+1}]$ **then**

— find the scalar α belonging to [0, 1] such as :

$U_k + \alpha [(x_{ik} - x_k) Px_k^{n+1} + (y_{ik} - y_k) Py_k^{n+1}] = U_{ik}^{n+1}$ or U_k^{n+1}

— limit the slopes :

$$Px_k^{n+1} := \alpha \, . \, Px_k^{n+1}$$
$$Py_k^{n+1} := \alpha \, . \, Py_k^{n+1}$$

end if
end for
(*N.B.* : one different value of α actually corresponds to each component of U).

Remarks :

1. A pathological case appears in some cases of one-dimensional flow. As a matter of fact, small oscillations, in the direction perpendicular to the flow, may create an exaggerate limitation of the slopes ($Px = Py = 0!$) and the order of the scheme decreases to 1. For a flow in the (Ox) direction (for instance), very small oscillations, physically insignificant, in the (Oy) direction may create a local extremum along (Oy), which leads, with the algorithm described above, to $\alpha = 0$ and then $Px = Py = 0$, even with a rectangular mesh. To alleviate this problem, we uncouple the limitation of the slopes on thre areas of the mesh where the elements sides are parallel to the axis of coordinates (so, in the case of a local extremum along (Oy), one has $Py = 0$ but it has no influence on the value of Px).

2. To improve the numerical stability of the scheme, we multiply the slopes, when they have to be limited, by a scalar coefficient β belonging to the interval [1/2, 1]. The « limit the slopes » part of the previous algorithm becomes then :

$$Px_k^{n+1} := \alpha \, . \, \beta \, . \, Px_k^{n+1}$$
$$Py_k^{n+1} := \alpha \, . \, \beta \, . \, Py_k^{n+1} \, .$$

3. The slopes have been computed on the « physical variables » $U = (\rho, u, v, P)$ and not on the « conservative variables » $F_0(U) = (\rho, \rho u, \rho v, E)$. Indeed, our numerical tests have shown to us that contact discontinuities are then better approximated. It may be explained as follows :

Theory indicates, that a contact discontinuity is observed for density and temperature but not for velocity and pressure. This contact discontinuity will then occur with each of the conservative variables $(\rho, \rho u, \rho v, E)$, but only with one of the physical variables (ρ, u, v, P). When computing slopes on the conservative variables, small oscillations on pressure and velocity appear. In order to prevent these oscillations, the slopes have to be more limited, thus decreasing accuracy in space. When computing slopes on the physical variables, it is easier to get closer to the true solution for u, v or P near the contact discontinuity of ρ.

4. This scheme has been compared with a « streamline diffusion with upwind and shock capturing finite element method » (see [Chévrier]). We have shown that, as in the present case (remark 3), it is better to do the shock capturing on the physical, rather than conservative, variables. However, the finite volume scheme is more accurate and less expensive than the finite element one.

5. QUASI-ONE-DIMENSIONAL MODEL PROBLEM (PLANE 2D SHOCK TUBE)

We consider a plane rectangular area (called « tube »), full of perfect gas (air), with a length of 100 m and a height of 5 m, partitionned off at its center (where $x = 50$ m) (*fig.* 3).

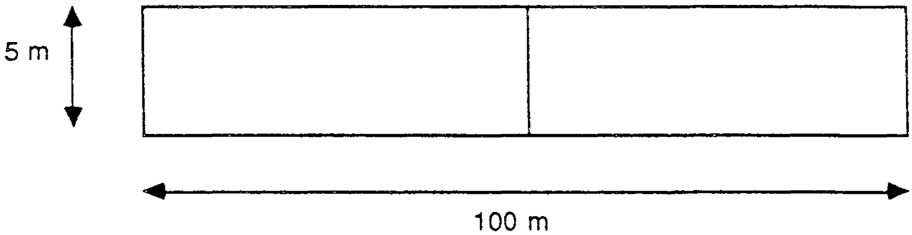


Figure 3.

In the left side of the tube (area 1) the initial conditions are the following :

gas velocity :	0 m/s
density :	12 kg/m^3
pressure :	10^6 Pa .

In the right side of the tube (area 2) the initial conditions are the following :

gas velocity :	0 m/s
density :	1.2 kg/m^3
pressure :	10^5 Pa .

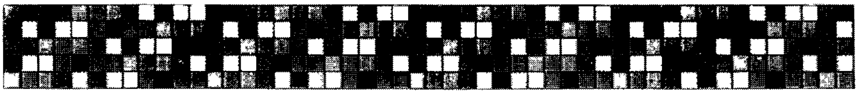
A the time $t = 0$ *s*, the central partition vanishes. One can see then the expansion of three waves : a rarefaction wave, a contact discontinuity, and a shock wave. Their analytical expression is well-known and can be calculated with the Rankine-Hugonniot relations ([Gilquin], [Smoller]).

We present the results obtained at $t = 0.06$ *s* with the Van Leer scheme described above, and with the first order Godunov scheme ($Px = Py = 0$ everywhere), along the longitudinal section of three different meshes :

1. regular mesh with 500 squares (CPU time Van Leer: 9 mn)
2. mesh with 592 irregular quadrilaterals (CPU time Van Leer: 17 mn)
3. mesh with 520 irregular triangles (CPU time Van Leer: 15 mn).

(For these calculations, the Godunov scheme is about 15 % faster than the Van Leer scheme).

5.1. Meshes (only the half is represented) (*fig.* 4)



Mesh with 500 squares

Mesh with 592 quadrilaterals

Mesh with 520 triangles

Figure 4.

5.2. Densities (*fig.* 5)

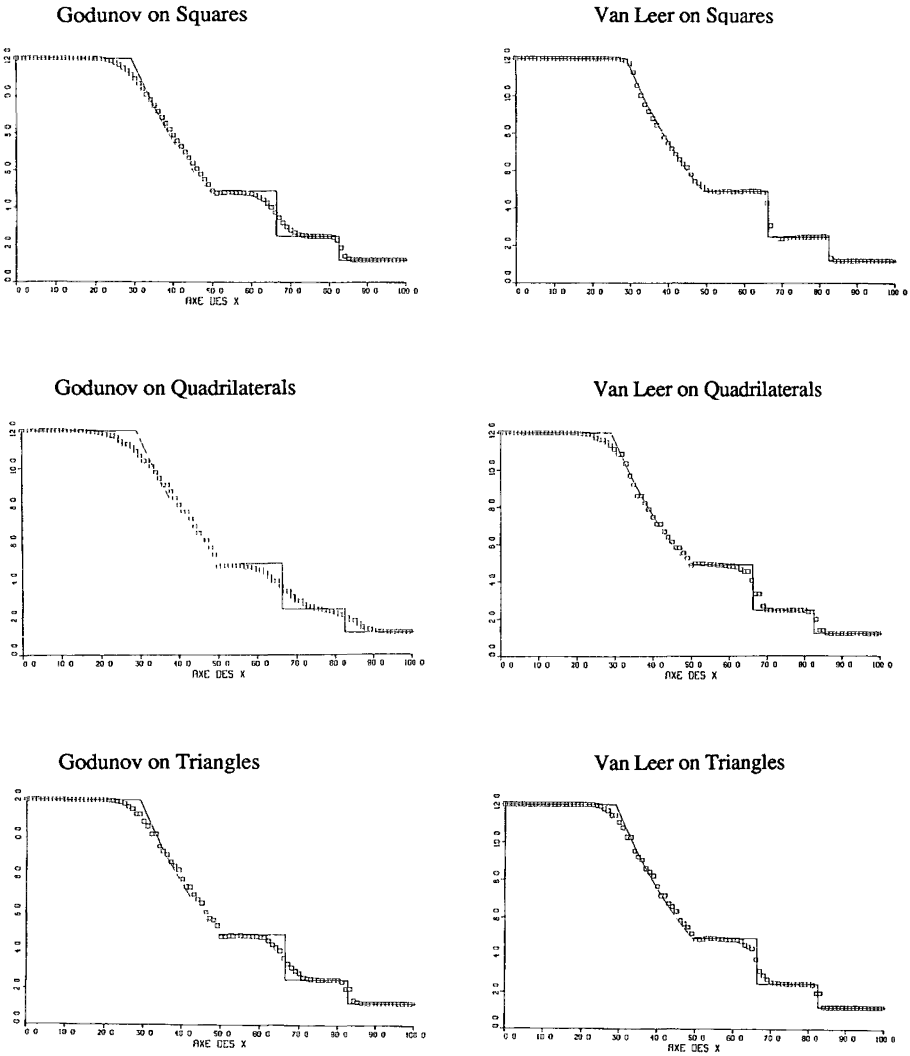


Figure 5.

5.3. Velocities (*fig.* 6)

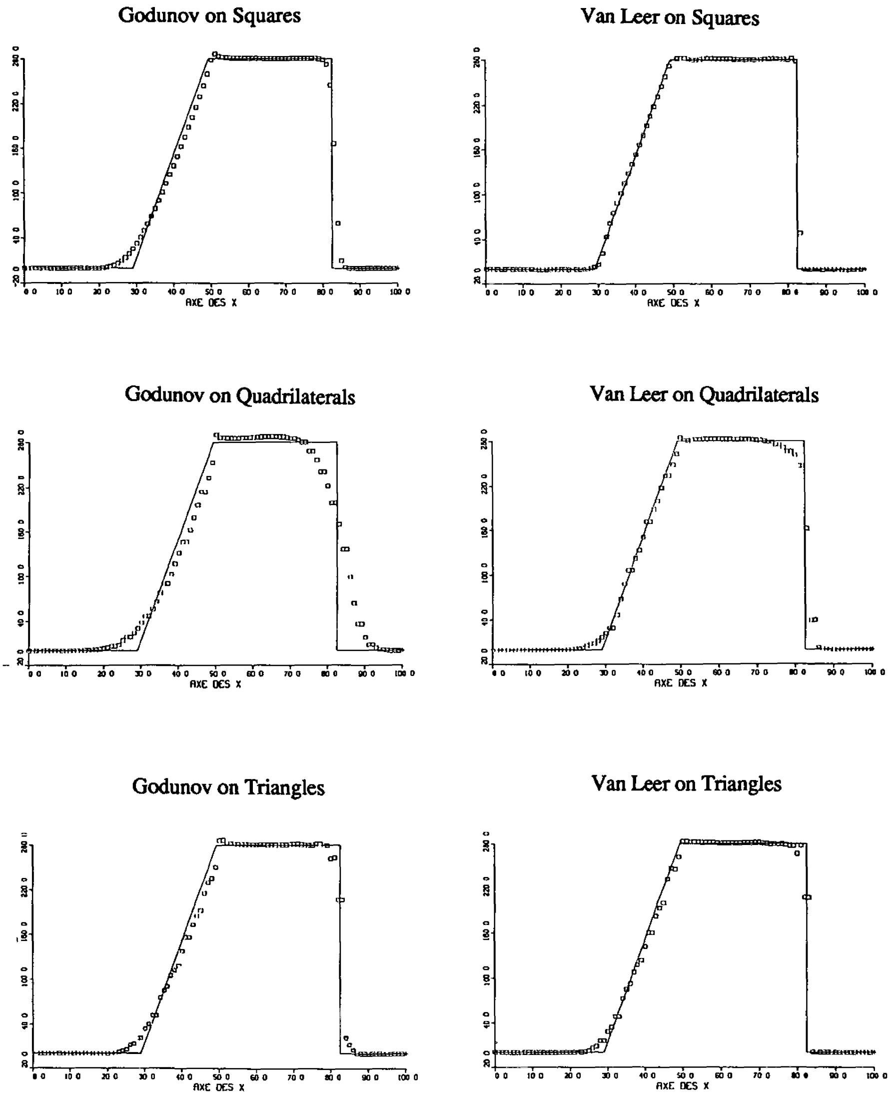


Figure 6.

5.4. **Pressures** (Van Leer only) (*fig.* 7)

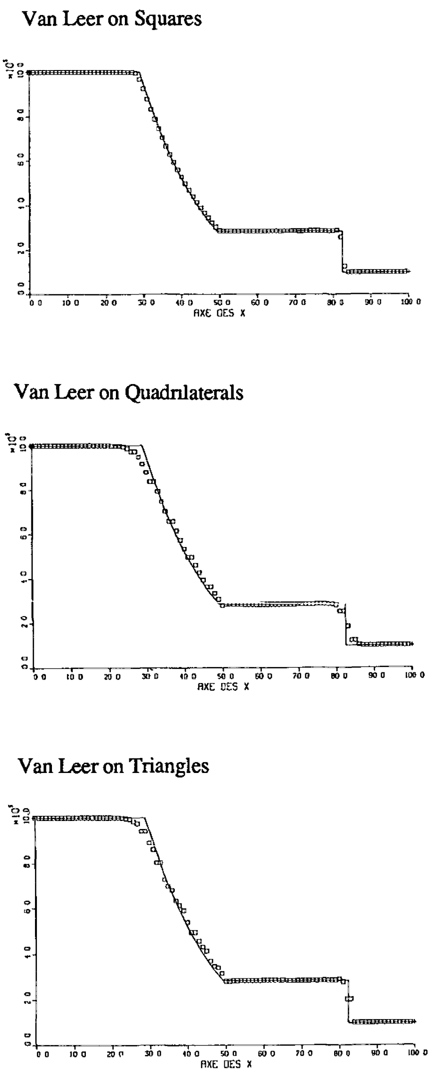
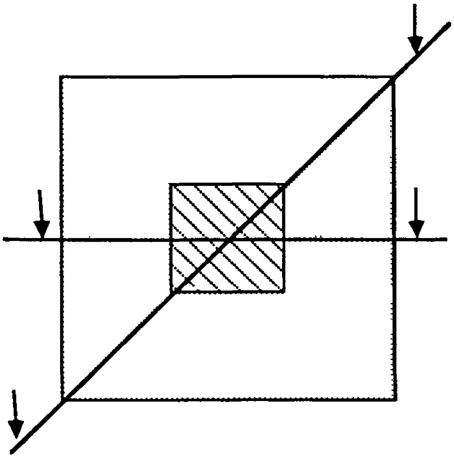


Figure 7.

6. FULL TWO-DIMENSIONAL MODEL PROBLEM

We consider a plane square area (dimensions 140 × 140 m, see figure below) with two sub-domains : a central square area (hachured, dimensions 40 × 40 m), and its complementary area (shaded), both full of perfect gas (air).



On the hachured area (inner square) the initial conditions are the following :

gas velocity : 0 m/s
density : 12 kg/m^3
pressure : 10^6 Pa .

On the shaded area the initial conditions are the following :

gas velocity : 0 m/s
density : 1.2 kg/m^3
pressure : 10^5 Pa .

We compare the computed solutions obtained at $t = 0.05$ s by using the Van Leer scheme described above, with three different meshes :

1. regular mesh with 3 600 squares (CPU time: 10 mn 09 s)
2. mesh with 3 600 irregular quadrilaterals (CPU time: 11 mn 34 s)
3. mesh with 3 834 irregular triangles (CPU time: 13 mn 37 s) .

Each of these solutions is presented along the median and the diagonal section

A computation made with a very refined regular mesh (9 216 squares) provides us with a better approximation of the exact solution, and will be used as a « reference solution »

6.1. Meshes (*fig* 8)

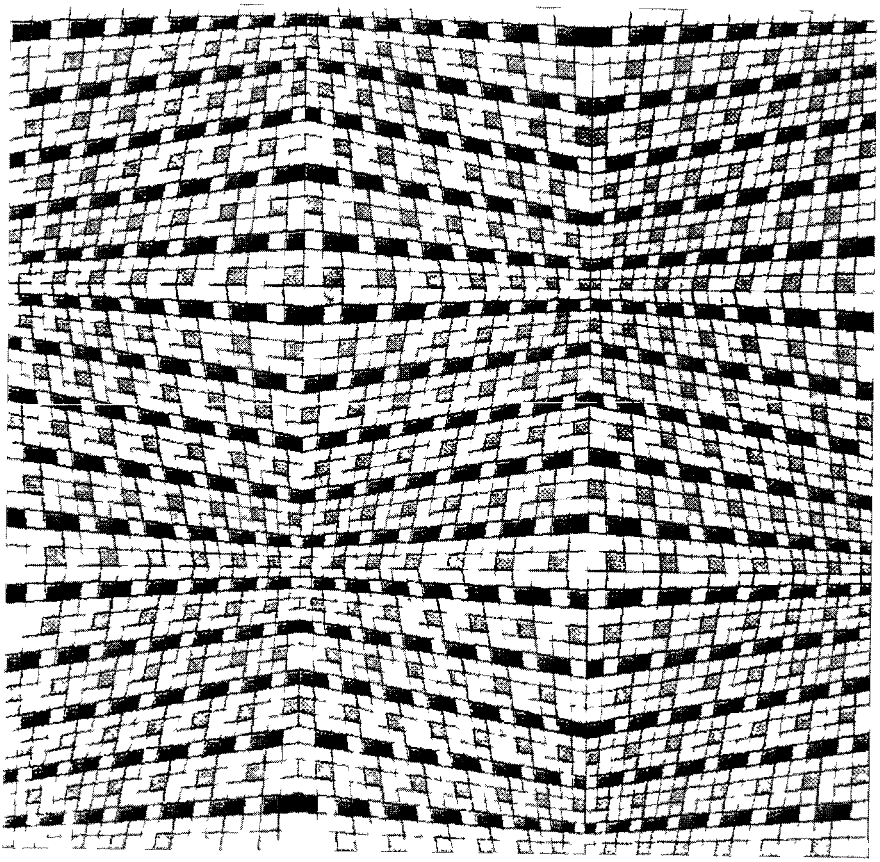


Figure 8 — Mesh with 3 600 quadrilaterals

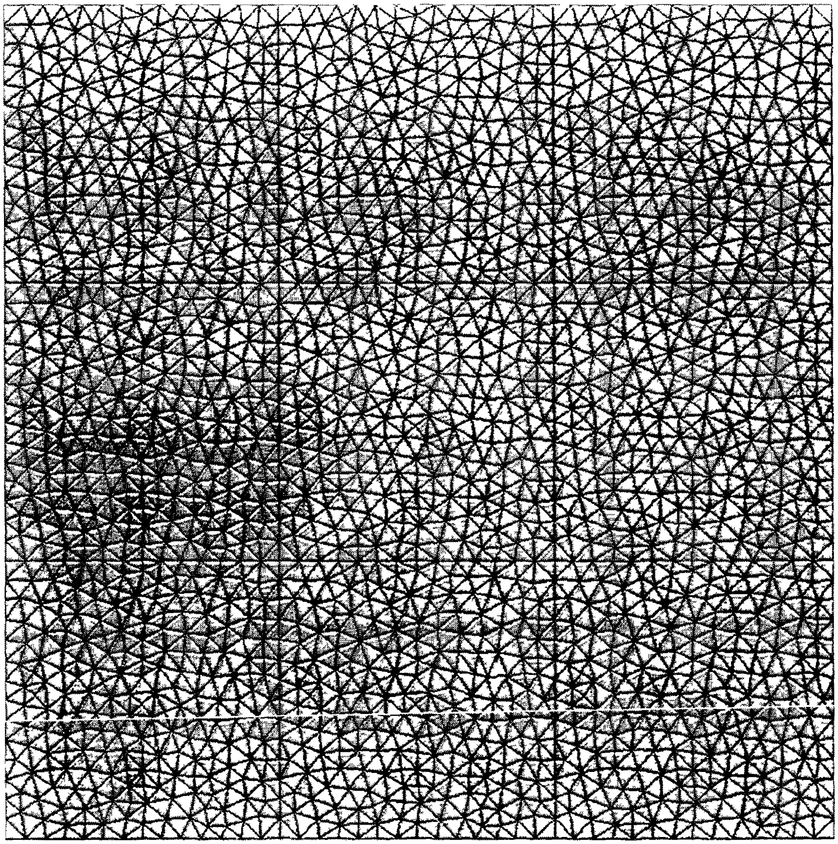
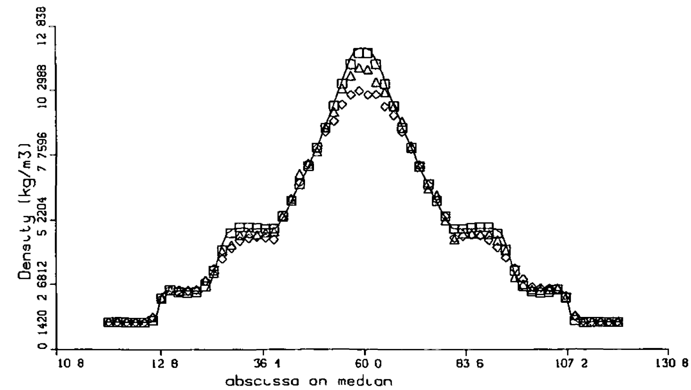


Figure 9. — Mesh with 3 834 triangles.

6.2. Densities (Van Leer only) (*fig* 10)

Median section



Diagonal section

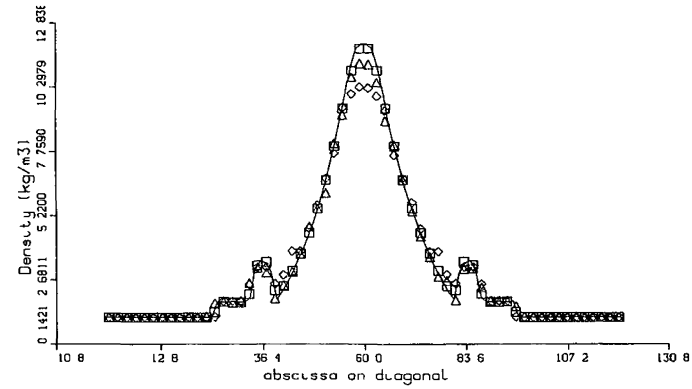
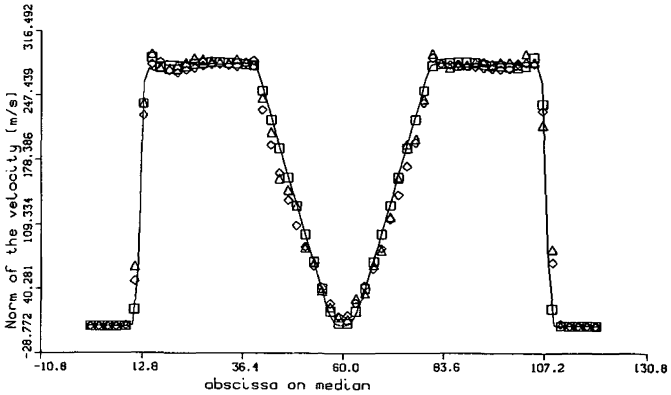


Figure 10 :

/ reference solution
□ solution obtained with mesh 1 (squares)
◇ solution obtained with mesh 2 (quadrilaterals)
△ solution obtained with mesh 3 (triangles)

6.3. Velocities (Van Leer only) (*fig.* 11)

Median section :



Diagonal section :

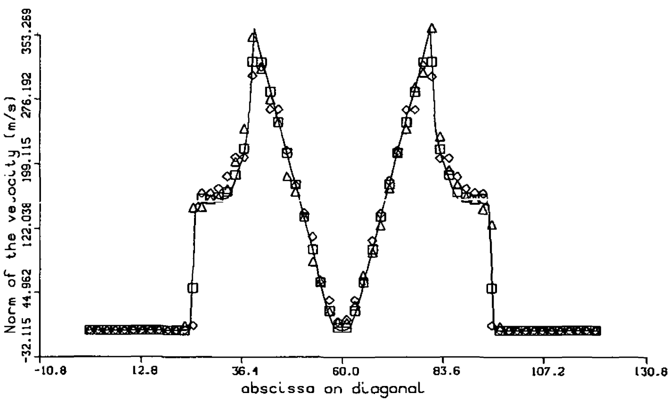
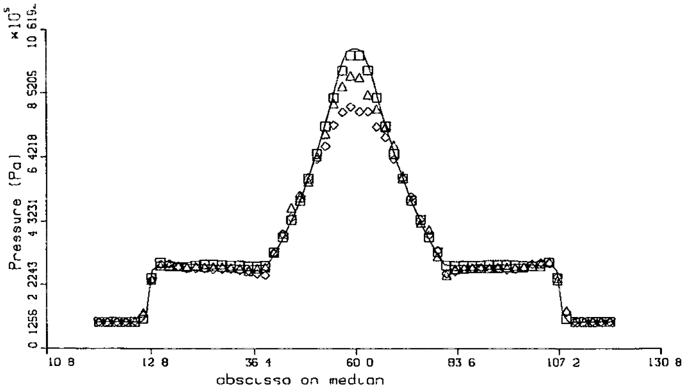


Figure 11 :

/ : reference solution
□ : solution obtained with mesh 1 (squares)
◇ : solution obtained with mesh 2 (quadrilaterals)
△ : solution obtained with mesh 3 (triangles).

6.4. Pressures (Van Leer only) (*fig.* 12)

Medıan sectıon



Dıagonal sectıon :

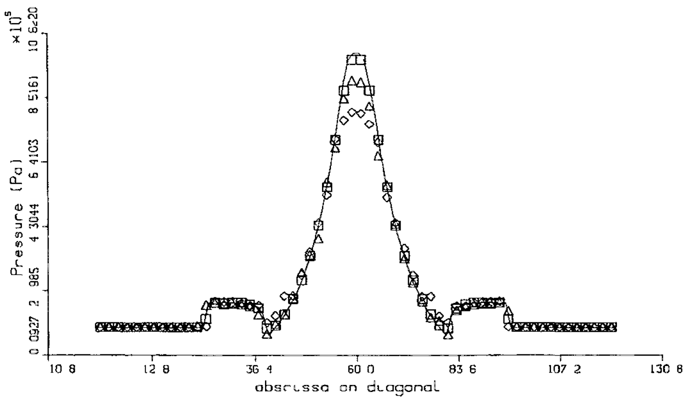


Figure 12 :

/ reference solutıon

□ solutıon obtaıned wıth mesh 1 (squares)

◇ solutıon obtaıned wıth mesh 2 (quadrılaterals)

△ solutıon obtaıned wıth mesh 3 (trıangles)

7. CONCLUSION

The numerical results of the classical shock tube show that, this Van Leer scheme is much more accurate than a first order Godunov scheme. On the regular mesh, only one point is necessary to describe the shock (instead of two) and only two points are necessary to describe the contact discontinuity (instead of eight or nine). On the unstructured meshes this superiority of Van Leer scheme is more significant.

The two dimensional problem shows the good behaviour of our scheme on unstructured meshes and especially on the triangulation.

REFERENCES

P. CHÉVRIER, *Simulation numérique de l'interaction arc électrique — écoulements gazeux dans les disjoncteurs,* Thèse de l'INP, Grenoble (1990).

P. COLLELA and H. M. GLAZ, Efficient Solution Algorithms for the Riemann Problem for Real Gases, *Journal of Computational Physics,* 1984 (preprint).

T. GALLOUËT, personal communication.

H. GILQUIN, *Analyse numérique d'un problème hyperbolique multidimensionnel en dynamique des gaz avec frontière mobile,* Thèse de Doctorat en Mathématiques Appliquées, Université de St-Etienne (France), 1984.

S. GODUNOV, *Résolution numérique des problèmes multidimensionnels de la dynamique des gaz,* Editions of Moscow, 1976.

F. JOUVE, P. LE FLOCH, *Une méthode de volumes finis d'ordre deux pour les équations d'Euler à deux dimensions d'espace,* preprint, 1991.

R. J. LEVEQUE, *Journal of Computational Physics, 78,* 1988, pp. 36-63.

P. L. ROE, Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes, *Journal of Computational Physics, 43,* 1981.

J. SMOLLER, *Shock Waves and Reaction-Diffusion Equations,* Springer-Verlag, 1983.

B. VAN LEER, Towards the ultimate conservative difference scheme II-IV-V, *Journal of Computational Physics, 14,* 1978, pp. 23, 32.

J. P. VILA, *Sur la théorie et l'approximation numérique de problèmes hyperboliques non linéaires,* Thèse de Doctorat en Mathématiques Appliquées, Université de Paris VI (France), 1986.