ESAIM: M2AN 59 (2025) 671–690
https://doi.org/10.1051/m2an/2025001

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

DOMAIN WALL DYNAMICS IN NOTCHED FERROMAGNETIC NANOWIRES

Gilles Carbou[1,*], Mustapha Jazar[2] and Mouna Kassan[1,2]

Abstract. We study the dynamics of magnetization reversals, known as walls, in a notched ferromagnetic nanowire. The model used is the Landau–Lifschitz equation in dimension 1, with a weight representing the notch. Despite the pinning properties of notches, we establish in this article that when the applied field tends to push the wall far away from the notch, pinning effects are negligible, and wall dynamics are asymptotically close to those in an unnotched wire.

Mathematics Subject Classification. 35K55, 35Q60.

Received June 21, 2024. Accepted January 9, 2025.

1. Introduction

1.1. The physical context

Ferromagnetic nanowires have become the subject of numerous studies in both physics (see [9, 11–13, 16]) and mathematics (see [4, 5, 7, 10, 17]). In such devices, magnetization spontaneously divides into uniformly magnetized regions called domains, separated by thin zones of magnetization reversal called domain walls (DW). This property enables digital data to be stored, the information being encoded by the positions of the DWs along the wire. In this context, DW stability is crucial for ensuring storage reliability, as an undesired displacement of a DW can deteriorate the data. To fix DW positions, notches are patterned along the wire (see [12, 18]). DWs pinning by the notches ensures the reliability of the storage mode without requiring energy consumption. Nevertheless, to bring the information to a read/write head, it can be necessary to unpin DWs. It is achieved by injecting an electric current or applying a magnetic field (see [9, 11, 12]).

In this paper, we consider a one-dimensional model of ferromagnetic notched nanowire obtained in [1] by asymptotic process. This model was used in [7] to highlight DW pinning effects of notches. Our goal is to describe in this model the behavior of a DW when subjected to a magnetic field and as it moves away from the notch. We establish that the DW dynamics become similar to those observed in the absence of a notch.

Afterwards, we denote the canonical basis in $\mathbb{R}^3$ by $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. The Euclidean scalar product, its associated norm and the cross product are denoted by $\cdot$, $|.|$ and $\times$ respectively.

Keywords and phrases. Ferromagnetic nanowire, Landau–Lifshitz equation, asymptotic stability.

[1] Universite de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, Pau, France.

[2] LaMA, Laboratoire de Mathématiques et Applications, Lebanese University, Tripoli, Lebanon.

*Corresponding author: gilles.carbou@univ-pau.fr

1.2. DW dynamics in unnotched nanowire

We consider the following one-dimensional model of ferromagnetic nanowire of infinite length and constant cross-section justified by asymptotic methods in [6, 15].

The wire is assimilated to the axis $\mathbb{R}\vec{e_1}$. The magnetization is described by the magnetic moment $m : \mathbb{R}_t^+ \times \mathbb{R}_x \longrightarrow \mathbb{R}^3$ satisfying the saturation constraint:

$$|m(t,x)| = 1 \ \text{ in } \mathbb{R}_t^+ \times \mathbb{R}_x, \tag{1.1}$$

so that m takes its values in S^2, the unit sphere of $\mathbb{R}^3$.

We assume that the wire is submitted to an applied magnetic field of the form $-h(t)\vec{e}_1$, where h is a time-depending continuous function. The variations of m are described by:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times (q(m) - h(t)\vec{e}_1) - m \times (m \times (q(m) - h(t)\vec{e}_1)) \\ q(m) = \partial_{xx} m + (m \cdot \vec{e}_1)\vec{e}_1. \end{cases} \tag{1.2}$$

We remark that system (1.2) is invariant by translation in the variable x and rotation around the axis $\mathbb{R}\vec{e}_1$, that is: if m satisfies (1.2), then for $\varphi \in \mathbb{R}$ and $\sigma \in \mathbb{R}$, $(t,x) \mapsto \mathbf{R}_\varphi m(t, x-\sigma)$ is also a solution, where

$$\mathbf{R}_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

We consider now the profile M_0 given by:

$$M_0(x) = \begin{pmatrix} \tanh x \\ \frac{1}{\cosh x} \\ 0 \end{pmatrix}.$$

This profile is a static solution of (1.2) for vanishing h and describes a DW connecting a left-hand-side $-\vec{e}_1$-domain and a right-hand-side $+\vec{e}_1$-domain. Stability and asymptotic stability modulo translation/rotation of the profile M_0 are established in [5].

For a non-zero h, the solution of (1.2) for initial data of the form $x \mapsto \mathbf{R}_{\varphi_0} M_0(x - \sigma_0)$ is given by:

$$\mathscr{M}_h(t,x) = \mathbf{R}_{(\varphi_0 - H(t))} M_0(x - \sigma_0 - H(t)), \quad \text{where } H(t) = \int_0^t h(s)\,\mathrm{d}s. \tag{1.3}$$

In [10], if h does not depend on time, it was shown that $\mathscr{M}_h$ is stable in the Lyapunov sense and asymptotically stable modulo translation-rotation if $|h| < 1$, and that $\mathscr{M}_h$ is linearly unstable if $|h| > 1$. In [8], the stability and asymptotic stability modulo rotation/translation of $\mathscr{M}_h$ are established for time-depending h, under a smallness assumption on $\|h\|_{L^\infty(\mathbb{R}_t^+)}$.

1.3. Model for notched nanowire

Now we consider a ferromagnetic nanowire of infinite length (assimilated to the real line) presenting a notch. We introduce the $\mathcal{C}^1$ function $s : \mathbb{R} \longrightarrow [\rho_1, 1] \subset \mathbb{R}^{*+}$, where $0 < \rho_1 < 1$, such that $s(x)$ is proportional to the cross-section area of the wire at the abscissa x. We assume that $s = 1$ outside $[-a, a]$. Note by s' the derivative of s with respect to x (Fig. 1).

In the static case, the energy associated to a magnetisation configuration $m : \mathbb{R}_x \longrightarrow S^2$ is given by:

$$\mathcal{E}(m) = \frac{1}{2}\int_{\mathbb{R}} s(x)|\partial_x m|^2 + \frac{1}{2}\int_{\mathbb{R}} s(x)\big(1 - |m(x)\cdot \vec{e}_1|^2\big)\,\mathrm{d}x.$$

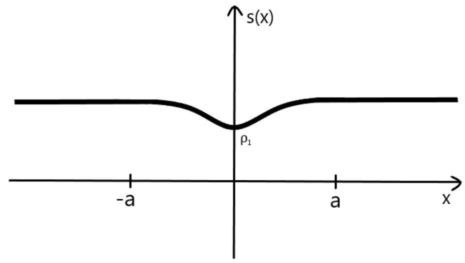

FIGURE 1. Function $s(x)$.

In the dynamical case, in the presence of an applied magnetic field of the form $H_a(t,x) = -h(t)\vec{e}_1$, variations of the magnetic moment $m : \mathbb{R}_t^+ \times \mathbb{R}_x \longrightarrow S^2$ are described by the following Landau–Lifschitz equation:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times H_{\text{eff}}(m) - \; m \times (m \times H_{\text{eff}}(m)) \text{ in } \mathbb{R}^+ \times \mathbb{R}, \\ H_{\text{eff}}(m) = \partial_{xx} m + \dfrac{s'}{s} \partial_x m + (m \cdot \vec{e_1})\vec{e_1} - h(t)\vec{e}_1. \end{cases} \tag{1.4}$$

We note that the model above is invariant under rotation around the $\mathbb{R}\vec{e}_1$ axis. Because of the notch, we no longer have translational invariance. In [7], in the case of a 0-symmetric notch (s is even) with s increasing on $[0,a]$, the authors establish the existence and asymptotic stability modulo rotation of static profiles describing a DW centered at the notch. This illustrates DW pinning effects of notches. For larger magnetic fields, it is shown in [7] that such stationary solutions no longer exist, which illustrates DW depinning by a sufficiently large applied field.

1.4. Main result

In this paper, we are interested in DW behavior after depinning, and we show that once away from the notch, the DW dynamics are close to those described by (1.3) for unnotched wires. We establish the following theorem:

Theorem 1.1. *Let $s : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a $\mathcal{C}^1$-function such that $s(x) = 1$ for $x \notin [-a,a]$ and $0 < \rho_0 \le s(x) \le 1$ for $x \in [-a,a]$. There exists $\delta_0 > 0$ such that for any continuous function $h : \mathbb{R}^+ \to \mathbb{R}$ satisfying*

$$\forall\, t, \ \ |h(t)| \le \delta_0, \tag{1.5}$$

and such that

$$\int_0^\infty e^{-H(t)}\, \mathrm{d}t < +\infty, \quad \text{where } H(t) = \int_0^t h(s)\, \mathrm{d}s, \tag{1.6}$$

then for any $\varepsilon > 0$, there exist σ^* and $\eta > 0$ such that the following holds: for all $\sigma_0 \ge \sigma^*$ and for all $\varphi_0 \in \mathbb{R}$, we consider $\mathscr{M}_h$ given by

$$\mathscr{M}_h(t,x) = \mathbf{R}_{(\varphi_0 - H(t))} M_0(x - \sigma_0 - H(t)).$$

Then all solution m of (1.4) such that $\|m(0,\cdot) - \mathscr{M}_h(0,\cdot)\|_{H^1(\mathbb{R})} \le \eta$ satisfies:

$$\forall\, t \in \mathbb{R}^+, \quad \|m(t,\cdot) - \mathscr{M}_h(t,\cdot)\|_{H^1(\mathbb{R})} \le \varepsilon.$$

Furthermore, there exists $(\varphi_\infty, \sigma_\infty) \in \mathbb{R}^2$ such that:

$$\|m(t,\cdot) - \mathbf{R}_{\varphi_\infty} \mathscr{M}_h(t, \cdot + \sigma_\infty)\|_{H^1(\mathbb{R})} \underset{t\to+\infty}{\longrightarrow} 0.$$

The paper is organized as follows. Section 2 is devoted to obtaining successive equivalent forms for equation (1.4). In particular, we describe the perturbations m of $\mathscr{M}_h$ satisfying the saturation constraint in a mobile frame so that the new unknown r takes its values in the flat space $\mathbb{R}^2$, and that m is close to $\mathscr{M}_h$ if and only if r is close to zero.

In Section 3, due to the system's invariance by rotation around the axis $\vec{e_1}$, and the quasi-invariance by translation when the wall is far from the notch, a two-parameter family R_Λ of quasi-solutions of the equation in r is obtained. Then we decompose r as $r = R_\Lambda + W$, where W is in the orthogonal of the linearized kernel of the system without notch used in [10]. The same technique was used in [2] to prove the stability of traveling waves in thin films and in [14] for radially symmetric traveling waves in reaction–diffusion equations.

In order to perform variational estimates in Section 5, we establish in Section 4 a uniform coercivity result for a one-parameter family of linear operators arising in the equation satisfied by W. For the convenience of the reader, technical estimates on nonlinear terms are postponed in Section 6.

2. Equivalent forms of the equation

Let $m : \mathbb{R}^+ \times \mathbb{R} \to S^2$ be a solution to the model (1.4). We define $M : \mathbb{R}^+ \times \mathbb{R} \to S^2$ as :

$$M(t,x) = \mathbf{R}_{(-\varphi_0 + H(t))} m(t, x + \sigma_0 + H(t)),$$

and $\mathfrak{a}_\star$ by:

$$\mathfrak{a}_\star(x) = \frac{s'}{s}(x + \sigma_0 + H(t)). \tag{2.1}$$

Then, M satisfies the saturation constraint, and we prove by a simple calculation that m is a solution for (1.4) if and only if M satisfies the following system:

$$\begin{cases} \dfrac{\partial M}{\partial t} = -M \times q(M) - M \times (M \times q(M)) + h(t)(\partial_x M + M \times (M \times \vec{e_1})) \\ \qquad -\mathfrak{a}_\star(x)(M \times \partial_x M + M \times (M \times \partial_x M)), \\ q(M) = \partial_{xx} M + (M \cdot \vec{e_1})\vec{e_1}. \end{cases} \tag{2.2}$$

Furthermore, m is close to $\mathscr{M}_h$ if and only if M is close to M_0.

We consider the following orthonormal frame $(M_0(x), M_1(x), M_2)$ with

$$M_0(x) = \begin{pmatrix} \tanh x \\ \dfrac{1}{\cosh x} \\ 0 \end{pmatrix}, \quad M_1(x) = \begin{pmatrix} \dfrac{1}{\cosh x} \\ -\tanh x \\ 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

In order to consider only perturbations M of M_0 that satisfy the saturation constraint, we describe these perturbations in the mobile frame by writing:

$$M(t,x) = M_0(x) + r_1(t,x)M_1(x) + r_2(t,x)M_2 + \mu(r(t,x))M_0(x), \tag{2.3}$$

where

$$\begin{aligned} r : \mathbb{R}^+ \times \mathbb{R} &\to \overline{B_2(0,1)} \subset \mathbb{R}^2 \\ (t,x) &\mapsto r(t,x) = (r_1(t,x), r_2(t,x)), \end{aligned}$$

and where $\mu(r) = \sqrt{1 - r_1^2 - r_2^2} - 1$. In order to avoid singularities of μ when $|r| = 1$, we will have to check in the calculations that for all t, $\|r(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{2}$, so that $r(t,x)$ remains far from the unit circle.

Plugging (2.3) into (2.2), we obtain that M satisfies (2.2) if and only if $r = (r_1, r_2)$ is a solution of the following system:

$$\frac{\partial r}{\partial t} = \mathfrak{p}_\star(x) + (\mathcal{J}\mathscr{L}_\star + h(t)\ell)r + \sum_{i=1}^{7} \mathscr{F}_i, \tag{2.4}$$

where:

– $\mathfrak{p}_\star$ is defined by

$$\mathfrak{p}_\star(x) = \frac{1}{\cosh(x)}\mathfrak{a}_\star(x)\begin{pmatrix}1\\0\end{pmatrix},$$

– $\mathcal{J}$ is given by

$$\mathcal{J} = \begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix},$$

– the two operators $\mathscr{L}_\star$ and ℓ are given by

$$\mathscr{L}_\star(r) = -\partial_{xx}r - \mathfrak{a}_\star(x)\partial_x r + g(x)r \quad \text{and} \quad \ell(r) = \partial_x r + \tanh(x)r, \tag{2.5}$$

with $g(x) = 2\tanh^2(x) - 1$.

The nonlinear terms $\mathscr{F}_i$, $i \in \{1,\dots,7\}$, are given by

$$\begin{aligned}&\mathscr{F}_1 = F_1(r)(\partial_{xx}r), \quad \mathscr{F}_2 = F_2(x,r)(\partial_x r), \quad \mathscr{F}_3 = F_3(r)(\partial_x r, \partial_x r), \quad \mathscr{F}_4 = F_4(x,r),\\ &\mathscr{F}_5 = h(t)F_5(x,r), \quad \mathscr{F}_6 = \mathfrak{a}_\star(x)F_6(r)(\partial_x r), \quad \mathscr{F}_7 = \mathfrak{a}_\star(x)F_7(x,r),\end{aligned}$$

where

$$\begin{aligned}
F_1(r) &= \begin{pmatrix}2\mu(r)+\mu^2(r)+r_1^2 & \mu(r)-r_1r_2\\ -\mu(r)+r_1r_2 & 2\mu(r)+\mu^2(r)+r_1^2\end{pmatrix} + \begin{pmatrix}-r_1(1+\mu(r))+r_2\\ r_1+r_2(1+\mu(r))\end{pmatrix}D\mu(r),\\
F_2(x,r) &= \frac{2}{\cosh(x)}\begin{pmatrix}r_1(1+\mu(r))+r_2 & 0\\ -r_1+r_2(1+\mu(r)) & 0\end{pmatrix} + \frac{2}{\cosh(x)}\begin{pmatrix}(1+\mu(r))^2+r_2^2\\ -(1+\mu(r))^2-r_1r_2\end{pmatrix}D\mu(r),\\
F_3(r) &= \begin{pmatrix}-r_1(1+\mu(r))+r_2\\ r_1-r_2(1+\mu(r))\end{pmatrix}D^2\mu(r),\\
F_4(x,r) &= \left(\frac{2\sinh(x)}{\cosh^2(x)}r_1+\mu(r)g(x)\right)\begin{pmatrix}-r_1(1+\mu(r))-r_2\\ r_1-r_2(1+\mu(r))\end{pmatrix} - \mu(r)g(x)r,\\
F_5(x,r) &= \begin{pmatrix}-\dfrac{\mu(r)}{\cosh(x)}(1+\mu(r))+\mu(r)r_1\tanh(x)-\dfrac{r_2^2}{\cosh(x)}\\ \dfrac{r_1r_2}{\cosh(x)}+\mu(r)r_2\tanh(x)\end{pmatrix},\\
F_6(r) &= \begin{pmatrix}2\mu(r)+\mu^2(r)+r_2^2 & \mu(r)-r_1r_2\\ -\mu(r)-r_1r_2 & 2\mu(r)+\mu^2(r)+r_1^2\end{pmatrix} + \begin{pmatrix}-r_1(1+\mu(r))-r_2\\ r_1-r_2(1+\mu(r))\end{pmatrix}D\mu(r),\\
F_7(x,r) &= \frac{1}{\cosh(x)}\begin{pmatrix}\mu^3(r)+3\mu(r)(1+\mu(r))+(r_1^2+r_2^2)(1+\mu(r))+r_1r_2\\ -r_1^2\end{pmatrix},
\end{aligned}$$

where $D\mu(r)$ and $D^2\mu(r)$ denote the first and second order total derivative of $r \mapsto \mu(r)$. We remark that

– $F_1 \in \mathcal{C}^\infty(B_2(0,1);\mathcal{M}_2(\mathbb{R}))$, and $F_1(x,r) = \mathcal{O}(|r|^2)$, where $\mathcal{M}_2(\mathbb{R})$ is the set of the 2×2 matrices.
– $F_2 \in \mathcal{C}^\infty(\mathbb{R}\times B_2(0,1);\mathcal{M}_2(\mathbb{R}))$, with $F_2(x,r) = \mathcal{O}(|r|)$.
– $F_3 \in \mathcal{C}^\infty(B_2(0,1);\mathcal{L}_2(\mathbb{R}^2;\mathbb{R}^2))$, where $\mathcal{L}_2(\mathbb{R}^2;R^2)$ is the set of the bilinear applications defined on $\mathbb{R}^2\times\mathbb{R}^2$ with values in $\mathbb{R}^2$. We have: $F_3(r) = \mathcal{O}(|r|)$.
– $F_4 \in \mathcal{C}^\infty(\mathbb{R}\times B_2(0,1);\mathbb{R}^2)$, and $F_4(r) = \mathcal{O}(|r|^2)$.
– $F_5 \in \mathcal{C}^\infty(\mathbb{R}\times B_2(0,1);\mathbb{R}^2)$, and $F_5(r) = \mathcal{O}(|r|^2)$.
– $F_6 \in \mathcal{C}^\infty(B_2(0,1);\mathcal{M}_2(\mathbb{R}))$, and $F_6(r) = \mathcal{O}(|r|)$.
– $F_7 \in \mathcal{C}^\infty(\mathbb{R}\times B_2(0,1);\mathbb{R}^2)$, and $F_7(r) = \mathcal{O}(|r|^2)$.

3. New coordinate system

We denote by $\langle \, | \, \rangle$ the $L^2(\mathbb{R})$ scalar product:

$$\langle u \,|\, v \rangle = \int_{\mathbb{R}} u(x) \cdot v(x) \, \mathrm{d}x,$$

where $\cdot$ is the multiplication in $\mathbb{R}$ if u and v are real-valued or the scalar product when they take their values in $\mathbb{R}^2$.

The operator $\mathscr{L}_\star$ is a perturbation the operator L given by:

$$L = -\partial_{xx} + g.$$

As proved in [3] (see Prop. 3.1.), L is self-adjoint with respect to the $L^2(\mathbb{R})$ inner product. In addition, it can be factorized as $L = \ell^* \circ \ell$, with $\ell = \partial_x + \tanh x$, so that L is positive and $\operatorname{Ker} L = \operatorname{Ker} \ell = \mathbb{R}\frac{1}{\cosh x}$. Moreover, the essential spectrum of L is $[1, +\infty[$ and has only zero as its eigenvalue. Indeed, if $Lu = \lambda u$, with $u \in H^2(\mathbb{R})$ and $\lambda \neq 0$, then by applying ℓ, we have:

$$\lambda \ell u = \ell \circ L u = \ell \circ \ell^* \circ \ell u,$$

so that ℓu is an eigenvector of $\ell \circ \ell^*$. However, $\ell \circ \ell^* = -\partial_{xx} + 1$, so this operator has no eigenvalue. This implies that $\ell u = 0$, so $\lambda = 0$.

Therefore, on $(\operatorname{Ker} L)^\perp$, $L \geq 1$:

$$\forall \, v \in H^2(\mathbb{R}), \; \left\langle v \,\middle|\, \frac{1}{\cosh x} \right\rangle = 0 \implies \langle Lv \,|\, v \rangle \geq \|v\|^2_{L^2(\mathbb{R})}. \tag{3.1}$$

Then, by invertibility of the matrix $\mathcal{J}$, the kernel of $\mathcal{J}L$ is the two-dimensional space generated by a_1 and a_2 with

$$a_1(x) = \begin{pmatrix} \dfrac{1}{\cosh(x)} \\ 0 \end{pmatrix}, \qquad a_2(x) = \begin{pmatrix} 0 \\ \dfrac{1}{\cosh(x)} \end{pmatrix}.$$

Equation (2.4) is invariant by rotation, and quasi-invariant by translation far from the notch. For $\Lambda = (\theta, \sigma) \in \mathbb{R}^2$, we denote by M_Λ the rotation/translation of the quasi-solution M_0:

$$M_\Lambda(x) = \mathbf{R}_\theta(M_0(x - \sigma)),$$

and we denote by R_Λ the coordinates of $M_\Lambda(x)$ in the basis $(M_1(x), M_2)$:

$$R_\Lambda(x) = \begin{pmatrix} M_\Lambda(x) \cdot M_1(x) \\ M_\Lambda(x) \cdot M_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\tanh(x-\sigma)}{\cosh x} - \cos\theta \dfrac{\tanh(x)}{\cosh(x-\sigma)} \\ \dfrac{-\sin\theta}{\cosh(x-\sigma)} \end{pmatrix}. \tag{3.2}$$

In [5], the authors establish that there exists a neighborhood $\mathcal{U} \subset L^2(\mathbb{R}; \mathbb{R}^2)$ of 0 and a neighborhood $\mathcal{V} \subset \mathbb{R}^2 \times L^2(\mathbb{R}; \mathbb{R}^2)$ of $(0, 0)$ such that for $r \in \mathcal{U}$, there exists one and only one couple $(\Lambda, W) \in \mathcal{V}$ such that $r = R_\Lambda + W$ where the components W_1 and W_2 of W are in $(\operatorname{Ker} L)^\perp$. In addition, W as the same regularity as r. Therefore, while $r(t)$ remains in $\mathcal{U}$, we decompose $r(t, \cdot)$ as:

$$r(t, x) = R_{\Lambda(t)}(x) + W(t, x), \tag{3.3}$$

where $(\Lambda(t), W(t, \cdot)) \in \mathcal{V} \subset \mathbb{R}^2 \times L^2(\mathbb{R}; \mathbb{R}^2)$, with $W(t, \cdot) \in (\operatorname{Ker}\, L)^\perp$, that is:

$$\forall \, j \in \{1, 2\}, \; \forall \, t, \quad \int_{\mathbb{R}} W(t, x) \cdot a_j(x) \, \mathrm{d}x = 0. \tag{3.4}$$

We introduce $\nu_0 > 0$ such that if $|\Lambda(t)| \leq \nu_0$ and $\|W(t,\cdot)\|_{H^1(\mathbb{R})} \leq \nu_0$, then:

$$R_{\Lambda(t)}(\cdot) + W(t,\cdot) \in \mathcal{V} \quad \text{and} \quad \|R_{\Lambda(t)}(\cdot) + W(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2}.$$

We substitute r by $R_\Lambda + W$ in (2.4) and we obtain that while $|\Lambda(t)| \leq \nu_0$ and $\|W(t,\cdot)\|_{H^1(\mathbb{R})} \leq \nu_0$, we have:

$$\begin{aligned} \partial_\Lambda R_\Lambda\left(\frac{\mathrm{d}\Lambda}{\mathrm{d}t}\right) + \frac{\partial W}{\partial t} = {} & \mathfrak{p}_\star(x) + (\mathcal{J}\mathscr{L}_\star + h(t)\ell)(W) + (\mathcal{J}\mathscr{L}_\star + h(t)\ell)(R_\Lambda) \\ & + F_1(R_\Lambda + W)(\partial_{xx}R_\Lambda + \partial_{xx}W) + F_2(x, R_\Lambda + W)(\partial_x R_\Lambda + \partial_x W) \\ & + F_3(R_\Lambda + W)(\partial_x R_\Lambda + \partial_x W, \partial_x R_\Lambda + \partial_x W) + F_4(x, R_\Lambda + W) \\ & + h(t)F_5(x, R_\Lambda + W) + \mathfrak{a}_\star(x)F_6(R_\Lambda + W)(\partial_x R_\Lambda + \partial_x W) \\ & \quad + \mathfrak{a}_\star(x)F_7(x, R_\Lambda + W). \end{aligned} \tag{3.5}$$

For a constant Λ, R_Λ is a static solution for equation (2.4) without the terms containing $\mathfrak{a}_\star(x)$. So we have:

$$\begin{aligned} (\mathcal{J}L + h(t)\ell)(R_\Lambda) + F_1(R_\Lambda)(\partial_{xx}R_\Lambda) + F_2(x, R_\Lambda)(\partial_x R_\Lambda) & \\ + F_3(R_\Lambda)(\partial_x R_\Lambda, \partial_x R_\Lambda) + F_4(x, R_\Lambda) + h(t)F_5(x, R_\Lambda) & = 0. \end{aligned} \tag{3.6}$$

In order to use Identity (3.6), we perform a Taylor expansion in the variable r for $F_1, \ldots, F_7$ on the following way. For a function F defined on $B_2(0, \frac{1}{2}) \subset \mathbb{R}^2$ with values in a vector space $\mathscr{W}$, we write:

$$F(R_\Lambda + W) = F(R_\Lambda) + \widetilde{F}(\Lambda, W)(W),$$

with

$$\widetilde{F}(\Lambda, W)(W) = \int_0^1 DF(R_\Lambda + yW)(W)\,\mathrm{d}y,$$

where we denote by D the total derivative in the variable r, so that $DF(r) \in \mathcal{L}(\mathbb{R}^2; \mathscr{W})$. Using these notations for $F_1, \ldots, F_7$, using (3.6), we have:

$$\partial_\Lambda R_\Lambda\left(\frac{\mathrm{d}\Lambda}{\mathrm{d}t}\right) + \frac{\partial W}{\partial t} = K(t, x, \sigma_0, \Lambda) + (\mathcal{J}\mathscr{L}_\star + h(t)\ell)(W) + \sum_{i=1}^{5} T_i, \tag{3.7}$$

where

$$K(t, x, \sigma_0, \Lambda) = \mathfrak{p}_\star(x) + \mathfrak{a}_\star(x)\Big(-\mathcal{J}\partial_x R_\Lambda + \; F_6(R_\Lambda)(\partial_x R_\Lambda) + F_7(x, R_\Lambda)\Big),$$

and

$$\begin{aligned} T_1 &= F_1(R_\Lambda + W)(\partial_{xx}W), \\ T_2 &= \widetilde{F}_1(\Lambda, W)(W)(\partial_{xx}R_\Lambda) + F_2(x, R_\Lambda + W)(\partial_x W) + \widetilde{F}_2(x, \Lambda, W)(W)(\partial_x R_\Lambda), \\ T_3 &= \widetilde{F}_3(\Lambda, W)(W)(\partial_x R_\Lambda, \partial_x R_\Lambda) + 2F_3(R_\Lambda + W)(\partial_x R_\Lambda, \partial_x W) + F_3(R_\Lambda + W)(\partial_x W, \partial_x W), \\ T_4 &= \widetilde{F}_4(\Lambda, W)(W) + h(t)\widetilde{F}_5(\Lambda, W)(W), \\ T_5 &= \mathfrak{a}_\star(x)\Big(\widetilde{F}_6(\Lambda, W)(W)(\partial_x R_\Lambda) + F_6(R_\Lambda + W)(\partial_x W) + \widetilde{F}_7(x, \Lambda, W)(W)\Big). \end{aligned}$$

We take the $L^2(\mathbb{R})$-inner product of (3.7) with a_1 and a_2. By the orthogonality condition on W, we have:

$$\left\langle \frac{\partial W}{\partial t} \,|\, a_1 \right\rangle = \left\langle \frac{\partial W}{\partial t} \,|\, a_2 \right\rangle = 0.$$

So, we obtain that

$$A(\Lambda)\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \overline{K}(t,\sigma_0,\Lambda) + \mathcal{N}(t,W) + \sum_{i=1}^{5}\overline{T_i}, \tag{3.8}$$

where

$$A(\Lambda) = \begin{pmatrix} \left\langle \dfrac{\partial R_\Lambda}{\partial\theta} \,\middle|\, a_1\right\rangle & \left\langle \dfrac{\partial R_\Lambda}{\partial\sigma} \,\middle|\, a_1\right\rangle \\ \left\langle \dfrac{\partial R_\Lambda}{\partial\theta} \,\middle|\, a_2\right\rangle & \left\langle \dfrac{\partial R_\Lambda}{\partial\sigma} \,\middle|\, a_2\right\rangle \end{pmatrix}, \qquad \overline{K}(t,\sigma_0,\Lambda) = \begin{pmatrix} \langle K(t,x,\sigma_0,\Lambda) \,|\, a_1\rangle \\ \langle K(t,x,\sigma_0,\Lambda) \,|\, a_2\rangle \end{pmatrix},$$

$$\mathcal{N}(t,W) = \begin{pmatrix} \langle \mathcal{J}\mathscr{L}_\star(W) + h(t)\ell(W) \,|\, a_1\rangle \\ \langle \mathcal{J}\mathscr{L}_\star(W) + h(t)\ell(W) \,|\, a_2\rangle \end{pmatrix}, \qquad \overline{T_i} = \begin{pmatrix} \langle T_i \,|\, a_1\rangle \\ \langle T_i \,|\, a_2\rangle \end{pmatrix}.$$

We observe that $A(0) = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, thus for sufficiently small Λ, we can invert the matrix $A(\Lambda)$. Even if it means reducing $\nu_0 > 0$, we assume that $(A(\Lambda))^{-1}$ is defined for all $\Lambda \in \overline{B_2(0,\nu_0)}$. Hence, while $|\Lambda(t)| \le \nu_0$, while $\|W(t,\cdot)\|_{H^1(\mathbb{R})} \le \nu_0$, we have:

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \mathcal{G}(t,\sigma_0,\Lambda,W), \tag{3.9}$$

where

$$\mathcal{G}(t,\sigma_0,\Lambda,W) = (A(\Lambda))^{-1}\left(\overline{K}(t,\sigma_0,\Lambda) + \mathcal{N}(t,W) + \sum_{i=1}^{5}\overline{T_i}\right).$$

By substituting the expression for $\frac{\mathrm{d}\Lambda}{\mathrm{d}t}$ into (3.7), we obtain:

$$\frac{\partial W}{\partial t} = \mathcal{J}\mathscr{L}_\star(W) + h(t)\ell(W) + F(t,x,\sigma_0,\Lambda,W), \tag{3.10}$$

where $F(t,x,\sigma_0,\Lambda,W)$, the nonlinear part of (3.10), is given by:

$$F(t,x,\sigma_0,\Lambda,W) = \sum_{i=1}^{5} T_i - \partial_\Lambda R_\Lambda(\mathcal{G}(t,\sigma_0,\Lambda,W)) + K(t,x,\sigma_0,\Lambda). \tag{3.11}$$

Therefore, (2.4) is equivalent to system (3.9) and (3.10), and r is close to zero in $H^1(\mathbb{R})$ if and only if (Λ, W) is close to $(0,0)$ in $\mathbb{R}^2 \times H^1(\mathbb{R})$. It is also important to note that the system (3.9) and (3.10) is valid as long as $|\Lambda(t)| \le \nu_0$ and $\|W\|_{H^1(\mathbb{R})} \le \nu_0$.

4. Uniform coercivity of the operator $\mathscr{L}_\star$

The operator $\mathscr{L}_\star$, defined in (2.5), depends on σ_0 and $H(t)$. We prove in this section that, when restricted to $(\operatorname{Ker} L)^\perp$, it is uniformly coercive with respect to $\sigma_0 + H(t)$ when this quantity is large enough.

For $\lambda \in \mathbb{R}$, we denote by $\langle \cdot \,|\, \cdot\rangle_\lambda$ the weighted scalar product given by:

$$\langle u \,|\, v\rangle_\lambda = \int_{\mathbb{R}} s(x+\lambda)u(x)v(x)\,\mathrm{d}x,$$

and by $\|\cdot\|_{L^2_\lambda}$ the associated norm. We denote by $\mathscr{L}_\lambda$ the operator:

$$\mathscr{L}_\lambda v = -\partial_{xx}v - \frac{s'}{s}(x+\lambda)\partial_x v + g(x)v.$$

We claim the following proposition:

Proposition 4.1. *There exists λ_0 such that for all $\lambda \geq \lambda_0$, for all $w \in H^2(\mathbb{R};\mathbb{R})$ such that $\left\langle w \,|\, \frac{1}{\cosh x} \right\rangle = 0$,*

$$\langle \mathscr{L}_\lambda w \,|\, w \rangle_\lambda \geq \frac{1}{4} \|w\|^2_{L^2_\lambda}.$$

Proof. We fix $x_0 > 0$ such that

$$\forall x \in \mathbb{R},\ |x| \geq x_0 \iff g(x) \geq \frac{1}{2}. \tag{4.1}$$

We consider a smooth non decreasing function $\psi : \mathbb{R} \longrightarrow [0,1]$ such that $\psi(x) = 0$ if $x \leq -\frac{2}{3}$ and $\psi(x) = 1$ if $x \geq -\frac{1}{3}$. We define $\phi : \mathbb{R} \longrightarrow [0,1]$ the smooth function defined by $\phi(x) = \sqrt{1 - (\psi(x))^2}$, so that $\psi(x)^2 + \phi(x)^2 = 1$.

For $\lambda \in \mathbb{R}$, we define ψ_λ and ϕ_λ by $\psi_\lambda(x) = \psi(\frac{x}{\lambda})$ and $\phi_\lambda(x) = \phi(\frac{x}{\lambda})$, so that

$$\psi_\lambda(x)^2 + \phi_\lambda(x)^2 = 1, \tag{4.2}$$

with supp $\psi_\lambda \subset [-\frac{2\lambda}{3}, +\infty[$ and supp $\phi_\lambda \subset]-\infty, -\frac{\lambda}{3}]$.

Using (4.2), we have:

$$\langle \mathscr{L}_\lambda w \,|\, w \rangle_\lambda = \left\langle \mathscr{L}_\lambda w \,|\, \psi_\lambda^2 w \right\rangle_\lambda + \left\langle \mathscr{L}_\lambda w \,|\, \phi_\lambda^2 w \right\rangle_\lambda.$$

Using that $\mathscr{L}_\lambda v = -\frac{1}{s(x+\lambda)} \partial_x (s(x+\lambda)\partial_x v) + g(x) v$, we have:

$$\left\langle \mathscr{L}_\lambda w \,|\, \psi_\lambda^2 w \right\rangle_\lambda = \tau_1 + \int_{\mathbb{R}} s(x+\lambda) g(x) |\psi_\lambda w|^2,$$

with

$$\tau_1 = \int_{\mathbb{R}} -\partial_x (s(x+\lambda)\partial_x w) \cdot \psi_\lambda(x)^2 w.$$

We have:

$$\begin{aligned}
\tau_1 &= \int_{\mathbb{R}} s(x+\lambda)\partial_x w \cdot \partial_x(\psi_\lambda(x)^2 w) \\
&= \int_{\mathbb{R}} s(x+\lambda)\partial_x w \cdot \psi_\lambda w \partial_x \psi_\lambda(x) + \int_{\mathbb{R}} s(x+\lambda)\partial_x w \cdot \partial_x(\psi_\lambda w)\psi_\lambda(x), \\
&= \int_{\mathbb{R}} s(x+\lambda)|\partial_x(\psi_\lambda w)|^2 - \int_{\mathbb{R}} s(x+\lambda) w \partial_x \psi_\lambda \cdot \partial_x(\psi_\lambda w) + \int_{\mathbb{R}} s(x+\lambda)\partial_x w \cdot w \psi_\lambda \partial_x \psi_\lambda(x), \\
&= \int_{\mathbb{R}} s(x+\lambda)|\partial_x(\psi_\lambda w)|^2 - \int_{\mathbb{R}} s(x+\lambda)|w|^2 |\partial_x \psi_\lambda|^2.
\end{aligned}$$

Thus, we have:

$$\left\langle \mathscr{L}_\lambda w \,|\, \psi_\lambda^2 w \right\rangle_\lambda = \int_{\mathbb{R}} s(x+\lambda)\big(|\partial_x(\psi_\lambda w)|^2 + g(x)|\psi_\lambda w|^2\big) - \int_{\mathbb{R}} s(x+\lambda)|w|^2|\partial_x \psi_\lambda|^2.$$

With the same calculation for the term in ϕ_λ, we obtain that:

$$\langle \mathscr{L}_\lambda w \,|\, w \rangle_\lambda = \langle \mathscr{L}_\lambda(\psi_\lambda w) \,|\, \psi_\lambda w \rangle_\lambda + \langle \mathscr{L}_\lambda(\phi_\lambda w) \,|\, \phi_\lambda w \rangle_\lambda - \int_{\mathbb{R}} s(x+\lambda)|w|^2\big(|\partial_x \psi_\lambda|^2 + |\partial_x \phi_\lambda|^2\big). \tag{4.3}$$

Estimate for $\psi_\lambda w$

We remark that the support of $\psi_\lambda w$ is included in $[-\frac{2\lambda}{3},+\infty[$, and that the support of $s'(\cdot+\lambda)$ is included in $[-a-\lambda,a-\lambda]$. Thus, for $\lambda\geq 3a$, $[-\frac{2\lambda}{3},+\infty[\cap[-a-\lambda,a-\lambda]=\emptyset$, so that $s(\cdot+\lambda)=1$ on the support of $\psi_\lambda W$. Therefore

$$\langle \mathscr{L}_\lambda(\psi_\lambda w)\,|\,\psi_\lambda w\rangle_\lambda=\langle L(\psi_\lambda w)\,|\,\psi_\lambda w\rangle.$$

In order to use the coercivity of L in Ker $L^\perp$, for $u\in H^2(\mathbb{R})$, we define $u^\perp$ by:

$$u^\perp=u-\frac{1}{2}\langle u\,|\,1/\cosh(x)\rangle\frac{1}{\cosh x},$$

and since $u^\perp\in(\operatorname{Ker}L)^\perp$, we have:

$$\langle Lu\,|\,u\rangle=\left\langle Lu^\perp\,|\,u^\perp\right\rangle\geq\|u^\perp\|^2_{L^2}.$$

Since by Pythagore:

$$\|u\|^2_{L^2(\mathbb{R})}=\|u^\perp\|^2_{L^2(\mathbb{R})}+\frac{1}{2}|\langle u\,|\,1/\cosh x\rangle|^2,$$

we obtain that:

$$\langle Lu\,|\,u\rangle\geq\|u\|^2_{L^2(\mathbb{R})}-\frac{1}{2}|\langle u\,|\,1/\cosh x\rangle|^2. \tag{4.4}$$

We apply (4.4) for $u=\psi_\lambda w$.

$$\langle\psi_\lambda w\,|\,1/\cosh x\rangle=\int_{\mathbb{R}}w\frac{1}{\cosh x}+\int_{\mathbb{R}}(\psi_\lambda-1)w\frac{1}{\cosh x}\cdot$$

The first term of the right-hand side vanishes by orthogonality condition $w\in(\operatorname{Ker}L)^\perp$. In addition,

$$\left|\int_{\mathbb{R}}(\psi_\lambda-1)w\frac{1}{\cosh x}\right|\leq\|w\|_{L^2(\mathbb{R})}\left\|(\psi_\lambda-1)\frac{1}{\cosh x}\right\|_{L^2(\mathbb{R})}.$$

Since:

$$\left\|(\psi_\lambda-1)\frac{1}{\cosh x}\right\|^2_{L^2(\mathbb{R})}=\int_{\mathbb{R}}|\psi_\lambda-1|^2\frac{1}{(\cosh x)^2}\leq\int_{-\infty}^{-\frac{\lambda}{3}}\frac{1}{(\cosh x)^2}\leq 1-\tanh\left(\frac{\lambda}{3}\right),$$

we obtain that:

$$\langle\mathscr{L}_\lambda(\psi_\lambda w)\,|\,\psi_\lambda w\rangle_\lambda\geq\|\psi_\lambda w\|^2_{L^2_\lambda}-\frac{1-\tanh(\frac{\lambda}{3})}{2}\|w\|^2_{L^2(\mathbb{R})}. \tag{4.5}$$

Estimate for $\phi_\lambda w$

We have:

$$\langle\mathscr{L}_\lambda(\phi_\lambda w)\,|\,\phi_\lambda w\rangle_\lambda=\int_{\mathbb{R}}s(x-\lambda)\big(|\partial_x(\phi_\lambda w)|^2+g(x)|\phi_\lambda w|^2\big).$$

We remark that $\operatorname{supp}\phi_\lambda\subset]-\infty,-\frac{\lambda}{3}]$, so if $\lambda\geq 3x_0$ (where x_0 is defined by (4.1)), then $g(x)\geq\frac{1}{2}$ on the support of $\phi_\lambda w$. Therefore,

$$\forall\,\lambda\geq 3x_0,\quad\langle\mathscr{L}_\lambda(\phi_\lambda w)\,|\,\phi_\lambda w\rangle_\lambda\geq\frac{1}{2}\|\phi_K w\|^2_{L^2_\lambda(\mathbb{R})}. \tag{4.6}$$

Estimate for the last term in (4.3)

We remark that $\partial_x \psi_\lambda = \frac{1}{\lambda}\psi'(\frac{x}{\lambda})$, so that:

$$\|\partial_x \psi_\lambda\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\lambda}.$$

The same estimate occurs for ϕ_λ, and we obtain that there exists a constant C such that:

$$\left|\int_{\mathbb{R}} s(x+\lambda)|w|^2\big(|\partial_x\psi_\lambda|^2 + |\partial_x\phi_\lambda|^2\big)\right| \leq \frac{C}{\lambda^2}\|w\|^2_{L^2_\lambda(\mathbb{R})}. \tag{4.7}$$

From (4.3), (4.5), (4.6) and (4.7), using also that for all λ,

$$\rho_1\|u\|^2_{L^2(\mathbb{R})} \leq \|u\|^2_{L^2_\lambda(\mathbb{R})} \leq \|u\|^2_{L^2(\mathbb{R})},$$

we obtain that for all λ,

$$\langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda \geq \|\psi_\lambda w\|^2_{L^2_\lambda} + \frac{1}{2}\|\phi_K w\|^2_{L^2_\lambda} - \left(\frac{1-\tanh\left(\frac{\lambda}{3}\right)}{2\rho_1} + \frac{C}{\lambda^2}\right)\|w\|^2_{L^2_\lambda}.$$

Since $\psi_\lambda^2 + \phi_\lambda^2 = 1$, since $\frac{1-\tanh(\frac{\lambda}{3})}{2\rho_1} + \frac{C}{\lambda^2}$ tends to zero when λ tends to $+\infty$, there exists λ_0 such that for all $\lambda \geq \lambda_0$, for all $w \in H^2(\mathbb{R};\mathbb{R})$ such that $\left\langle w \,|\, \frac{1}{\cosh x}\right\rangle = 0$,

$$\langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda \geq \frac{1}{4}\|w\|^2_{L^2_\lambda}. \tag{4.8}$$

This concludes the proof of Proposition 4.1. ☐

Remark 4.1. For $w \in H^2(\mathbb{R})$, we have:

$$\langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda = \int_{\mathbb{R}} s(x+\lambda)\big(|\partial_x w|^2 + g(x)|w|^2\big),$$

so we can give a sense to $\langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda$ for $w \in H^1(\mathbb{R})$, and by density argument, we can extend Inequality (4.8) to functions w in $H^1(\mathbb{R}) \cap \frac{1}{\cosh x}^{\perp}$.

By Cauchy–Schwartz inequality in (4.8), we obtain that for $\lambda \geq \lambda_0$, for all $w \in H^2(\mathbb{R};\mathbb{R}) \cap (1/\cosh x)^\perp$, we have:

$$\frac{1}{4}\|w\|^2_{L^2_\lambda} \leq \|\mathscr{L}_\lambda w\|_{L^2_\lambda}\|w\|_{L^2_\lambda},$$

so that:

$$\|w\|_{L^2_\lambda} \leq 4\|\mathscr{L}_\lambda w\|_{L^2_\lambda}. \tag{4.9}$$

In addition, by Cauchy–Schwartz inequality and using (4.9), we have:

$$\langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda \leq 4\|\mathscr{L}_\lambda w\|^2_{L^2_\lambda}. \tag{4.10}$$

We obtain also the following corollary:

Corollary 4.1. *There exist α_1 and α_2, with $0 < \alpha_1 < \alpha_2$, there exists λ_0 such that for all $\lambda \geq \lambda_0$, for all $w \in H^2(\mathbb{R};\mathbb{R})$ such that $\left\langle w \,|\, \frac{1}{\cosh x}\right\rangle = 0$,*

$$\begin{aligned} \alpha_1\|w\|_{H^1(\mathbb{R})} &\leq \langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda^{\frac{1}{2}} \leq \alpha_2\|w\|_{H^1(\mathbb{R})},\\ \alpha_1\|w\|_{H^2(\mathbb{R})} &\leq \|\mathscr{L}_\lambda w\|_{L^2_\lambda} \leq \alpha_2\|w\|_{H^2(\mathbb{R})}. \end{aligned} \tag{4.11}$$

Proof. First, we have:

$$\|\partial_x w\|^2_{L^2(\mathbb{R})} \le \frac{1}{\rho_1}\|\partial_x w\|^2_{L^2_\lambda} \le \frac{1}{\rho_1}\Big(\langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda + \|g\|_{L^\infty(\mathbb{R})}\|w\|^2_{L^2_\lambda}\Big).$$

So, using (4.9) and (4.10), we obtain that:

$$\|\partial_x w\|^2_{L^2(\mathbb{R})} \le \frac{20}{\rho_1}\|\mathscr{L}_\lambda w\|^2_{L^2_\lambda}. \tag{4.12}$$

Writing that $\partial_{xx} w = -\mathscr{L}_\lambda w + g(x)w - \frac{s'}{s}(x+\lambda)\partial_x w$, we obtain that:

$$\begin{aligned}\|\partial_{xx} w\|^2_{L^2(\mathbb{R})} &\le \frac{1}{\rho_1}\|\partial_{xx} w\|^2_{L^2_\lambda},\\ &\le \frac{1}{\rho_1}\left(\|\mathscr{L}_\lambda w\|^2_{L^2_\lambda} + \|g\|_{L^\infty(\mathbb{R})}\|w\|^2_{L^2_\lambda} + \left\|\frac{s'}{s}\right\|_{L^\infty(\mathbb{R})}\|\partial_x w\|_{L^2_\lambda}\right),\\ &\le K\|\mathscr{L}_\lambda w\|^2_{L^2_\lambda}\end{aligned} \tag{4.13}$$

where K does not depend on $\lambda \ge \lambda_0$.

From (4.9), (4.12) and (4.13), we obtain that there exists a constant K independent of $\lambda \ge \lambda_0$ such that for all w in $(\operatorname{Ker} L)^\perp$,

$$\|w\|_{H^1(\mathbb{R})} \le K\langle \mathscr{L}_\lambda w \,|\, w\rangle_\lambda^{\frac12} \quad \text{and} \quad \|w\|_{H^2(\mathbb{R})} \le K\|\mathscr{L}_\lambda w\|_{L^2_\lambda}. \tag{4.14}$$

From (4.14), we obtain the existence of α_1. The existence for α_2 is straightforward. □

5. Variational estimates and proof of Theorem 1.1

In the remainder of this paper, we assume that:

$$\|h\|_{L^\infty(\mathbb{R}^+)} \le 1 \quad \text{and} \quad \forall\, t \ge 0,\ \sigma_0 + H(t) \ge \lambda_0. \tag{5.1}$$

We will perform variational estimates on equations (3.9) and (3.10). We equip $L^2(\mathbb{R})$ with the weighted scalar product (see [7]) given by: for $u, v \in L^2(\mathbb{R})$,

$$\langle u \,|\, v\rangle_\star = \int_{\mathbb{R}} s(x+\sigma_0+H(t))u(x)v(x)\,\mathrm{d}x = \langle u \,|\, v\rangle_{\sigma_0+H(t)}, \tag{5.2}$$

associated with the norm $\|\cdot\|_{L^2_\star(\mathbb{R})} = \|\cdot\|_{L^2_{\sigma_0+H(t)}}$ defined by

$$\|u\|_{L^2_\star(\mathbb{R})} = \left(\int_{\mathbb{R}} s(x+\sigma_0+H(t))|u(x)|^2\,\mathrm{d}x\right)^{\frac12}. \tag{5.3}$$

Using (4.11), we define $\nu_1 > 0$ such that $\nu_1 \le \nu_0$ and such that for all $W \in H^1(\mathbb{R};\mathbb{R}^2) \cap (\operatorname{Ker} L)^\perp$,

$$\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac12} \le \nu_1 \Longrightarrow \|W\|_{H^1(\mathbb{R})} \le \nu_0.$$

Therefore, while $|\Lambda(t)| \le \nu_1$ and $\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac12} \le \nu_1$, the system (3.9) and (3.10) is valid.

The right-hand-side terms in (3.9) and (3.10) are estimated in the following proposition.

Proposition 5.1. *Under assumption (5.1), there exists a constant C_1 such that while $|\Lambda(t)| \le \nu_1$, while $\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac12} \le \nu_1$,*

$$\begin{aligned}\|F(t,\cdot,\sigma_0,\Lambda,W)\|_{L^2_\star} &\le C_1\|\mathscr{L}_\star W\|_{L^2_\star}\Big(|\Lambda| + \langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac12}\Big) + C_1\exp(-\sigma_0 - H(t)),\\ |\mathcal{G}(t,\sigma_0,\Lambda,W)| &\le C_1\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac12} + C_1\exp(-\sigma_0 - H(t)).\end{aligned} \tag{5.4}$$

For the convenience of the reader, the proof of Proposition 5.1 is postponed in Section 6.

5.1. Variational estimates for W

We observe that $s(-\partial_{xx} - \frac{s'}{s}\partial_x) = -\partial_x(s\partial_x)$, which implies that the operator $\mathscr{L}_\star$ is self-adjoint with respect to the weighted scalar product $\langle\cdot\,|\,\cdot\rangle_\star$. Moreover, for all $u \in H^2(\mathbb{R})$, we have

$$\langle \mathscr{L}_\star u\,|\,u\rangle_\star = \int_{\mathbb{R}} s(x+\sigma_0+H(t))\big(|\partial_x u|^2 + g(x)|u|^2\big), \tag{5.5}$$

where $g(x) = 2\tanh^2 x - 1$. Now, we take the scalar product in $L^2_\star(\mathbb{R})$ of (3.10) with $\mathscr{L}_\star W$. We obtain:

$$\left\langle \frac{\partial W}{\partial t}\,|\,\mathscr{L}_\star W\right\rangle_\star = -\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})} + h(t)\langle \ell(W)\,|\,\mathscr{L}_\star W\rangle_\star + \langle F(t,x,\sigma_0,\Lambda,W)\,|\,\mathscr{L}_\star W\rangle_\star. \tag{5.6}$$

We notice that

$$\left\langle \frac{\partial W}{\partial t}\,|\,\mathscr{L}_\star W\right\rangle_\star = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle W\,|\,\mathscr{L}_\star W\rangle_\star - \frac{1}{2}h(t)\int_{\mathbb{R}} s'(x+\sigma_0+H(t))\big(|\partial_x W|^2 + g(x)|W|^2\big).$$

Since s' is bounded, there exists a constant C_2 such that:

$$h(t)\int_{\mathbb{R}} s'(x+\sigma_0+H(t))\big(|\partial_x W|^2 + g(x)|W|^2\big) \le C_2\|h\|_{L^\infty}\|W\|^2_{H^1(\mathbb{R})},$$

therefore, using (4.11), there exists a constant C_3 such that

$$h(t)\int_{\mathbb{R}} s'(x+\sigma_0+H(t))\big(|\partial_x W|^2 + g(x)|W|^2\big) \le C_3\|h\|_{L^\infty}\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})}. \tag{5.7}$$

By the Cauchy–Schwarz inequality,

$$|\langle \ell(W)\,|\,\mathscr{L}_\star W\rangle_\star| \le \|\ell(W)\|_{L^2_\star(\mathbb{R})}\|\mathscr{L}_\star W\|_{L^2_\star(\mathbb{R})},$$

and since $\|\ell(W)\|_{L^2_\star(\mathbb{R})} \le C\|W\|_{H^1(\mathbb{R})}$, using (4.11), there exists a constant C_4 such that:

$$|\langle \ell(W)\,|\,\mathscr{L}_\star W\rangle_\star| \le \|\ell(W)\|_{L^2_\star(\mathbb{R})}\|\mathscr{L}_\star W\|_{L^2_\star(\mathbb{R})} \le C_4\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})}, \tag{5.8}$$

In addition, by using Proposition 5.1, we obtain:

$$\begin{aligned}|\langle F(t,x,\sigma_0,\Lambda,W)\,|\,\mathscr{L}_\star W\rangle_\star| \le{}& C_1\Big(|\Lambda| + \langle \mathscr{L}_\star W\,|\,W\rangle_\star^{\frac{1}{2}}\Big)\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})}\\ &+ C_1\exp(-(\sigma_0+H(t)))\|\mathscr{L}_\star W\|_{L^2_\star(\mathbb{R})}.\end{aligned} \tag{5.9}$$

Therefore, combining (5.6)–(5.9), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{L}_\star W\,|\,W\rangle_\star + 2\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})} \le{}& C_1\exp(-(\sigma_0+H(t)))\|\mathscr{L}_\star W\|_{L^2_\star(\mathbb{R})}\\ &+ \Big(2(C_3+C_4)\|h\|_{L^\infty} + C_1\Big(|\Lambda| + (\langle \mathscr{L}_\star W\,|\,W\rangle_\star)^{\frac{1}{2}}\Big)\Big)\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})}.\end{aligned}$$

By Young inequality, we obtain that:

$$C_1\exp(-(\sigma_0+H(t)))\|\mathscr{L}_\star W\|_{L^2_\star(\mathbb{R})} \le \frac{1}{4}\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})} + 2(C_1)^2\exp(-2(\sigma_0+H(t))).$$

Therefore, under assumption (5.1), we obtain that there exist $K_1 > 0$ and $K_2 > 0$ such that while $|\Lambda| \le \nu_1$ and $\langle \mathscr{L}_\star W\,|\,W\rangle_\star^{\frac{1}{2}} \le \nu_1$, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{L}_\star W\,|\,W\rangle_\star + \Big(2 - \frac{1}{4} - K_1\Big(|\Lambda| + \langle \mathscr{L}_\star W\,|\,W\rangle_\star^{\frac{1}{2}} + \|h\|_{L^\infty}\Big)\Big)\|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})}&\\ \le K_2\exp(-2(\sigma_0+H(t))).&\end{aligned} \tag{5.10}$$

We assume that $\|h\|_{L^\infty} \le \delta_1 := \min\{1, \frac{1}{4K_1}\}$. We denote by $\eta_1 = \min(\nu_1, \frac{1}{4K_1})$. Then, while $|\Lambda| \le \eta_1$ and $\langle \mathscr{L}_\star W \mid W\rangle_\star^{\frac{1}{2}} \le \eta_1$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{L}_\star W \mid W\rangle_\star + \|\mathscr{L}_\star W\|^2_{L^2_\star(\mathbb{R})} \le K_2 \exp(-2(\sigma_0 + H(t))).$$

Therefore, using (4.10), we obtain that while $|\Lambda| \le \eta_1$ and $\langle \mathscr{L}_\star W \mid W\rangle_\star^{\frac{1}{2}} \le \eta_1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{L}_\star W \mid W\rangle_\star + \frac{1}{4}\langle \mathscr{L}_\star W \mid W\rangle_\star \le K_2 \exp(-2(\sigma_0 + H(t))). \tag{5.11}$$

We consider the solution of the Cauchy problem:

$$\begin{cases} Y' + \dfrac{1}{4}Y = K_2 \exp(-2(\sigma_0 + H(t))), \\ Y(0) = \langle \mathscr{L}_\star W(0,\cdot) \mid W(0,\cdot)\rangle_\star, \end{cases} \tag{5.12}$$

given by:

$$Y(t) = \langle \mathscr{L}_\star W(0,\cdot) \mid W(0,\cdot)\rangle_\star \exp\left(-\frac{t}{4}\right) + K_2 \exp(-2\sigma_0) \int_0^t \exp\left(\frac{s-t}{4} - 2H(s)\right) \mathrm{d}s. \tag{5.13}$$

By (5.11), $\langle \mathscr{L}_\star W \mid W\rangle_\star$ is a sub-solution for (5.12), so that: for all $t \ge 0$, while $|\Lambda| \le \eta_1$ and $\langle \mathscr{L}_\star W \mid W\rangle_\star^{\frac{1}{2}} \le \eta_1$, we have:

$$\langle \mathscr{L}_\star W \mid W\rangle_\star(t) \le Y(t). \tag{5.14}$$

We remark that for all $t \ge 0$,

$$\sqrt{Y(t)} \le \sqrt{\langle \mathscr{L}_\star W(0,\cdot) \mid W(0,\cdot)\rangle_\star} \exp\left(-\frac{t}{8}\right) + \sqrt{K_2} \exp(-\sigma_0)\exp(-H(t))\left(\int_0^t \exp\left(\frac{s-t}{4} + 2H(t) - 2H(s)\right)\mathrm{d}s\right)^{\frac{1}{2}}.$$

We remark that for $0 \le s \le t$,

$$H(t) - H(s) \le \int_s^t |h(\tau)|\,\mathrm{d}\tau \le (t-s)\|h\|_{L^\infty(\mathbb{R}^+)}.$$

Therefore,

$$\int_0^t \exp\left(\frac{s-t}{4} + 2H(t) - 2H(s)\right)\mathrm{d}s \le \int_0^t \exp\left(2\|h\|_{L^\infty} - \frac{1}{4}\right)(t-s)\,\mathrm{d}s.$$

Thus, there exists K_3 such that if $\|h\|_{L^\infty} \le \frac{1}{16}$,

$$\sqrt{Y(t)} \le \sqrt{\langle \mathscr{L}_\star W(0,\cdot) \mid W(0,\cdot)\rangle_\star} \exp\left(-\frac{t}{8}\right) + K_3 \exp(-\sigma_0)\exp(-H(t)). \tag{5.15}$$

So that:

$$\int_0^{+\infty} \sqrt{Y(t)}\,\mathrm{d}t \le 8\sqrt{\langle \mathscr{L}_\star W(0,\cdot) \mid W(0,\cdot)\rangle_\star} + K_3 \exp(-\sigma_0) \int_0^{+\infty} \exp(-H(t)). \tag{5.16}$$

5.2. Estimate on Λ

According to (3.9), (5.4) and (5.14), for all $t \geq 0$, while $|\Lambda| \leq \eta_1$ and $\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac{1}{2}} \leq \eta_1$, we have:

$$\left|\frac{\mathrm{d}\Lambda}{\mathrm{d}t}\right| = |\mathcal{G}(t,\sigma_0,\Lambda,W)| \leq C_1\sqrt{Y(t)} + C_1\exp(-(\sigma_0 + H(t))). \tag{5.17}$$

We define $\mathscr{H}$ by:

$$\mathscr{H} = \int_0^{+\infty} \exp(-H(t))\,\mathrm{d}t.$$

By integration of (5.17), by Estimate (5.16), under the assumption $\|h\|_{L^\infty} \leq \min\{\frac{1}{16},\delta_1\}$, we obtain that while $|\Lambda| \leq \eta_1$ and $\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac{1}{2}} \leq \eta_1$, we have:

$$\begin{aligned} |\Lambda(t)| &\leq |\Lambda(0)| + C_1\int_0^{+\infty}\sqrt{Y(\tau)}\,\mathrm{d}\tau + C_1\exp(-\sigma_0)\int_0^{+\infty}\exp(-H(\tau))\,\mathrm{d}\tau, \\ &\leq |\Lambda(0)| + C_1\left(8\sqrt{\langle \mathscr{L}_\star W(0,\cdot)\,|\,W(0,\cdot)\rangle_\star} + K_3\mathscr{H}\exp(-\sigma_0)\right) + C_1\exp(-\sigma_0)\mathscr{H}. \end{aligned}$$

Therefore, under the assumption $\|h\|_{L^\infty} \leq \min\{\frac{1}{16},\delta_1\}$, there exists a constant K_4 such that while $|\Lambda| \leq \eta_1$ and $\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac{1}{2}} \leq \eta_1$, we have:

$$|\Lambda(t)| \leq |\Lambda(0)| + K_4\sqrt{\langle \mathscr{L}_\star W(0,\cdot)\,|\,W(0,\cdot)\rangle_\star} + K_4\mathscr{H}\exp(-\sigma_0). \tag{5.18}$$

5.3. End of the proof

We define δ_0 by $\delta_0 = \min\{\frac{1}{16},\delta_1\}$. We assume that $\|h\|_{L^\infty} \leq \delta_0$ and that $\mathscr{H} = \int_0^{+\infty}\exp(-H(t))\,\mathrm{d}t < +\infty$. We remark that these asumptions imply that $H(t)$ tends to $+\infty$ when t tends to $+\infty$, so that $H(t)$ is bounded by below on $\mathbb{R}^+$:

$$\exists\, H^- \in \mathbb{R}^+, \qquad \forall\, t \geq 0,\; -H^- \leq H(t).$$

We fix $\bar{\sigma}$ such that

$$\lambda_0 + H^- \leq \bar{\sigma}, \quad K_3 e^{H^-}e^{-\bar{\sigma}} \leq \frac{\eta_1}{3} \quad \text{and} \quad K_4\mathscr{H}e^{-\bar{\sigma}} \leq \frac{\eta_1}{4} \tag{5.19}$$

(we recall that λ_0 is given by Prop. 4.1). We fix $\bar{\nu} > 0$ such that

$$\bar{\nu} \leq \frac{\eta_1}{4} \quad \text{and} \quad K_4\bar{\nu} \leq \frac{\eta_1}{4}\cdot \tag{5.20}$$

We assume that:

$$\sigma_0 \geq \bar{\sigma}, \quad |\Lambda(0)| \leq \bar{\nu} \quad \text{and} \quad \langle \mathscr{L}_\star W(0,\cdot)\,|\,W(0,\cdot)\rangle_\star^{\frac{1}{2}} \leq \bar{\nu}. \tag{5.21}$$

We remark that for all t, $\sigma_0 + H(t) \geq \lambda_0$. Let us prove that for all $t \geq 0$,

$$|\Lambda(t)| \leq \eta_1 \quad \text{and} \quad \langle \mathscr{L}_\star W(t,\cdot)\,|\,W(t,\cdot)\rangle_\star^{\frac{1}{2}} \leq \eta_1. \tag{5.22}$$

If it is not the case, we consider the first time T such that:

$$|\Lambda(T)| = \eta_1 \quad \text{or} \quad \langle \mathscr{L}_\star W(T,\cdot)\,|\,W(T,\cdot)\rangle_\star^{\frac{1}{2}} = \eta_1. \tag{5.23}$$

Since at initial time, $|\Lambda(0)| \le \bar{\nu} < \eta_1$ and $\langle \mathscr{L}_\star W(0,\cdot) \,|\, W(0,\cdot)\rangle_\star^{\frac{1}{2}} \le \nu < \eta_1$, $T > 0$. For all time $t \in [0,T]$, we are in the domain of validity for (5.14) and (5.18), so using also (5.15), (5.19) and (5.20), we have:

$$\begin{aligned}\sqrt{\langle \mathscr{L}_\star W \,|\, W\rangle_\star}(t) &\le \sqrt{\langle \mathscr{L}_\star W(0,\cdot) \,|\, W(0,\cdot)\rangle_\star}\exp\left(-\frac{t}{8}\right) + K_3\exp(-\sigma_0)\exp(-H(t))\\ &\le \sqrt{\langle \mathscr{L}_\star W(0,\cdot) \,|\, W(0,\cdot)\rangle_\star} + K_3\exp(-\sigma_0)\exp(H^-)\\ &\le \frac{2\eta_1}{3}\end{aligned}$$

and

$$|\Lambda(t)| \le \frac{3\eta_1}{4},$$

which leads to a contradiction with (5.23). Thus, (5.22) holds for all $t \ge 0$.

Therefore, under assumption (5.21), the following holds for all $t \ge 0$:

$$\begin{cases}\langle \mathscr{L}_\star W \,|\, W\rangle_\star^{\frac{1}{2}}(t) \le \sqrt{\langle \mathscr{L}_\star W(0,\cdot) \,|\, W(0,\cdot)\rangle_\star}\exp\left(-\frac{t}{8}\right) + K_3\exp(-\sigma_0)\exp(-H(t)),\\ |\Lambda(t)| \le |\Lambda(0)| + K_4\sqrt{\langle \mathscr{L}_\star W(0,\cdot) \,|\, W(0,\cdot)\rangle_\star} + K_4\mathscr{H}\exp(-\sigma_0),\\ \left|\dfrac{\mathrm{d}\Lambda}{\mathrm{d}t}\right| \le C_1\sqrt{Y(t)} + C_1\exp(-\sigma_0)\exp(-H(t)).\end{cases}.$$

The first two estimates imply the stability of $(0,0)$ for the system coupling (3.9) and (3.10). The third estimate implies that $\Lambda(t)$ has a limit when t tends to $+\infty$. This concludes the proof of Theorem 1.1. It remains to prove Proposition 5.1.

6. Proof of Proposition 5.1

6.1. Estimates of K and $\mathcal{N}$

From (3.2), there exists a constant C such that for all $\Lambda \in \mathbb{R}^2$ such that $|\Lambda| \le \nu_0$, we have:

$$\forall x \in \mathbb{R}, \qquad |R_\Lambda(x)| + |\partial_x R_\Lambda(x)| + |\partial_{xx} R_\Lambda(x)| \le \frac{C}{\cosh x}|\Lambda|. \tag{6.1}$$

In particular, we obtain that

$$\|R_\Lambda\|_{L^\infty(\mathbb{R})} + \|\partial_x R_\Lambda\|_{L^\infty(\mathbb{R})} + \|\partial_{xx} R_\Lambda(x)\|_{L^\infty(\mathbb{R})} \le C|\Lambda|.$$

In addition, while $|\Lambda(t)| \le \nu_0$ and $\|W(t,\cdot)\|_{H^1(\mathbb{R})} \le \nu_0$, we have:

$$\|R_{\Lambda(t)} + W(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C\big(|\Lambda(t)| + \|W(t,\cdot)\|_{H^1(\mathbb{R})}\big).$$

Since the support of s' is included in $[-a,a]$, the support of $x \mapsto \mathfrak{a}_\star(x)$ is included in $[-a-\sigma_0-H(t), a-\sigma_0-H(t)]$, so that for all $\sigma_0 \ge a$ and all t such that $H(t) \ge a$,

$$|\mathfrak{p}_\star(x)| \le \|\frac{s'}{s}\|_{L^\infty}\frac{1}{\cosh(x)}\chi_{[-a-\sigma_0-H(t),a-\sigma_0-H(t)]}.$$

Now, there exists a constant C such that

$$\forall x \in \mathbb{R}, \quad \forall K \in \mathbb{R}, \qquad \left|\frac{\chi_{[-a-K,a-K]}}{\cosh x}\right| \le Ce^{-K}. \tag{6.2}$$

Therefore, there exists a constant C such that for all $\sigma_0 \geq a$, for all $t \geq 0$ such that $H(t) \geq a$ and all $x \in \mathbb{R}$,

$$\|\mathfrak{p}_\star\|_{L^2(\mathbb{R})} \leq C e^{-\sigma_0 - H(t)}. \tag{6.3}$$

In addition, using (6.1), using that $F_6(r) = \mathcal{O}(|r|)$ and $F_7(r) = \mathcal{O}(|r|^2)$, we obtain that there exists a constant C such that for all $x \in \mathbb{R}$, while $|\Lambda(t)| \leq \nu_0$, we have:

$$|-\mathcal{J}\partial_x R_\Lambda + F_6(R_\Lambda)(\partial_x R_\Lambda) + F_7(x, R_\Lambda)| \leq \frac{C}{\cosh x},$$

so that, with the same arguments as above, there exists a constant C such that for all σ_0, for all $t \geq 0$ and all $x \in \mathbb{R}$, while $|\Lambda(t)| \leq \nu_0$

$$\|K(t, \cdot, \sigma_0, \Lambda)\|_{L^2(\mathbb{R})} \leq C_1 e^{-\sigma_0 - H(t)}. \tag{6.4}$$

By Cauchy–Schwartz inequality, we obtain that there exists C such that for all $\sigma_0 \geq a$, for all $t \geq 0$ such that $H(t) \geq a$, while $|\Lambda(t)| \leq \nu_0$,

$$|\overline{K}(t, \sigma_0, \Lambda)| \leq C e^{-\sigma_0 - H(t)}. \tag{6.5}$$

We recall that $\mathcal{N}(t, W)$ is defined by $\mathcal{N}(t, W) = \mathcal{N}_1(t, W) + \mathcal{N}_2(t, W)$, with:

$$\mathcal{N}_1(t, W) = \begin{pmatrix} \langle \mathcal{J}\mathscr{L}_\star(W) \,|\, a_1 \rangle \\ \langle \mathcal{J}\mathscr{L}_\star(W) \,|\, a_2 \rangle \end{pmatrix}, \qquad \mathcal{N}_2(t, W) = \begin{pmatrix} \langle h(t)\ell(W) \,|\, a_1 \rangle \\ \langle h(t)\ell(W) \,|\, a_2 \rangle \end{pmatrix}$$

where $\mathscr{L}_\star = L - \mathfrak{a}_\star(x)\partial_x$. For $j \in \{1, 2\}$, we have

$$\langle \mathcal{J} L(W) \,|\, a_j \rangle = \left\langle L(W) \,\Big|\, \frac{1}{\cosh x} \mathcal{J}^T e_j \right\rangle = \left\langle W \,\Big|\, L\Big(\frac{1}{\cosh x}\Big) \mathcal{J}^T e_j \right\rangle = 0,$$

so,

$$\langle \mathcal{J}\mathscr{L}_\star(W) \,|\, a_j \rangle = -\langle \mathcal{J}\mathfrak{a}_\star(x)\partial_x W \,|\, a_j \rangle.$$

Therefore, with arguments used to prove Inequality (6.3), we have:

$$|\langle \mathcal{J}\mathscr{L}_\star(W) \,|\, a_j \rangle| \leq C \|\mathfrak{a}_\star(x) \frac{1}{\cosh x}\|_{L^2(\mathbb{R})} \|W\|_{H^1(\mathbb{R})} \leq C e^{-\sigma_0 - H(t)} \|W\|_{H^1(\mathbb{R})}.$$

So,

$$|\mathcal{N}_1(t, W)| \leq C e^{-\sigma_0 - H(t)} \|W\|_{H^1(\mathbb{R})}. \tag{6.6}$$

On the other hand,

$$|\mathcal{N}_2(t, W)| \leq C |h(t)| \|W\|_{H^1(\mathbb{R})} \|a_j\|_{L^2(\mathbb{R})} \leq K |h(t)| \|W\|_{H^1(\mathbb{R})}. \tag{6.7}$$

6.2. Estimate of the nonlinear part

From the expressions of the F_i's, we obtain that there exists a constant C such that for all $r \in \overline{B}_2(0, \frac{1}{2})$ and for all $x \in \mathbb{R}$, we have:

$$\begin{aligned} |F_1(r)| + |F_2(x, r)| + |DF_4(x, r)| + |DF_5(r)| + |F_6(r)| + |DF_7(x, r)| \leq C|r|, \\ |DF_1(r)| + |DF_2(x, r)| + |F_3(r)| + |DF_3(r)| + |DF_6(r)| \leq C. \end{aligned} \tag{6.8}$$

Therefore, there exist universal constants C such that for all Λ and W such that $|\Lambda| \leq \nu_0$ and $\|W\|_{H^1(\mathbb{R})} \leq \nu_0$, we have:

$$\|T_1\|_{L^2(\mathbb{R})} \leq \|F_1(R_\Lambda + W)\|_{L^\infty(\mathbb{R})} \|\partial_{xx} W\|_{L^2(\mathbb{R})}$$

$$
\begin{aligned}
&\leq C\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big)\|W\|_{H^2(\mathbb{R})}.\\
\|T_2\|_{L^2(\mathbb{R})} &\leq \Big\|\widetilde{F}_1(\Lambda, W)\Big\|_{L^\infty(\mathbb{R})}\|W\|_{L^2(\mathbb{R})}\|\partial_{xx}R_\Lambda\|_{L^\infty(\mathbb{R})} + \|F_2(x, R_\Lambda + W)\|_{L^\infty(\mathbb{R})}\|\partial_x W\|_{L^2(\mathbb{R})}\\
&\quad + \Big\|\widetilde{F}_2(x, \Lambda, W)\Big\|_{L^\infty(\mathbb{R})}\|W\|_{L^2(\mathbb{R})}\|\partial_x R_\Lambda\|_{L^\infty(\mathbb{R})}\\
&\leq C\|W\|_{L^2(\mathbb{R})}|\Lambda| + C\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big)\|W\|_{H^1(\mathbb{R})} + C|\Lambda|\|W\|_{L^2(\mathbb{R})}.\\
\|T_3\|_{L^2(\mathbb{R})} &\leq \Big\|\widetilde{F}_3(\Lambda, W)\Big\|_{L^\infty(\mathbb{R})}\|W\|_{L^2(\mathbb{R})}\|\partial_x R_\Lambda\|^2_{L^\infty(\mathbb{R})}\\
&\quad + 2\|F_3(R_\Lambda + W)\|_{L^\infty(\mathbb{R})}\|\partial_x R_\Lambda\|_{L^\infty(\mathbb{R})}\|\partial_x W\|_{L^2(\mathbb{R})}\\
&\quad + \|F_3(R_\Lambda + W)\|_{L^\infty(\mathbb{R})}\|\partial_x W\|_{L^2(\mathbb{R})}\|\partial_x W\|_{L^\infty(\mathbb{R})}\\
&\leq C|\Lambda|\|W\|_{L^2(\mathbb{R})} + C|\Lambda|\|W\|_{H^1(\mathbb{R})} + C\|W\|_{H^1(\mathbb{R})}\|W\|_{H^2(\mathbb{R})}.\\
\|T_4\|_{L^2(\mathbb{R})} &\leq \Big\|\widetilde{F}_4(\Lambda, W)\Big\|_{L^\infty(\mathbb{R})}\|W\|_{L^2(\mathbb{R})} + |h(t)|\Big\|\widetilde{F}_5(\Lambda, W)\Big\|_{L^infty(\mathbb{R})}\|W\|_{L^2(\mathbb{R})}\\
&\leq C\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big)\|W\|_{L^2(\mathbb{R})}.\\
\|T_5\|_{L^2(\mathbb{R})} &\leq \|\mathfrak{a}_\star(\cdot)\|_{L^\infty(\mathbb{R})}\bigg(\Big\|\widetilde{F}_6(\Lambda, W)\Big\|_{L^\infty(\mathbb{R})}\|W\|_{L^2(\mathbb{R})}\|\partial_x R_\Lambda\|_{L^\infty(\mathbb{R})}\\
&\quad + \|F_6(R_\Lambda + W)\|_{L^\infty(\mathbb{R})}\|\partial_x W\|_{L^2(\mathbb{R})} + \Big\|\widetilde{F}_7(x, \Lambda, W)\Big\|_{L^\infty(\mathbb{R})}\|W\|_{L^2(\mathbb{R})}\bigg)\\
&\leq C\|W\|_{L^2(\mathbb{R})}|\Lambda| + C\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big)\|W\|_{H^1(\mathbb{R})}.
\end{aligned}
$$

Hence, there exists a constant C_2 such that while $|\Lambda| \leq \nu_0$ and $\|W\|_{H^1(\mathbb{R})} \leq \nu_0$, we have:

$$\forall i \in \{1, \dots, 5\}, \qquad \|T_i\|_{L^2(\mathbb{R})} \leq C_2\|W\|_{H^2(\mathbb{R})}\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big). \tag{6.9}$$

We claim that there exists a constant C_3 such that while $|\Lambda| \leq \nu_0$ and $\|W\|_{H^1(\mathbb{R})} \leq \nu_0$, we have:

$$\forall i \in \{1, \dots, 5\}, \qquad |\overline{T_i}| \leq C_3\|W\|_{H^1(\mathbb{R})}\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big). \tag{6.10}$$

Since $\overline{T_i}$ is the $L^2(\mathbb{R})$-scalar product of T_i with $\frac{1}{\cosh x}$, the estimates performed for T_i, $i = 2, 4, 5$, yield such estimates by Cauchy–Schwarz inequality.

Concerning $\overline{T_1}$, we have:

$$
\begin{aligned}
\langle T_1 \,|\, a_j\rangle &= \langle F_1(R_\Lambda + W)(\partial_{xx}W) \,|\, a_j\rangle\\
&= \langle \partial_{xx}W \,|\, (F_1(R_\Lambda + W))^* a_j\rangle,\\
&\qquad \text{where we denote by } A^* \text{ the transpose of a matrix } A,\\
&= -\big\langle \partial_x W \,|\, \partial_x\big((F_1(R_\Lambda + W))^* a_j\big)\big\rangle,\\
&= -\big\langle \partial_x W \,|\, D(F_1)^*(R_\Lambda + W)(\partial_x R_\Lambda + \partial_x W)a_j + (F_1(R_\Lambda + W))^*\partial_x a_j\big\rangle,
\end{aligned}
$$

so that

$$
\begin{aligned}
|\langle T_1 \,|\, a_j\rangle| &\leq \|\partial_x W\|_{L^2(\mathbb{R})}\bigg(\Big\|D(F_1)^\perp(R_\Lambda + W)\Big\|_{L^\infty(\mathbb{R})}\Big(\|\partial_x R_\Lambda + \partial_x W\|_{L^2(\mathbb{R})}\Big)\|a_j\|_{L^\infty(\mathbb{R})}\\
&\quad + \Big\|F_1(R_\Lambda + W)^\perp\Big\|_{L^2(\mathbb{R})}\|\partial_x a_j\|_{L^\infty(\mathbb{R})}\bigg)\\
&\leq C\|W\|_{H^1(\mathbb{R})}\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big).
\end{aligned}
$$

Concerning $\overline{T_3}$, we have:

$$|\langle T_3 \,|\, a_j\rangle| \leq \Big\|\widetilde{F}_3(\Lambda, W)(W)(\partial_x R_\Lambda, \partial_x R_\Lambda)\Big\|_{L^2(\mathbb{R})}\|a_j\|_{L^2(\mathbb{R})}$$

$$
\begin{aligned}
&+ \|2F_3(R_\Lambda + W)(\partial_x R_\Lambda, \partial_x W)\|_{L^2(\mathbb{R})} \|a_j\|_{L^2(\mathbb{R})} \\
&+ \|F_3(R_\Lambda + W)(\partial_x W, \partial_x W)\|_{L^1(\mathbb{R})} \|a_j\|_{L^\infty(\mathbb{R})} \\
\leq\ & \left\| \widetilde{F}_3(\Lambda, W) \right\|_{L^\infty(\mathbb{R})} \|W\|_{L^2(\mathbb{R})} \|\partial_x R_\Lambda\|^2_{L^\infty(\mathbb{R})} \|a_j\|_{L^2(\mathbb{R})} \\
&+ 2\|F_3(R_\Lambda + W)\|_{L^\infty(\mathbb{R})} \|\partial_x R_\Lambda\|_{L^\infty(\mathbb{R})} \|\partial_x W\|_{L^2(\mathbb{R})} \|a_j\|_{L^2(\mathbb{R})} \\
&+ \|F_3(R_\Lambda + W)\|_{L^\infty(\mathbb{R})} \|\partial_x W\|_{L^2(\mathbb{R})} \|\partial_x W\|_{L^2(\mathbb{R})} \|a_j\|_{L^\infty(\mathbb{R})} \\
\leq\ & C\|W\|_{H^1(\mathbb{R})}(|\Lambda| + \|W\|_{H^1}).
\end{aligned}
$$

Therefore, as shown previously, and by the Young inequalities, we obtain that there exist a constant C such that while $|\Lambda(t)| \leq \nu_0$ and $\|W(t)\|_{H^1(\mathbb{R})} \leq \nu_0$, then:

$$\|F(t, x, \sigma_0, \Lambda, W)\|_{L^2(\mathbb{R})} \leq C\|W\|_{H^2(\mathbb{R})}\big(|\Lambda| + \|W\|_{H^1(\mathbb{R})}\big) + C \exp(-(\sigma_0 + H(t))). \tag{6.11}$$

and,

$$|\mathcal{G}(t, \sigma_0, \Lambda, W)| \leq C\|W\|_{H^1(\mathbb{R})}\big(\|h\|_{L^\infty} + \|W\|_{H^1(\mathbb{R})}\big) + C \exp(-(\sigma_0 + H(t))). \tag{6.12}$$

All the previous inequalities in this subsection are true under the condition $|\Lambda(t)| \leq \nu_0$ and $\|W\|_{H^1(\mathbb{R})} \leq \nu_0$. We conclude the proof of Proposition 5.1 using the norms'equivalence results in Corollary 4.1.

7. Conclusion

For a small enough applied field, it is proven in [7] that the DW is pinned at the notch. It is also proven that the DW can be depinned by applying a large enough magnetic field. In the present paper, when a small magnetic field is applied, we establish that DW dynamics are less and less affected by the notch as the DW moves away from it.

To demonstrate our result, we need assumptions about the applied field. Concerning the smallness assumption (1.5), inspired by Jizzini [10], we can conjecture that any δ_0 in $[0, 1[$ would allow us to conclude. We remark that the condition $\delta_0 < 1$ is necessary to ensure that the magnetization almost equal to $+e_1$ in the right-hand-side domain will not be destabilized by a contrary applied field.

With regard to assumption (1.6), although certainly not optimal, it reflects the fact that the applied field must be strong enough to "control" the DW dynamics. Indeed, we can conjecture that in the absence of a magnetic field, even when far from the notch, the DW remains subject to its attraction, and will therefore move closer to it. Quantifying this long-range interaction and describing the DW dynamics it generates remain open problems.

Funding

This work is supported by the ANR project MOSICOF, ANR-21-CE40-0004.

Data availability statement

The research data associated with this article are included in the article.

References

[1] A. Al Sayed, G. Carbou and S. Labbé, Asymptotic model for twisted bent ferromagnetic wires with electric current. *Z. Angew. Math. Phys.* **70** (2019) 1–15.

[2] A.L. Bertozzi, A. Münch, M. Shearer and K. Zumbrun, Stability of compressive and undercompressive thin film travelling waves. *Eur. J. Appl. Math.* **12** (2001) 253–291.

[3] G. Carbou, Stability of static walls for a three-dimensional model of ferromagnetic material. *J. Math. Pures Appl.* **93** (2010) 183–203.

[4] G. Carbou, Domain walls dynamics for one dimensional models of ferromagnetic nanowires. *Diff. Integral Equ.* **26** (2013) 201–236.

[5] G. Carbou and S. Labbé, Stability for static walls in ferromagnetic nanowires. *Discrete Continuous Dyn. Syst. – B* **6** (2006) 273–290.

[6] G. Carbou and S. Labbé, Stabilization of walls for nano-wires of finite length. *ESAIM Control Optim.* **18** (2012) 1–21.

[7] G. Carbou and D. Sanchez, Stabilization of walls in notched ferromagnetic nanowires. hal-01810144 (2018).

[8] R. Côte and R. Ignat, Asymptotic stability of precessing domain walls for the Landau–Lifshitz–Gilbert equation in a nanowire with Dzyaloshinskii–Moriya interaction. *Comm. Math. Phys.* **401** (2023) 2901–2957.

[9] S. Fukami, M. Yamanouchi, S. Ikeda and H. Ohno, Depinning probability of a magnetic domain wall in nanowires by spin-polarized currents. *Nat. Commun.* **4** (2013) 2293.

[10] R. Jizzini, Optimal stability criterion for a wall in a ferromagnetic wire in a magnetic field. *J. Differ. Equ.* **250** (2011) 3349–3361.

[11] G. Malinowski, O. Boulle and M. Kläui, Current-induced domain wall motion in nanoscale ferromagnetic elements. *Mater. Sci. Eng. R* **72** (2011) 159–187.

[12] S.S.P. Parkin, M. Hayashi and L. Thomas, Magnetic domain-wall racetrack memory. *Science* **320** (2008) 190–194.

[13] O.V. Pylypovskyi, D.D. Sheka, V.P. Kravchuk, K.V. Yershov, D. Makarov and Y. Gaididei, Rashba torque driven domain wall motion in magnetic helices. *Sci. Rep.* **6** (2016) 23316.

[14] V. Roussier, Stability of radially symmetric travelling waves in reaction–diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004) 341–379.

[15] D. Sanchez, Behaviour of the Landau–Lifschitz equation in a ferromagnetic wire. *Math. Methods Appl. Sci.* **32** (2009) 167–205.

[16] D.D. Sheka, V.P. Kravchuk, K.V. Yershov and Y. Gaididei, Torsion-induced effects in magnetic nanowires. *Phys. Rev. B* **92** (2015) 054417.

[17] K. Takasao, Stability of travelling wave solutions for the Landau–Lifshitz equation. *Hiroshima Math. J.* **41** (2011) 367–388.

[18] H.Y. Yuan and X.R. Wang, Domain wall pinning in notched nanowires. *Phys. Rev. B* **89** (2014) 054423.