SUPPLEMENTARY MATERIAL FOR THE ARTICLE A variant of the Raviart-Thomas method to handle smooth domains using straight-edged triangles

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APPENDIX I - Independence of h of the continuity constant for Poisson problems in the polygon Ω_h equal to the union of the straight-edged triangles in meshes of a curved domain Ω

In this Appendix we establish that the continuity constant $C_{s,h}$ fulfilling the equation (39) in the paper admits a strictly positive lower bound C_s independent of h.
First of all we consider the case where Ω is convex. Since in this case Ω_h is also convex for every h and sufficiently close to Ω, Γ and Γ_h can be respectively expressed in polar coordinates (r, θ) and (r_h, θ) with the same origin, say $O \in \Omega$. We know that $r_h = rR_h(\theta)/R(\theta)$ where $R(\theta)$ and $R_h(\theta)$ are the radial coordinates of the boundaries Γ and Γ_h of Ω and Ω_h at the azimuthal coordinate θ. We refer to Figure 2 for an illustration of both polar coordinate systems, together with some pertaining quantities exploited in the following lemmata:

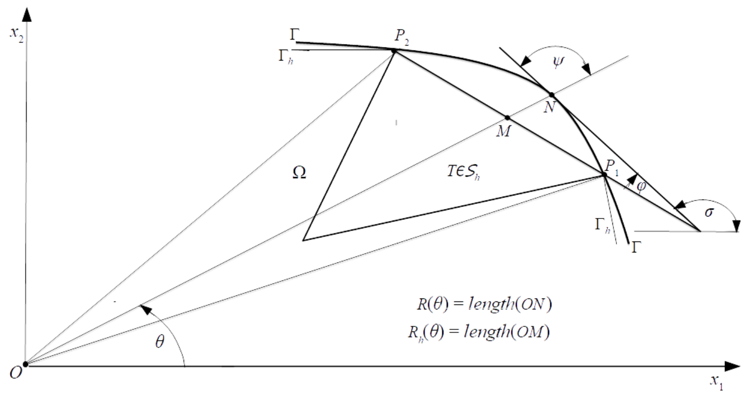

Figure 1: A portion of Ω and Ω_h and the attached polar coordinate systems with origin $O \in \Omega$.

Lemma 0.1 *Let $\rho_h(\theta) := R_h(\theta)/R(\theta)$ for $\theta \in [0, 2\pi)$. Denoting by τ' the derivative with respect to θ of any function τ of θ, provided h is sufficiently small it holds:*

1. *There exists a constant C^- independent of h such that*

$$C^- \leq \rho_h(\theta) \leq 1 \; \forall \theta \in [0, 2\pi). \tag{1}$$

2. *There exists another constant C' independent of h such that*

$$\rho'_h(\theta) \leq C'h \; \forall \theta \in [0, 2\pi). \tag{2}$$

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PROOF. As for (1), we first note that $R_h(\theta) \leq R(\theta)$ for all θ. On the other hand we clearly have $R_h(\theta) + h \geq R(\theta)\ \forall \theta$. Therefore $\rho_h(\theta) \geq 1 - h/R_{min}$, where $R_{min} := \min_{\theta \in [0,2\pi)} R(\theta)$. Therefore under the very reasonable assumption that $h \leq R_{min}/2$ we can take $C^- = 1/2$.
Next we turn our attention to (2). Referring to Figure 2, first of all we consider a generic triangle T in $\mathcal{S}_h$. In Figure 2 $e_T \subset \Gamma_h$ is the edge of T whose ends are vertices P_1 and P_2 of T.
According to well known properties, we may write:

$$\tan \psi = R(\theta)/R'(\theta), \tag{3}$$

where ψ is the angle between the polar radius ON and the tangent to Γ at N.
Similarly we have

$$\tan \psi_h = R_h(\theta)/R'_h(\theta), \tag{4}$$

where ψ_h is the angle between the polar radius OM and the segment e_T.
It happens that $\psi_h = \sigma - \theta$, where σ is the angle between the cartesian axis Ox and e_T. Moreover $\psi = \psi_h - \varphi$, where φ is the angle between the tangent to Γ at N and the same edge of triangle T; skipping details for the sake of brevity, the angles ψ, σ and φ are oriented in a coherent counterclockwise sense, according to the value of θ.

We have

$$\rho'_h(\theta) = \frac{R'_h(\theta)}{R(\theta)} - \frac{R_h(\theta)}{R(\theta)}\frac{R'(\theta)}{R(\theta)} = \rho_h(\theta)\left[\frac{R'_h(\theta)}{R_h(\theta)} - \frac{R'(\theta)}{R(\theta)}\right]. \tag{5}$$

Taking into account (3) and (4), (5) yields

$$\rho'_h(\theta) = \rho_h(\theta)[(\tan\psi_h)^{-1} - \tan\psi)^{-1} = \rho_h(\theta)\{[\tan(\sigma-\theta)]^{-1} - [\tan(\sigma-\varphi-\theta)]^{-1}\}. \tag{6}$$

On the other hand, it is easy to see that

$$[\tan(\sigma-\theta)]^{-1} - [\tan(\sigma-\varphi-\theta)]^{-1} = \frac{-\sin\varphi}{\sin\psi_h \sin\psi}. \tag{7}$$

Plugging (7) into (6) we obtain

$$|\rho'_h(\theta)| = |\sin\varphi|\rho_h(\theta)\sqrt{1+\tan^{-2}(\psi+\varphi)}\sqrt{1+\tan^{-2}\psi}. \tag{8}$$

Next we develop the first square root in (8) to rewrite it as

$$|\rho'_h(\theta)| = |\sin\varphi|\rho_h(\theta)\sqrt{1+\frac{(1-\tan\psi\tan\varphi)^2}{(\tan\psi+\tan\varphi)^2}}\sqrt{1+\tan^{-2}\psi}. \tag{9}$$

Recalling (3) and owing to the smallness of φ, (9) yields

$$\begin{aligned} |\rho'_h(\theta)| &\leq |\tan\varphi|\frac{\sqrt{\tan^2\psi+\tan^2\varphi+1+\tan^2\psi\tan^2\varphi}\sqrt{1+\tan^{-2}\psi}}{|\tan\psi+\tan\varphi|} \\ &= |\tan\varphi|\frac{\sqrt{1+\tan^2\varphi}(1+\tan^{-2}\psi)}{|1+\tan^{-1}\psi\tan\varphi|} \end{aligned} \tag{10}$$

Since $\tan\psi = R(\theta)/R'(\theta)$, setting $Q_R := \max_{\theta\in[0,2\pi)} \frac{|R'(\theta)|}{R(\theta)}$ and recalling that $|\tan\varphi| \leq C_\Gamma h$ (cf. equation (9) of the article), after straightforward manipulations we obtain from (10)

$$|\rho'_h(\theta)| \leq C_\Gamma h \frac{(1+Q_R^2)\sqrt{1+C_\Gamma^2 h^2}}{1 - C_\Gamma Q_R h} \tag{11}$$

as long as $h < \frac{1}{C_\Gamma Q_R}$. Finally, for $h \leq \frac{1}{2C_\Gamma Q_R}$, (2) holds with $C' = \frac{1+3Q_R^2+2Q_R^4}{2Q_R^2}$. ■

Lemma 0.2 *Let $\mathcal{A}$ be an n-component tensor of any order defined in Ω_h and $\tilde{\mathcal{A}}$ be its counterpart defined in Ω by the rule $\tilde{\mathcal{A}}(r,\theta) = \mathcal{A}(r_h,\theta)\ \forall(r,\theta) \in \Omega$. Assuming that both $\tilde{\mathcal{A}}$ and $\mathcal{A}$ are referred to the same cartesian frame with origin $O \in \Omega$, provided h is not too large, for all $\mathcal{T}_h \in \mathcal{F}$ and for every $p \in [1,\infty)$ it holds*

$$(C^-)^{2/p}\|\tilde{\mathcal{A}}\|_{0,p} \leq \|\mathcal{A}\|_{0,p,h} \leq \|\tilde{\mathcal{A}}\|_{0,p}\ \forall \mathcal{A} \in [L^p(\Omega_h)]^n. \tag{12}$$

PROOF. We have

$$\|\mathcal{A}\|^p_{0,p,h} := \int_0^{2\pi}\left[\int_0^{R_h(\theta)} |\mathcal{A}(r_h,\theta)|^p r_h dr_h\right] d\theta = \int_0^{2\pi}[\rho_h(\theta)]^2\left[\int_0^{R(\theta)}|\tilde{\mathcal{A}}(r,\theta)|^p r dr\right] d\theta. \tag{13}$$

Then using (1) the result follows.■

Next we introduce the space $W := \{w|\ w \in H^1(\Omega_h)$ with $w = 0$ on $\Gamma_{0,h}$ if $length(\Gamma_0) > 0$ or $\int_{\Omega_h} w = 0$ otherwise$\}$ together with $\tilde{W} := \{\tilde{w}|\ \tilde{w} \in H^1(\Omega)$ with $\tilde{w} = 0$ on Γ_0 if $length(\Gamma_0) > 0$ or $\int_\Omega \tilde{w} = 0$ otherwise$\}$. Now, recalling Proposition 5.2, given $v \in \bar{V}$, we express the Poisson problem (38) in the article in variational form using polar coordinates, that is:

$$\int_0^{2\pi}\int_0^{R_h(\theta)}\left[\frac{\partial z}{\partial r_h}\frac{\partial w}{\partial r_h} + \frac{1}{r_h^2}\frac{\partial z}{\partial \theta}\frac{\partial w}{\partial \theta}\right] r_h dr_h d\theta = \int_0^{2\pi}\int_0^{R_h(\theta)} vw\ r_h dr_h d\theta\ \forall w \in W. \tag{14}$$

Let us operate a change of variables in the partial derivatives appearing in (14).
We have $(\partial\mathcal{A}/\partial r_h;\partial\mathcal{A}/\partial\theta)^T = J_h(\partial\tilde{\mathcal{A}}/\partial r;\partial\tilde{\mathcal{A}}/\partial\theta)^T$, where J_h is the Jacobian matrix of the transformation from (r_h,θ) into (r,ω) with $\omega = \theta$, expressed by

$$J_h = \begin{bmatrix} \rho_h(\theta) & \rho_h'(\theta)r \\ 0 & 1 \end{bmatrix}. \tag{15}$$

Therefore, after straightforward calculations, we come up with

$$\begin{aligned}&\int_0^{2\pi}\int_0^{R_h(\theta)}\left[\frac{\partial z}{\partial r_h}\frac{\partial w}{\partial r_h} + \frac{1}{r_h^2}\frac{\partial z}{\partial \theta}\frac{\partial w}{\partial \theta}\right] r_h dr_h d\theta = \int_0^{2\pi}\rho_h^2\int_0^{R(\theta)}\tilde{v}\tilde{w}\ r dr d\theta = \\ &\int_0^{2\pi}\int_0^{R(\theta)}\left\{[1+(\rho_h')^2]\frac{\partial\tilde{z}}{\partial r}\frac{\partial\tilde{w}}{\partial r} - \frac{1}{r}\rho_h\rho_h'\left[\frac{\partial\tilde{z}}{\partial r}\frac{\partial\tilde{w}}{\partial\theta} + \frac{\partial\tilde{z}}{\partial\theta}\frac{\partial\tilde{w}}{\partial r}\right] + \frac{1}{r^2}\rho_h^2\frac{\partial\tilde{z}}{\partial\theta}\frac{\partial\tilde{w}}{\partial\theta}\right\} r dr d\theta\ \forall\tilde{w} \in \tilde{W}.\end{aligned} \tag{16}$$

Proposition 0.3 *The function $\tilde{z}$ fulfilling (16) is the unique solution of the following mixed second order elliptic equation:*

$$\begin{cases} -\nabla\cdot(\tilde{A}\nabla\tilde{z}) = \rho_h^2\tilde{v} \text{ in } \Omega \\ \tilde{z} = 0 \text{ on } \Gamma_0 \text{ if } length(\Gamma_0) > 0 \text{ and } \int_\Omega \rho_h^2\tilde{z} = 0 \text{ otherwise} \\ \partial(\tilde{A}\nabla\tilde{z})/\partial n = 0 \text{ on } \Gamma_1, \end{cases} \tag{17}$$

where, $\tilde{A}$ is a symmetric second order tensor expressed as the matrix $\tilde{A}_{|polar}$ in the frame attached to the pair of polar unit vectors $(\mathbf{e}_r;\mathbf{e}_\theta)$:

$$\tilde{A}_{|polar} = \begin{bmatrix} 1+(\rho_h')^2 & \rho_h\rho_h' \\ \rho_h\rho_h' & \rho_h^2 \end{bmatrix}$$

Furthermore for a certain $s \in (2,4]$ there exits a mesh-independent constant $\tilde{C}_s$ such that

$$\|\nabla\tilde{z}\|_{0,s} \leq \tilde{C}_s\|\tilde{v}\|_0. \tag{18}$$

PROOF. First we note that, in case $length(\Gamma_0) = 0$, $\tilde{z}$ differs by an additive constant from any solution $\bar{z}$ of the following problem:

$$\begin{cases} -\nabla \cdot [\tilde{A}\nabla \bar{z}] = \bar{v} := \rho_h^2 \tilde{v} \text{ in } \Omega \\ \partial(\tilde{A}\nabla \bar{z})/\partial n = 0 \text{ on } \Gamma_1, \end{cases} \tag{19}$$

where both $\bar{v}$ and $\bar{z}$ belong to $L_0^2(\Omega)$. Therefore the properties of $\nabla \tilde{z}$ to be exploited below are the same as those of $\nabla \bar{z}$ in case $length(\Gamma_0) = 0$.
By a straightforward calculation we infer that the eigenvalues of $\tilde{A}$ are both strictly positive and the smallest one is bounded below by $\lambda_h := \rho_h^2/[1 + \rho_h^2 + (\rho_h')^2]$. Thus, recalling (1) and (2), the uniform coercivity (i.e. independently of h) of the bilinear form associated with (17) is guaranteed. Indeed,

$$(\tilde{A}\nabla \bar{w}, \nabla \tilde{w}) \geq \tilde{\lambda} \|\nabla \tilde{w}\|_0^2 \; \forall \tilde{w} \in \tilde{W},$$

with $\tilde{\lambda} = (C^-)^2/[2 + (C')^2] \leq \lambda_h$, since $h < 1$ by assumption. Furthermore, owing to well known inequalities of the Friedrichs-Poincaré type (see e.g. [18]), there exists a strictly positive constant η depending only on Ω such that $\|\nabla \tilde{w}\|_0 \geq \eta \|\tilde{w}\|_1 \; \forall \tilde{w} \in \tilde{W}$. Hence the uniform coercivity constant $\tilde{\alpha}$ of the operator $-\nabla \cdot [\tilde{A}\nabla(\cdot)]$ holds with $\tilde{\alpha} = \tilde{\lambda}\eta$, that is,

$$(\tilde{A}\nabla \tilde{w}, \nabla \tilde{w}) \geq \tilde{\alpha} \|\tilde{w}\|_1^2, \; \forall \tilde{w} \in \tilde{W}.$$

Similarly, we can easily prove that the above bilinear form is uniformly continuous, since the largest eigenvalue of $\tilde{A}$ is bounded above by $[1 + \rho_h^2 + (\rho_h')^2]$, or yet by $2 + (C')^2$, according to (1) and (2).
Now resorting to [35], we infer from the above considerations on the uniform boundedness from above and below of the elliptic operator associated with either second order problem (17) or (19), that the solution $\tilde{z}$ enjoys the property (18). Indeed, the constant $\tilde{C}_s$ depends only on $\tilde{\alpha}$ (cf. [35]). ■

Proposition 0.4 *There exists a constant C_s independent of h such that the solution z of problem (38) in the article satisfies,*

$$\|\nabla z\|_{s,h} \leq C_s \|v\|_{0,h}. \tag{20}$$

PROOF. Expressing both z and $\tilde{z}$ in polar coordinates and recalling the matrix J_h given by (15), first we note that $\nabla z_{|(\mathbf{e}_r;\mathbf{e}_\theta)}(r_h, \theta) = J_h \nabla \tilde{z}_{|(\mathbf{e}_r;\mathbf{e}_\theta)}(r, \theta)$.
Then from (12), it follows that

$$\|\nabla z\|_{0,s,h}^s \leq \sqrt{2} \| |\widetilde{\nabla z}| \|_{0,s}^s = \sqrt{2} \| |J_h \nabla \tilde{z}| \|_{0,s}^s. \tag{21}$$

Denoting by $\|J_h\|$ the spectral norm of J_h, after straightforward calculations (21) further yields,

$$\|\nabla z\|_{0,s,h} \leq 2^{1/(2s)} \| \|J_h\| |\nabla \tilde{z}| \|_{0,s} \leq \hat{C}_s(\Omega) \|\nabla \tilde{z}\|_{0,s}, \tag{22}$$

where $\hat{C}_s(\Omega) = (2s)^{-1} \sqrt{2 + (C')^2 diam(\Omega)^2}$.
Finally, since $\rho_h \leq 1$, combining (12) with (18) and taking into account (22), we obtain

$$\|\nabla z\|_{0,s,h} \leq \hat{C}_s(\Omega) \tilde{C}_s \|\tilde{v}\|_0 \leq \hat{C}(\Omega) \tilde{C}_s C^- \|v\|_{0,h}, \tag{23}$$

and the result follows with $C_s := \hat{C}_s(\Omega) \tilde{C}_s C^-$. ■

We next examine in main lines how Proposition 0.4 extends to the case where Ω is not convex.
The key to the problem is an invertible mapping $\Upsilon_h \in [W^{1,\infty}(\Omega)]^2$ from Ω onto Ω_h, together with related equations (24) given hereafter.
Actually we can consider that in the convex case such a mapping is the one based on the representation of Ω (resp. Ω_h) in polar coordinates $(r;\theta)$ (resp. $(r_h;\theta)$). Here the same result can be achieved, by subdividing Ω (resp. Ω_h) into a finite number υ of non overlapping star-shaped sub-domains Ω_ι (resp. $\Omega_{h,\iota}$), in such a way that each one of them can be represented by a local system of polar coordinates

$(r_\iota;\theta_\iota))$ (resp. $(r_{h,\iota};\theta_\iota))$, $\iota = 1,\ldots,\upsilon$. After lengthy but in all natural calculations, we can establish two analogs of (1) and (2) in the form of the following pair of estimates:

$$\left\{ \begin{array}{l} \|\Upsilon_h - \mathcal{I}\|_{0,\infty} \leq C_{0,\Upsilon} h \\ |\Upsilon_h|_{1,\infty} \leq C_{1,\Upsilon} h, \end{array} \right. \tag{24}$$

where $\mathcal{I}$ is the identity operator on Ω and $C_{0,\Upsilon}$ and $C_{1,\Upsilon}$ are two constants independent of h. More precisely, letting (24) play the role of (1)-(2) the proof of an analog of Proposition 0.4 for the non convex case follows in a rather straightforward manner.

Incidentally, it is noteworthy that, if $R_\iota(\theta_\iota)$ (resp. $R_{h,\iota}(\theta_\iota)$) is the local radial coordinate of the boundary of Ω_ι (resp. $\Omega_{h,\iota}$), at the intersection of the boundary of a star-shaped sub-domain, say Ω_ϵ, among those Ω is subdivided into, with the boundary of any other such a sub-domain, we necessarily have $R_\epsilon(\theta_\epsilon) = R_{h,\epsilon}(\theta_\epsilon)$ for all pertinent values of θ_ϵ.

APPENDIX II - A numerical verification

The authors refer to [30] for a thorough numerical experimentation of the method studied in this work in the case where $k \leq 1$ and $k = 1$, including comparisons with the classical (i.e. polygonal) approach. Nevertheless, just to validate our analysis we report below numerical results obtained for $k = 1$ taking a non convex domain. In doing so our aim here is to highlight that optimal second order convergence rates do apply to the case of a mixed Poisson problem with Dirichlet conditions prescribed on a concave boundary portion Γ_0, which confirms the predictions given at the end of Paragraph III) in Subsection 6.2.

More specifically, we checked the performance of the formulation (7) of the article taking Raviart-Thomas discretization spaces RT_k for $k = 1$, referred to here as the P-G (for Petrov-Galerkin) RT_1 method, by solving a test-problem in an annulus with inner radius $r_i = 1/2$ and outer radius $r_e = 1$, for an exact solution given by $u(x, y) = (r^2 - 2rr_e + 2r_ir_e - r_i^2)/2$, where $r = \sqrt{x^2 + y^2}$. This function satisfies $u = 0$ on Γ_0, namely, the circle given by $r = r_i$ and $\partial u/\partial r = 0$ on Γ_1, that is, along the circle given by $r = r_e$. We computed with a quasi-uniform family of meshes for a quarter annulus, constructed for a quarter unit disk with $2L^2$ triangles for $L = 2^m$, by removing the $L^2/2$ triangles fully contained in the disk with radius $1/2$. For simplicity we set $h = 1/L$.
In the upper part of Table 1 we supply the approximation errors of u, $\mathbf{p}$ and $\nabla \cdot \mathbf{p}$ measured in the norm of $L^2(\Omega_h)$, taking $m = 2, 3, 4, 5, 6$. Although we did not study this kind of errors, in order to further illustrate the properties of the Petrov-Galerkin approximation (7) of the article, in the lower part of Table 1 we display the maximum errors of the computed DOFs for both u_h and $\mathbf{p}_h$, represented in the form of a mesh-dependent semi-norm $| \cdot |_{0,\infty,h}$.

h	$\longrightarrow$	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\parallel u_h - u \parallel_{0,h}$	$\longrightarrow$	0.28440E-2	0.70676E-3	0.17641E-3	0.44086E-4	0.11021E-4
$\parallel \mathbf{p}_h - \mathbf{p} \parallel_{0,h}$	$\longrightarrow$	0.38543E-2	0.97914E-3	0.24598E-3	0.61577E-4	0.15400E-4
$\parallel \nabla \cdot (\mathbf{p}_h - \mathbf{p}) \parallel_{0,h}$	$\longrightarrow$	0.82685E-2	0.21250E-2	0.53575E-3	0.13424E-3	0.33579E-4
$\|u_h - u\|_{0,\infty,h}$	$\longrightarrow$	0.12974E-1	0.31851E-2	0.79104E-3	0.19721E-3	0.49252E-4
$\|\mathbf{p}_h - \mathbf{p}\|_{0,\infty,h}$	$\longrightarrow$	0.66908E-2	0.17059E-2	0.44211E-3	0.11378E-3	0.29004E-4

Table 1: The P-G RT_1 method: errors for a test-problem in a non convex curved domain

The predicted second order convergence in norm of our method in all the three senses can be observed in Table 1. The same rate also applies to the computed DOFs of both u_h and $\mathbf{p}_h$.