REVUE FRANÇAISE D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE, SÉRIE ROUGE

J. NIEDEREICHHOLZ

On the connection between the theory of signal flow graphs and the theory of directed graphs

Revue française d'informatique et de recherche opérationnelle, série rouge, tome 4, n^{o} 2 (1970), p. 3-9.

<http://www.numdam.org/item?id=M2AN_1970__4_2_3_0>

R.I.R.O.
(4e année, R-2, 1970, p. 3-9)

ON THE CONNECTION BETWEEN THE THEORY OF SIGNAL FLOW GRAPHS AND THE THEORY OF DIRECTED GRAPHS

par J. NIEDEREICHHOLZ (1)

Résumé. — *La représentation des systèmes linéaires par des graphes de transfert de Mason* [7] *était complétée par le développement des graphes de transfert de Coates* [5] *pour lesquels Desoer* [6] *a démontré l'optimalité de la fonction de transfert de Coates. Ces deux types de graphes de transfert peuvent être combinés par le concept des N-graphes de Chow et Cassignol* [3]. *L'article suivant démontre la connexion de ces trois types de graphes de transfert avec les graphes matriciels linéaires et finalement avec les bipartitions des graphes de la théorie générale des graphes.*

1. Introduction

A square matrix of the order n $A = [a_{ij}]$ can always be represented by a flow graph in the following way :

A node k $(k = 1(1)n)$ is assigned to each row (or column) and an arc (i, j) is directed node from i to node j with the associated weight a_{ij} $(a_{ij} \neq 0)$.

Using the notation of the digraph theory we introduce the three-tuple $D(V, A, f)$ as a flow graph were V is a node-set, A a set of arcs or directed edges and f a mapping function from A to the complex field with $f((i, j)) = a_{ij}$ for all $i, j, \in V$ [1]. For the development of the unifying concept of matrix graphs some further definitions of the theory of directed and undirected graphs need to be introduced :

An arc (i, j) is positively incident with its initial node i and negatively incident with its terminal node j. The positive degree of i $p^+(i)$ is defined as the number of arcs that are positively incident with node i, analogously the negative degree $p^-(i)$ is the number of arcs having i as terminal node. For an undirected graph $G(V, E, f)$ the degree of node i $p(i)$ indicates the number of edges incident with i. A simple node of G is a node joined by

(1) Ingénieur en chef, Centre de calcul à l'université de Karlsruhe, département R.O.

only one edge, whereas an isolated node is a node joined by no edges. A simple graph in our sense contains only simple nodes. This is not the definition commonly used in graph theory where a simple graph is a graph having no loops and no parallel edges.

A digraph D is regular of degree k or k-regular if $p^+(i) = p^-(i) = k$ for all $i \in V$. An undirected graph G is regular of degree k if $p(i) = k$ for all $i \in V$. A spanning subgraph of D is a subgraph with the same node-set as D and a m-factor of D is a regular subgraph of degree m. A matching subgraph of G is a spanning subgraph of G that is regular of degree one. It forms a simple graph.

An undirected graph $B(V_1', V_2', E', f')$ is said to be a bipartite graph if its nodes can be partioned into two disjoint node-sets V_1', V_2' in such a way that an edge (i', j') connects node $i' \in V_2'$. with node $j' \in V_1'$ A directed bipartite graph $D(V_1', V_2', A', f')$ is defined analogously with the additional characteristic that the arc-set A$'$ can be classified into two groups, one directed from V_1' to V_2' and another from V_2' to V_1'.

If $B(V_1', V_2', E', f')$ corresponds to a flow graph $D(V, A, f)$, then a subgraph S_D of D is an n-factor if the corresponding subgraph S_B of S_D in B is a regular graph of degree n and a subgraph S_D of D is a 1-factor or connection [6] if the corresponding subgraph S_B of S_D in B is a matching subgraph.

A matrix can always be associated directly with a bipartite graph and a flow graph $D(V, A, f)$ always corresponds to a bipartite graph $B(V_1', V_2', E', f')$ in the following way : To the node-set V of D one constructs a one-to-one correspondence to V_2' with $V_1' = V$ for each edge (i', j') or (j', i') if $(i, j) \in A$ with $f'((i', j')) = f'((j', i')) = f((i, j))$ [9].

2. Matrix graphs

2.1 Conventions

A matrix can be represented by a matrix graph in the following way :

a) The rows and columns of the matrix are represented by nodes. The column-nodes are placed in one level and the row-nodes in another level below of this.

b) Each matrix element $a_{ij}(a_{ij} \neq 0)$ is represented by the weight of an edge connecting the i-th row-node with the j-th column-node.

This matrix graph actually forms a bipartite graph [4].

We now introduce internal nodes which are nodes in a level between the column- and the row-level. A cross-point is an intersection of two edges. It is always possible to deform a graph in such a way that a cross-point is formed by only two edges, what we call a binary cross-point.

Using the well known addition rule respectively multiplication rule for the weights of parallel respectively serial edges, internal nodes can be eliminated [7].

2.2 Determinants

The determinant of a square matrix $A = [a_{ij}]$ of order n is given by

$$\det A = \sum_{\{j\}} P_{\{j_1 j_2 .. j_n\}} \prod_{k=1}^{n} a_{kj_k} \tag{1}$$

with the permutation symbol

$$P_{\{j_1 j_2 \dots j_n\}} = (-1)^r$$

where r is the number of transpositions in the permutation $\{j\}$ of the first n integers from their natural order. The case of no transposition corresponds to the simple graph in figure 1.1.

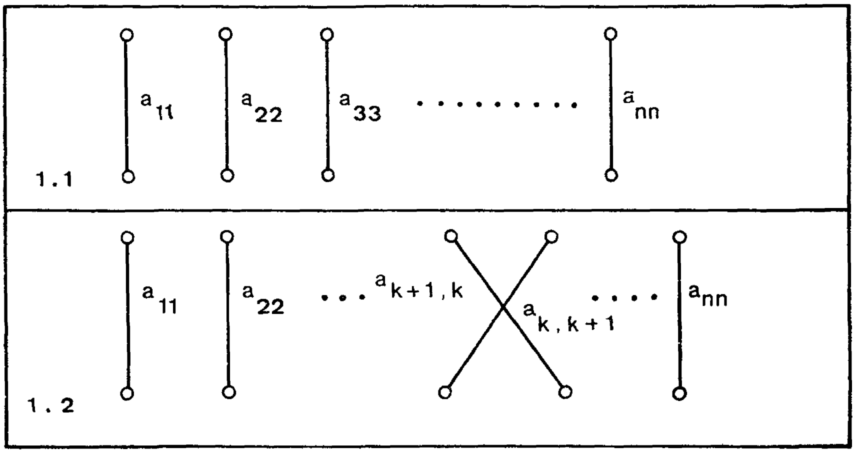

Figure 1. — Simple graphs of transpositions as bipartite graphs

The case of one transposition with

$$(-1)^1 a_{11} a_{22} \dots\dots a_{k+1,k} a_{k,k+1} \cdots a_{nn}$$

corresponds to the simple graph with one binary cross-point of figure 1.2. We notice that the number of transpositions in the permutation $\{j\}$ equals the number of binary cross-points in the equivalent simple graph.

Now we can state the following rules for the evaluation of the determinant of a square matrix :

a) Design the corresponding matrix graph.

b) Design all possible simple subgraphs of the matrix graph containing the whole node-set V of the matrix graph.

c) Evaluate the determinant using the following relation :

$$\det A = \sum_k (-1)^{c_k} \cdot P_k. \tag{2}$$

with

P_k : the product of the weights associated to all the edges of the simple subgraph k. The summation involves all possible subgraphs.

C_k : the sum of the weights of the cross-points of the simple subgraph k. The weight of a cross-point is given by C_2 (e) where e is the number of edges at this cross-point.

2.3 Simultaneous Linear Algebraic Équations

We turn now to a set of n linear algebraic equations in $n + 1$ variables

$$\sum_{j=0}^{n} a_{ij}x_j = 0 \quad , \quad i = 1(1)n \tag{3}$$

with

x_i : an unknown variable, $i = 1(1)n$, $i \neq j$

x_j : a known independent variable.

The gain formula for (3) is given by

$$\frac{x_i}{x_j} = (-1)^{i-j} \frac{\sum_k (-1)^{Ck} P_k(\bar{i})}{\sum_k (-1)^{Ck} P_k(\bar{j})}, \; i = 1(1)n, \; i \neq j \tag{4}$$

with

$P_k(\bar{i})$: the product of the weight associated to all the edges of the simple subgraph k after the exclusion of column-node i of the original matrix graph.

An extended proof of (4) is given in [10]. Now we are prepared to deal with the Null-node graphs.

3. Null-node graphs

We can convert a matrix graph into a Null-node graph by the following steps :

a) The column-nodes of a matrix graph become the variables x_0, x_1, ..., x_n of the matrix equation $AX = N$.

b) The row-nodes of a matrix graph become the n null-variables N_i of the right hand side of the matrix equation.

The solution of (3) can be obtained by N-graphs using the following relation :

$$\frac{x_i}{x_k} = (-1)^{i-k} \frac{\sum_k (-1)^{Cr} P_r(\bar{x}_i)}{\sum_m (-1)^{Cm} P_m(\bar{x}_k)} \tag{5}$$

with

x_k : a known variable

$P_r(\bar{x}_i)$: the product of the weights associated to all the edges of the r-th subgraph of the Null-node graph were node x_i and all edges incident with it are deleted. All nodes must be simple.

With the result of the Null-node graph gain formula (5) we turn to the development of the Coates' gain formula for flow graphs.

4. Flow Graphs

For the conversion of an Null-node graph with n null-nodes and $n+1$ variables into a flow graph with no null-nodes, we take as a source-node for instance x_0 and coincide the other nodes in pairs $(N_i\, x_i)$, $i = 1(1)n$, equivalent with the connection of edges of unit value directed from node N_i to node x_i. The gain formula (5) for x_n yields the following relation [10] :

$$\frac{x_n}{x_0} = \frac{\sum_r (-1)^{C_r} P_r(\bar{x}_n)}{\sum_m (-1)^{n-C_m} P_m(\bar{x}_0)} \tag{7}$$

with

$P_r(\bar{x}_n)$: the subgraph with node x_n devoid of all incident edges and all other nodes having exactly one incident edge.

The conversion of these subgraphs to flow graphs by the addition of an unit edge yields one source-node x_0 and one sink-node x_n only, while exactly one arc is positively incident and negatively incident with each other node. The subgraphs of the numerator in (7) yield the one-connections of Desoer [6] of the corresponding flow graph, while the subgraphs of the denominator yield the connections of the flow graph with the source-node not deleted.

We can show the following equivalence [10] :

If C_r is even (odd) then the number of loops l_r in the corresponding r-th subgraph in the numerator is also even (odd) and if n-C_m is even (odd) the number of loops l_m in the corresponding m-th subgraph in the denominator is even (odd).

Using this correspondence we get the Coates, gain formula for flow graphs :

$$\frac{x_n}{x_0} = \frac{\sum_r (-1)^{l_r} C_r(G_{0-n})}{\sum_m (-1)^{l_m} C_m(G_0)} \tag{8}$$

with

$C_r(G_{0-n})$: the product of the weights associated to all the arcs of the r-th one-connection of the numerators transformation of the N-graph formula to a flow graph.

$C_m(G_0)$: the product of the weights associated to all the arcs of the m-th connection of the denominators transformation of the N-graph formula to a flow graph.

With this result we turn to the Mason-gain formula for signal-flow graphs.

5. Signal-Flow Graphs

A flow graph can be transformed to a signal-flow graph by the following steps [2] :

a) Add a weight of 1 to the weight of each existing self-loop.

b) Add a self-loop with a weight of 1 to each node devoid of a self-loop except the source-node.

c) Break the source-node into n source-nodes. Denote them by the weights of the arcs positively incident with the old source-node. Multiply these weights by their negative reciprocals to get the new weights of the arcs positively incident with the n source-nodes.

It is also possible to develop a signal-flow graph and the Mason-gain formula from the N-graph gain formula after a rearrangement of the set of n independent equations in $n + 1$ variables in such a way that all edges $(N_i x_i)$, $i = 1(1)n$ are verified with a weight of (-1), which is always possible [4]. The N-graph gain formula yields the following relation [10] :

$$\frac{x_n}{x_0} = \frac{\sum_r (-1)^{Cr-Er} P_r'(\bar{x}_n)}{1 + \sum_m (-1)^{n-Cm+Em} P_m'(\bar{x}_0)} \qquad (9)$$

with

E_k : the number of edges $(N_i x_i)$ in the k-th subgraph

$P_k'(\bar{x}_i)$: the product of the weights in the k-th subgraph without the weights of the edges $(N_i x_i)$, which have been reversed to get unit value.

This is the famous gain formula of signal-flow graphs by Mason [7], well known to systems analysts not only in physical sciences but also in economics, operations research and statistics [8].

REFERENCES

[1] Busacker (R. G.) et Saaty (T. L.). *Finite graphs and networks.* New York, McGraw-Hill, 1965.

[2] Chan (S. P.). « Topology : A shorcut to understanding systems », *Electronics,* 1967, 40, 82-94.

[3] Chow (Y.) et Cassignol (E.). *Sur les graphes de transfert d'un système linéaire.* Compt. rend. Acad. Sci., 1961, 251, 1365-1367.

[4] Chow (Y.) et Roe (P. H.). « A note on matrix graphs », *IEEE Trans.*, CT-10, 1963, 308-130.

[5] Coates (C. L.). Flow-graph solutions of linear algebraic equations », *IRE Trans.*, CT-6, 1959, 170-187.

[6] Desoer (C. A'). « The optimum formula for the gain of a flow graph or a simple derivation of Coates' formula », *Proc. IRE*, 1960, 48, 833-889.

[7] Mason (S. J.). « Feedback theory-someproperties of signal-flow graphs », *Proc. IRE*, 1953, 41, 1144-1156.

[8] Niedereichholz (J.). *Grundzüge einer Systemanalyse ökonomischer Modelle mittels Flussgraphen, Jahrbücher für Nationalökonomie und Statistik*, 1969, 183, 1.

[9] Ore (O.). « Graphs and matching theorems », *Duke Math. J.*, 1955, 22, 625-639.

[10] Roe (P. H.). *State equations of dynamic systems*, Univ. of Waterloo, Waterloo, Ontario, 1962.