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AN ALGORITHM FOR OPTIMAL SYNTHESIS IN CONTROL PROBLEMS

by Stefan MIRICĂ (1)

Abstract. — *An algorithm for an optimal synthesis is described and its effectiveness is shown on two examples.*

1. INTRODUCTION

The algorithm proposed in this paper was suggested by the R. Isaacs' technique and the results from [10] and [11] concerning admissible and optimal synthesis for a class of control problems and differential games.

We ought to point out that the algorithm may be considered as a generalisation and in the same time a justification of Isaacs' technique.

The algorithm consists in the « backward integration » (with some special « final » conditions) of the Hamiltonian system which defines in [10], [11] the dual variables.

R. Isaacs uses in [8] a technique to construct optimal synthesis for many examples of differential games and control problems. This technique consists in the backward integration of the characteristic system of a partial differential equation — the fundamental equation.

As we may easily observe, the dual trajectories defined in [10] and [11] eoincide with the characteristique curves from [8] in the particular case considcred by Isaacs.

The Isaacs' technique can be applied only to the control problems (and differential games) for that the terminal manifold is a surface that is a diffe-

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rentiable manifold of dimension $n - 1$ if the phase space is of dimension n. The algorithm proposed in this paper is applicable also to control problems for that the terminal manifold is k-dimensional where $0 \leqslant k \leqslant n - 1$ and hence it represent a generalisation of the Isaacs' technique.

The properties of the admissible synthesis proved in [10], [11] allow us to describe in a precise manner all the operations of the algorithm and especially the technique in the large. The algorithm is rigurously divided in « steps », « routines », « subroutines » and « operations » and this allows to apply it in a sufficiently automatic manner.

Moreover, the definition of the admissible synthesis and the sufficient conditions for its optimality represent rigurous criteria for optimality of the obtained synthesis. From this point of view this algorithm represent a justification of R. Isaacs' technique.

We ought to notice that the consideration in [11] of the control systems on differentiable manifolds suggested a basic idea of the algorithm : to work in the cotangent manifold of the phase space and to project the results on the phase space by the cotangent bundle. In this case — the global one — the dual variables are to be considered in the cotangent space and there exist some curves on this space — the dual trajectories — that are projected by cotangent bundle on the « marked trajectories » (the trajectories generated by the admissible synthesis on the phase space).

To understand and to justify the operations of the algorithm we present shortly in the section 2 the definition and some properties of the admissible synthesis proved in [10] and [11].

In the section 3 we present the algorithm and we prove that if the algorithm is working for a control problem then we obtain the optimal synthesis.

In the section 4 we apply the algorithm to two examples of control problems solved in [9], [6] by other methods and we obtain the same results.

The algorithm may be formulated and may be applied to control systems on differentiable manifolds. In this paper was preffered the local case — control systems in which the phase space is an open domain of a real euclidian space — because of the frequency of such problems for the applications and because in this case the main features become more understandable.

For differential games the algorithm can be applied in the same manner as for the control problems ([8], [10]).

We note that although the algorithm is described as a typical one for the use of the computers, the use of computers to construct the optimal synthesis is not matematically justified because of the absence of some results concerning the « stability » of the optimal synthesis to variations of the data of the control problem and to computing errors. This remains an important open problem.

2. THE ADMISSIBLE SYNTHESIS. DEFINITION AND PROPERTIES

We consider an open domain $G \subset R^n$ — called *phase space*, a set $U \subset R^p$ which is supposed to be a closed set, called *control space* and a C^1 — map $f : G \times U \to R^n$ which defines the « parametrized » differential system :

$$\frac{dx}{dt} = f(x, u) \quad , \quad x \in G \quad , \quad u \in U. \tag{2.1}$$

A differentiable manifold $\mathcal{C} \subset \bar{G}$ of dimension k, $0 \leqslant k \leqslant n - 1$, called *terminal manifold*, is also given.

We say that $S = (G, U, f, \mathcal{C})$ is a *control system* on G.

An *admissible control* related to the initial point $x_0 \in G$ is a vector valued piecewise continuous function $u : [0, t_u] \to U$ such that the « controlled » differential system :

$$\frac{dx}{dt} = f(x, u(t)) = \bar{f}_u(x, t) \tag{2.2}$$

has the solution $\varphi(\cdot ; x_0)$ which remains in G and intersects $\mathcal{C}$ in a finite time (that is there exists $t_1 \in [0, t_u]$ such that $\varphi(t ; x_0) \in G \backslash \mathcal{C}$ for $0 \leqslant t < t_1$ and $x_1 = \varphi(t_1 ; x_0) \in \mathcal{C}$). The curve φ is called *admissible trajectory*.

If two other C^1-functions, $g : \mathcal{C} \to R$ and $f^0 : G \times U \to R$ are given, for each admissible control u we may define the real number :

$$P(u) = P(\varphi) = g(x_1) + \int_0^{t_1} f^0(\varphi(t ; x_0), u(t))\, dt, \tag{2.3}$$

called the *performance of the control u*. If $\mathcal{U}_x$ is the set of all admissible controls related to the point $x \in G$ and $\mathcal{U} = \bigcup_{x \in G} \mathcal{U}$ then the relation (2.3) defines a map $P : \mathcal{U} \to R$ called the *performance of the system S*.

We say that the pair (S, P) represent a *preferential control system on G*.

An admissible control $\tilde{u}_x \in \mathcal{U}_x$ is an *optimal control* (related to the point $x \in G$) if the following inequality holds for any $u \in \mathcal{U}_x$:

$$P(\tilde{u}_x) \leqslant P(u) \tag{2.4}$$

Generally speaking, and admissible synthesis is a map $v : G \to U$ such that the « synthetised » differential system :

$$\frac{dx}{dt} = f(x, v(x)) = \bar{f}(x) \tag{2.5}$$

has a solution $\varphi_x(\varphi_x(0) = x)$ at every point $x \in G$ such that $v^0\varphi_x$ is an admissible control. The most simple examples show that the optimal synthesis is a piecewise smooth map and hence the differential system (2.5) is a right hand side discontinuous one. But it is impossible to study such differential systems without the explicite description of the discontinuity set of the function f and without the explanation of the behavior of the solutions of (2.5) on this set.

One of the most general hypothesis in this sense was proposed by V. G. Boltyanskii ([4] [5]) : the synthesis (the « regular synthesis ») is a C^1-map on G exept a singular set which is a « piecewise smooth set ».

In [10], [11] is defined an admissible synthesis by omitting the condition that the marked trajectory of the regular synthesis of Boltyanskii satisfy the maximum principle. For such synthesis a set of properties are proved and this allow by adding the maximum principle or the functional equation of the dynamic programming to deduce that the synthesis is optimal. In this way it is proved that the maximum principle (or the dynamic programming principle) assures the optimality of the Boltyanskii's regular synthesis.

To define the admissible synthesis we need the notions of « curvilinear polyhedron » and « piecewise smooth set » ([4], [5]) :

Definition 2.1

Let $K \subset R^s$ be a convex, bounded, closed, s-dimensional polyhedron, $V \subset R^s$ an open neighborhood of K and $\varphi : V \to G$ a C^1-map, injection at the points of K and such that

$$\operatorname{rank}\left(\frac{\partial \varphi^i}{\partial x^j}(x^1, \ldots, x^s)\right) = s$$

for any $(x^1, \ldots, x^s) \in K$.

Then the set $L = \varphi(K) \subset G$ is a *curvilinear polyhedron* in G of dimension s.

Definition 2.2

The set $M \subset G$ is a *piecewise smooth set od dimension* $s \leqslant n$ if the following conditions hold :

(1) M is a union of curvilinear polyhedra in G ;

(2) every compact subset of G intersects only a finite number of such polyhedra ;

(3) there exists in M a s-dimensional polyhedron and the others are of dimension $\leqslant s$.

As is observed in [4], p. 256, every closed in $G \subset R^n$ smooth surface of dimension less then n is a piecewise smooth set.

Let $N, P^k, P^{k+1}, \ldots, P^{n-1} \subset G$ be piecewise smooth sets, P^i is of dimension i, $i = k, k+1, \ldots n-1$, N is of dimension less then n and

$$\mathcal{T} \subset P^k \subset P^{k+1} \subset \ldots \subset P^{n-1} \subset G.$$

We denote $P^{k-1} = \mathcal{T}$, $P^n = G$.

Definition 2.3

The sets $N, P^k, P^{k+1}, \ldots, P^{n-1}$ and the map $v : G \to U$ represent an *admissible synthesis* if the following requirements are fulfilled :

A. (i). The connected components of the sets

$$P^i \setminus (P^{i-1} \cup N), i = k, k+1, \ldots, n,$$

are differentiable manifolds in G of dimension i ; we call them i-dimensional cells. The connected components of the target set $\mathcal{T} = P^{k-1}$ are also called k-dimensional cells.

(ii) The restriction $v_{\mathfrak{c}} = v|\mathfrak{c}$ is a C^1-map from the cell $\mathfrak{c}$ to U. Moreover, there exists a neighborhood $\tilde{\mathfrak{c}} \subset G$ of the closure $\bar{\mathfrak{c}}$ of the cell $\mathfrak{c}$ and a smooth extension $\tilde{v}_{\mathfrak{c}} : \tilde{\mathfrak{c}} \to U$ of the map $v_{\mathfrak{c}}$.

B. Every cell is either of type I or of type II :

(i) The n-dimensional cells are of type I, the k-dimensional ones of type II.

(ii) If $\mathfrak{c}$ is a i-dimensional cell of type I then from any point $x \in \mathfrak{c}$ a unique solution φ_x of the differential system (2.5) (for which the right hand side is discontinuous) starts.

There exists a unique (i-1)-dimensional cell $\Pi(\mathfrak{c})$ (of type I or II) such that the solution φ_x leaves $\mathfrak{c}$ after a finite time and reaches $\Pi(\mathfrak{c})$ transversally (nontangently), that is, in the incidence point $x' = \varphi_x(t') \in \Pi(\mathfrak{c})$, the vector $\lim_{t \nearrow t'} f(\varphi_x(t))$ does not belong to the tangent space $T_{x'}\Pi(\mathfrak{c})$.

(iii) If $\mathfrak{c}$ is a i-dimensional cell of type II and $\mathfrak{c} \not\subset \mathcal{T}$ then there exists a unique $(i+1)$-dimensional cell $\Sigma(\mathfrak{c})$ of type I such that from any point $x \in \mathfrak{c}$ a unique solution of the system (2.5) entering $\Sigma(\mathfrak{c})$ and having in $\mathfrak{c}$ only the point x starts. Moreover, the set $\mathfrak{c}' = \mathfrak{c} \cup \Sigma(\mathfrak{c})$ is a differentiable manifold possibly with boundary and $v_{\mathfrak{c}'}$ is a C^1-map.

C. (i) Every solution of the system (2.5) reaches $\mathcal{T}$ transversally, in a finite time and intersects only a finite number of cells.

(ii) From the points in N may start several solutions of (2.5). The solutions of (2.5) starting at points in N do not remain in N but enter a cell of type I.

We call the solution φ_x of (2.5) *marked trajectory* through the point $x \in G$.

If $x \in G \backslash N$ then φ_x is unique. If t_F is the first moment when the curve φ_x reaches $\mathcal{T}$ and $x_F = \varphi_x(t_F)$ then for the point $x \in G$ and for the marked trajectory φ_x we define the real number :

$$P(x, \varphi_x) = g(x_F) + \int_0^{t_F} f^0(\varphi_x(t), v(\varphi_x(t)))\, \mathrm{d}t \tag{2.6}$$

D. The number $P(x, \varphi_x)$ is the same for any marked trajectory starting at $x \in N$. The function $W : G \to R$ defined by :

$$W(x) = P(x, \varphi_x) \tag{2.7}$$

is continuous and we call it *the value* of the synthesis.

Let us enumerate the properties of the admissible synthesis which justify the operations of the algorithm for the construction of the optimal synthesis.

1. To obtain differentiability properties for the solutions of the discontinuous differential system (2.5) we use the extensions $\tilde{v}_{\mathfrak{c}} : \tilde{\mathfrak{c}} \to U$ of the restrictions $v_{\mathfrak{c}}$ for all the cells of the admissible synthesis.

Indeed we define the maps $\tilde{f}_{\mathfrak{c}} : \tilde{\mathfrak{c}} \to R^n$:

$$\tilde{f}_{\mathfrak{c}}(x) = f(x, \tilde{v}_{\mathfrak{c}}(x)) \tag{2.8$_{(\mathfrak{c})}$}$$

which defines the differential systems :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{f}_{\mathfrak{c}}(x) \quad , \quad x \in \tilde{\mathfrak{c}}. \tag{2.9$_{(\mathfrak{c})}$}$$

It is easy to show that the solutions of $(2.9)_{(\mathfrak{c})}$ that pass through the points in $\mathfrak{c}$ coincide with the corresponding marked trajectories.

2. From the definition of the admissible synthesis we deduce that the marked trajectory enter the cell $\mathfrak{c}$ of type I in the following two manners :

— either there exists a cell $\mathfrak{c}_0$ of type II such that from any point in $\mathfrak{c}_0$ starts a marked trajectory which enters $\mathfrak{c}$ and hence $\mathfrak{c} = \Sigma(\mathfrak{c}_0)$;

— or the marked trajectory reaches $\mathfrak{c}$ from another cell of type I. In what follows we denote by $\mathfrak{c}'$ the submanifold which is either the cell $\mathfrak{c}$ of type I or the union $\mathfrak{c}_0 \cup \mathfrak{c}$ if there exists a cell $\mathfrak{c}_0$ of type II such that $\mathfrak{c} = \Sigma(\mathfrak{c}_0)$.

3. Analising the way that the marked trajectories leave the cell $\mathfrak{c}$ of type I we deduce that for any point $x \in \mathfrak{c}'$ there exists a real number $\tau(x) > 0$ such

that the curve φ_x reaches the cell $\Pi(\mathfrak{c})$ at the moment $\tau(x)$, that is we have :

$$\chi(x) = \varphi_x(\tau(x)) \in \Pi(\mathfrak{c}) \tag{2.10}$$

and $\varphi_x(t) \in \mathfrak{c}'$ for $0 \leqslant t < \tau(x)$.

If we denote $\tilde{\Pi}(\mathfrak{c}) = \Pi(\mathfrak{c}) \cap \tilde{\mathfrak{c}}$ then we have two maps :

$$\tau : \mathfrak{c}' \to R \quad , \quad \chi : \mathfrak{c}' \to \Pi(\mathfrak{c}) \tag{2.11}$$

which satisfy the property (2.10).

4. Using the maximal flow $\tilde{\psi}_{\mathfrak{c}} : \tilde{D}_{\mathfrak{c}} \subset R \times \tilde{\mathfrak{c}} \to \tilde{\mathfrak{c}}$ of the differential system $(2.9)_{(\mathfrak{c})}$ we may prove that the maps (2.11) are of class C^1.

5. Let consider the cells $\mathfrak{c}_1 = \mathfrak{c}, \mathfrak{c}_2, \ldots \mathfrak{c}_q$, of type I through passes every marked trajectory starting in $\mathfrak{c}'$ and such that $\Pi(\mathfrak{c}_q) \subset \mathcal{T}$.

From the definition of the admissible synthesis it follows that every marked trajectory passes from the cell $\mathfrak{c}_{i-1}$ either directly to the cell $\mathfrak{c}_i$ if $\Pi(\mathfrak{c}_{i-1})$ is of type I (and hence $\mathfrak{c}_i = \Pi(\mathfrak{c}_{i-1})$) or by « crossing » the cell $\Pi(\mathfrak{c}_{i-1})$ of type II and then $\mathfrak{c}_i = \Sigma(\Pi(\mathfrak{c}_{i-1}))$.

For every such a cell $\mathfrak{c}_i$ we obtain : the submanifold $\mathfrak{c}'_i$ (which is the union $\mathfrak{c}_i \cup \Pi(\mathfrak{c}_{i-1})$ or even the cell $\mathfrak{c}_i$) ; the neighborhood $\tilde{\mathfrak{c}}_i$; the map $\tilde{f}_i : \tilde{\mathfrak{c}}_i \to R^n$ (and hence the system $(2.9)_{(i)}$) and the maps :

$$\tau^i : \mathfrak{c}'_i \to R \quad , \quad \mathfrak{X}^i : \mathfrak{c}'_i \to \Pi(\mathfrak{c}_i) \tag{2.11$_{(i)}$}$$

with the properties (2.10).

6. On the other hand, for every point $x \in \mathfrak{c}' = \mathfrak{c}'_1$, the marked trajectory φ_x reaches the cell $\Pi(\mathfrak{c}_j)$ at the moment $\tau_j(x)$, $j = 1, 2, \ldots q$. If we denote :

$$\mathfrak{X}_j(x) = \varphi_x(\tau_j(x)) \quad , \quad j = 1, 2, \ldots, q, \tag{2.12}$$

we obtain the maps

$$\tau_j : \mathfrak{c}' \to R \quad , \quad \mathfrak{X}_j : \mathfrak{c}' \to \tilde{\Pi}(\mathfrak{c}_j) \quad , \quad j = 1, 2, \ldots, q \tag{2.13}$$

which satisfy the conditions :

$$\mathfrak{X}_j(x) = \varphi_x(\tau_j(x)) \quad , \quad \varphi_x(t) \in \mathfrak{c}'_j \text{ for } \tau_{j-1}(x) \leqslant t < \tau_j(x) \tag{2.14}$$

Since the marked trajectory φ_x is unique we have :

$$\varphi_x(t) = \varphi_{\mathfrak{X}_{j-1}(x)}(t - \tau_{j-1}(x)) \quad \text{for} \quad t \in [\tau_{j-1}(x), \tau_j(x)] \tag{2.15}$$

and $j = 2, 3, \ldots q$. If we define the maps $\tau_0 : \mathfrak{c}' \to R$, $\mathfrak{X}_0 : \mathfrak{c}' \to \mathfrak{c}'$

$$\tau_0(x) = 0 \quad , \quad \mathfrak{X}_0(x) = x \quad , \quad x \in \mathfrak{c}' = \mathfrak{c}'_1, \tag{2.16}$$

then we have :

$$(2.17)\qquad \begin{cases} \tau_j(x) = \tau^j(\mathfrak{X}_{j-1}(x)) + \tau_{j-1}(x) \\ \mathfrak{X}_j(x) = \mathfrak{X}^j(\mathfrak{X}_j(x)) \end{cases} \quad , \qquad j = 1, 2, \ldots, q,$$

and hence the maps τ_j, $\mathfrak{X}_j$ are also of class C^1.

7. For every cell $\mathfrak{c}$ of type I the function $\tilde{f}^0_{\mathfrak{c}} : \tilde{\mathfrak{c}} \to R$ given by :

$$(2.18)\qquad \tilde{f}^0_{\mathfrak{c}}(x) = f^0(x, \tilde{v}_{\mathfrak{c}}(x)) \quad , \qquad x \in \tilde{\mathfrak{c}}$$

is of class C^1 and hence the function $\tilde{H}_{\mathfrak{c}} : \tilde{\mathfrak{c}} \times R^n \to R$ given by :

$$(2.19)\qquad \tilde{H}_{\mathfrak{c}}(x, \lambda) = \tilde{f}^0_{\mathfrak{c}}(x) + \lambda \cdot \tilde{f}_{\mathfrak{c}}(x) \quad , \quad x \in \tilde{\mathfrak{c}} \subset G \quad , \quad \lambda \in R^n$$

is also of class C^1.

Since the first n equations of the Hamiltonian system :

$$(2.20)_{(\mathfrak{c})}\qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial \tilde{H}_{\mathfrak{c}}}{\partial \lambda}(x, \lambda) = \tilde{f}_{\mathfrak{c}}(x) \\ \dfrac{d\lambda}{dt} = -\dfrac{\partial \tilde{H}_{\mathfrak{c}}}{\partial x}(x, \lambda) = -\dfrac{\partial \tilde{f}^0_{\mathfrak{c}}}{\partial x}(x) - \lambda \dfrac{\partial \tilde{f}_{\mathfrak{c}}}{\partial x}(x) \end{cases}$$

do not depend on λ and represent a differential system which coincides with $(2.9)_{(\mathfrak{c})}$, we deduce that at every point (x, λ) from $\tilde{\mathfrak{c}} \times R^n$ there exists a unique solution $\Phi_{\mathfrak{c},(x,\lambda)} = (\tilde{\psi}_{\mathfrak{c},x}, \eta_{\mathfrak{c},x\lambda})$ of the system $(2.20)_{(\mathfrak{c})}$ which is defined on the whole interval of definition $(t^-_{\mathfrak{c}}(x), t^+_{\mathfrak{c}}(x))$ of the maximal solution $\tilde{\psi}_{\mathfrak{c},x}$ of the system $(2.9)_{(\mathfrak{c})}$. Moreover, the function $\tilde{H}_{\mathfrak{c}}$ is a first integral for the system $(2.20)_{(\mathfrak{c})}$ that is $\tilde{H}_{\mathfrak{c}}(\psi_{\mathfrak{c},x}(t), \eta_{\mathfrak{c},x\lambda}(t)) =$ constant for $t \in (t^-_{\mathfrak{c}}(x), t^+_{\mathfrak{c}}(x))$.

8. We consider again the cells $\mathfrak{c} = \mathfrak{c}_1, \mathfrak{c}_2, \ldots \mathfrak{c}_q$, of type I, through which pass all the marked trajectories starting in $\mathfrak{c}'_1$, the maps $\tau_i : \mathfrak{c}' \to R$, $\mathfrak{X}_i : \mathfrak{c}' \to \Pi(\mathfrak{c}_i)$ which satisfy (2.14). Since $\Pi(\mathfrak{c}_q) \subset \mathcal{T}$ we have $t_q(x) = t_F(x)$, $\mathfrak{X}_q(x) = x_F(x)$ for $x \in \mathfrak{c}'$.

If $\Pi(\mathfrak{c}_i)$ (and hence $\tilde{\Pi}(\mathfrak{c}_i)$ is of dimension k_i, $k_i \geqslant k$, $k_i \leqslant n-1$ $i = 1, 2, \ldots q$) and if we have the following parametric representation :

$$(2.21)\qquad x' = \chi_{(i)}(s^1, s^2, \ldots s^{k_i})$$

for x' in a coordinate neighborhood of the point $\mathfrak{X}_i(x) \in \Pi(\mathfrak{c}_i)$ we may prove :

Proposition

For every marked trajectory $\varphi_x : [0, t_F] \to G$ which starts from the point $x \in G \backslash N$ there exists a vector $\lambda(x) \in R^n$ and a curve $\eta_{\lambda(x)} : [0, t_F] \to R^n$ with the following properties :

(i) $\eta_{\lambda(x)}(0) = \lambda(x)$ and for $t \in (\tau_{i-1}(x), \tau_i(x))$ the curve $(\varphi_x, \eta_{\lambda(x)})$ is a solution of the system $(2.20)_{(i)}$;

(ii) $\eta_{\lambda(x)}$ is continuous to the right hand side and its one sided limits at the points $t = \tau_i(x)$, $\lambda_i^- = \eta_{\lambda(x)}(\tau_i(x) - 0)$, $i = 1, 2, \dots q$, satisfy the following relations :

$$(2.22) \qquad \begin{cases} \lambda_q^- \dfrac{\partial \chi_{(q)}}{\partial s^j}(s^1, s^2, \dots s^k) = \dfrac{\partial h}{\partial s^j}(s^1, s^2, \dots s^k), j = 1, 2, \dots k \\ \tilde{H}_q(\chi_{(q)}(s^1, s^2, \dots s^k), \lambda_q^-) = 0 \end{cases}$$

$$(2.23)_{(i)} \qquad \begin{cases} \lambda_i^- \dfrac{\partial \chi_{(i)}}{\partial s^j}(s^1, s^2, \dots s^{k_i}) = \lambda_i^+ \dfrac{\partial \chi(i)}{\partial s^j}(s^1, s^2, \dots s^{k_i}) \\ \tilde{H}_i(\chi_{(i)}(s^1, s^2, \dots s^{k_i}), \lambda_i^-) = 0 \end{cases}$$

where $i = 1, 2, \dots q - 1$ and $h(s^1, s^2, \dots s^k)$ is the local representative of the function g in the considered neighborhood of the point $\mathfrak{X}_q(x) = x_F(x) \in \mathcal{C}$ and $\lambda_i^+(x) = \eta_{\lambda(x)}(\tau_i(x))$, $i = 1, 2, \dots q - 1$.

In particular we have $\tilde{H}_1(x, \lambda(x)) = f^0(x, v(x)) + \lambda(x) \cdot f(x, v(x)) = 0$.

9. For every $x \in G \backslash N$ the value of the functional to minimize along the marked trajectory φ_x is given by :

$$(2.24) \qquad W(x) = g(\mathfrak{X}_q(x)) + \sum_{i=1}^{q} \int_{\tau_{i-1}(x)}^{\tau_i(x)} \tilde{f}_i^0(\varphi_x(t)) \mathrm{d}t$$

and it is proved that the restriction $W_{\mathfrak{c}} = W|_{\mathfrak{c}}$ is of class C^1 and verifies the relation :

$$(2.25) \qquad \frac{\partial}{\partial s^j} W(\chi_{(\mathfrak{c})}(s^1, s^2, \dots s^r)) = \lambda(x) \frac{\partial \chi_{\mathfrak{c}}}{\partial s^j}(s^1, s^2, \dots s^r)$$

for $j = 1, 2, \dots r$, where r is the dimension of the cell $\mathfrak{c}$ and $x' = \chi_{(\mathfrak{c})}(s^1, s^2, \dots s^r)$ is the parametric representation of in a coordinate neighborhood of the point $x = \chi_{(\mathfrak{c})}(s_0^1, \dots s_0^r) \in \mathfrak{c}$.

In particular, when $\mathfrak{c} \subset G \setminus M$, $M = \left(\bigcup_{i=k-1}^{n-1} P^i \right) \cup N$, that is $\mathfrak{c}$ is a cell of the maximum dimension n, then we obtain :

$$(2.26) \qquad \frac{\partial W}{\partial x} = \lambda(x)$$

and $\lambda(x)$ is uniquely determined. From 8. it follows that :

$$(2.27) \qquad \frac{\partial W}{\partial x}(x) \cdot f(x, v(x)) + f^0(x, v(x)) = 0, x \in G \setminus M.$$

10. If we denote

$$g_i = W \,|\, \Pi_{(\mathfrak{c}_i)} = W_{\Pi(\mathfrak{c}_i)},\ i = 1, 2, \dots q - 1,\ g_q = g$$

then from 9. we obtain :

$$\lambda_i^+ \cdot \frac{\partial \chi_{(i)}}{\partial s^j}(s^1, s^2, \dots s^{k_i}) = \frac{\partial}{\partial s^j} g_i(\chi_{(i)}(s^1, s^2, \dots s^{k_i}) = \frac{\partial}{\partial s^j} h_i(s^1, \dots s^{k_i})$$

for $j = 1, 2, \dots k_i$, where $h_i(s^1, s^2, \dots s^{k_i})$ is the local representative of the function g_i in a neighborhood of the point $\mathfrak{X}_i(x)$ on $\Pi(\mathfrak{c}_i)$. Hence the formulae (2.22), (2.23) from 8. may be written in a unitary manner :

$$\lambda_i^- \frac{\partial \chi_{(i)}}{\partial s^j}(s^1, s^2, \dots s^{k_i}) = \frac{\partial h_i}{\partial s^j}(s^1, s^2, \dots s^{k_i}) \tag{2.28}$$

for $i = 1, 2, \dots q$ where $h_q = h$ and $g_q = g$.

11. Using the Boltyanskii's lemmas and the properties of the admissible synthesis we may prove the following necessary ans sufficient condition for optimality of the admissible synthesis in the form of dynamic programming principle :

Theorem 1

The marked trajectories (the controls generated by the admissible synthesis) are optimals if and only if for every point $x \in G \backslash M$ the following inequality holds :

$$\frac{\partial W}{\partial x}(x) \cdot f(x, u) + f^0(x, u) \geqslant \frac{\partial W}{\partial x}(x) \cdot f(x, v(x)) + f^0(x, v(x)) = 0 \tag{2.29}$$

for any $u \in U$.

12. The same condition may be stated in a certain form of the maximum (minimum) principle of Pontryagin : if we define the function

$$\mathcal{H} : G \times U \times R^n \to R \text{ by :}$$

$$\mathcal{H}(x, u, \lambda) = f^0(x, u) + \lambda \cdot f(x, u), \tag{2.30}$$

we observe that $\mathcal{H}(x, v(x), \lambda) = \tilde{H}_{\mathfrak{c}}(x, \lambda)$ for $x \in \tilde{\mathfrak{c}}$. Using 2.26 and (2.27) and the theorem from 11. we obtain :

Theorem 2

The marked trajectories are optimal if and only if for every marked tratectory $\varphi_x : [0, t_F] \to G$, $x \in G \backslash M$, we have for every $t \in [0, t_F]$:

$$\mathcal{H}(\varphi_x(t), u, \eta_{\lambda(x)}(t)) \geqslant \mathcal{H}(\varphi_x(t), v(\varphi_x(t)), \eta_{\lambda(x)}(t) = 0$$

for any $u \in U$.

3. THE ALGORITHM FOR THE CONSTRUCTION OF THE OPTIMAL SYNTHESIS

The algorithm proposed in this section contains three « steps » every step on his side containing several « operations ».

The first step contains preliminary operations which deal with the whole phase space G.

The second step which is « the main routine » is worked many times for different « data » of the problem. This step contains two « cycles » and represent what usually is called a *routine.*

The third step contains the operations to verify some conditions that must be satisfied by synthesis obtained.

Some of the operations of the algorithm (even in the step I or II) cannot be worked if some conditions are not satisfied. These operations were labeled with small latine letters a, b, ... Some of these conditions are very strong : if they are not satisfied then algorithm does not work for our problem. The other are less strong in the sense that they are satisfied if we restrict the phase space to a subset of G.

The passage from an operation to another is made either nonconditioned, in the natural order of the operations, or conditioned, that is we must pass to an operation or to another if a condition or another is satisfied.

THE ALGORITHM

The step I

a) **I-1.** We define the map $\mathcal{H} : G \times U \times R^n \to R$:

$$\mathcal{H}(x, u, \lambda) = f^0(x, u) + \lambda \cdot f(x, u)$$

or

$$\mathcal{H}(x^1, x^2, \dots x^n, u^1, u^2, \dots u^p, \lambda_1, \dots \lambda_n) = f^0(x^1, \dots x^n, u^1, \dots u^p) + \sum_{j=1}^{n} \lambda_j f^j(x^1, x^2, \dots x^n, u^1, u^2, \dots u^p).$$

For every $(x, \lambda) \in G \times R^n$ we are looking for $\min_{u \in U} \mathcal{H}(x, u, \lambda)$.

Condition (*a*). There exists a map $\bar{u} : G \times R^n \to \mathcal{P}(U)$ (where $\mathcal{P}(U)$ is the family of all subsets of U) such that :

$$\min_{u \in U} \mathcal{H}(x, u, \lambda) = \mathcal{H}(x, \bar{u}(x, \lambda), \lambda) = \bar{H}(x, \lambda) \text{ for every } (x, \lambda) \tag{3.1}$$

in $G \times R^n$.

If U is a compact subset of R^p, condition (a) is satisfied.

If there exist points in G for which $\bar{u}(x, \lambda) \subset U$ with the property (3.1) does not exist then we may restrict our problem to an open subset $G' \subset G$ for which the condition (a) is satisfied.

If there is no such G' we stop : the algorithm does not work for our problem ; otherwise we pass to the next operation.

b) **I-2**

Condition (*b*). We verify that function $\bar{H} : G \times R^n \to R$ obtained by the operation I-1 is a piecewise smooth function (of class C^1) that is, there exist the open sets $U_1^*, U_2^*, \ldots, U_m^* \subset G \times R^n$ such that $\bigcup_{j=1}^{m} (\bar{U}_j^*) = \overline{G \times R^n}$ and $\bar{H}_j = \bar{H} \mid_{U_j^*}$ is of class C^1 for $j = 1, 2, \ldots m$.

If this condition is not satisfied we stop : the algorithm does not work ; otherwise we pass to the next operation.

We denote by $\tilde{H}_j$ the extension of class C^1 of the map $\bar{H}_j$ to a neighborhood $\tilde{U}_j^*$ of the closure $\bar{U}_j^*, j = 1, 2, \ldots, m$.

The step II

We take by definition $\mathcal{C}_0^{(0)} = \mathcal{C}, g_0^{(0)} = g, k_0 = k = \dim (\mathcal{C}_0^{(0)})$.

We suppose that we have determined by recurrence : (α) the nonnegative integers $k_i \geqslant k$, $m_i \geqslant 0$, for $i = 0, 1, 2, \ldots \mathbf{I} - 1$, and for $i = 0, 1, \ldots \mathbf{I}$, the integers $m_{i,r} \geqslant 1$, $n_{i,r} \geqslant 1$ where $r = 0, 1, \ldots m_i$, when $i = 0, 1, \ldots \mathbf{I} - 1$, and $r = 0, 1, \ldots q - 1$ when $i = \mathbf{I}$; (β) the differentiable manifolds $\mathcal{C}_r^{(i)} \subset G$ of dimension k_i for $r = 0, 1, \ldots m_i$ when $i = 0, 1, \ldots \mathbf{I} - 1$ and for $r = 0, 1, \ldots q$ when $i = \mathbf{I}$; (γ) the C^1-functions $g_r^{(i)} : \mathcal{C}_r^{(i)} \to R$ for $\gamma = 0, 1, \ldots m_i$ when $i = 0, 1, \ldots \mathbf{I} - 1$, and for $r = 0, 1, \ldots, q$ when $i = \mathbf{I}$;

We suppose that for every pair $(\mathcal{C}_r^{(i)}, g_r^{(i)})$ we have determined :

(i) the differentiable manifolds $\mathfrak{S}_{r,p,1}^{(i),\alpha} \subset G$ of dimension $k_i + 1$ possibly with boundary $\partial \mathfrak{S}_{r,p,1}^{(i),\alpha}$;

(ii) the differentiable manifolds $\mathfrak{S}_{r,p,2}^{(i),\alpha} \subset G$ of dimension k_i such that the union $\mathfrak{S}_{r,p}^{(i),\alpha} = \mathfrak{S}_{r,p,1}^{(i),\alpha} \cup \mathfrak{S}_{r,p,2}^{(i),\alpha}$ is a differentiable manifold with boundary $\mathfrak{S}_{r,p,2}^{(i),\alpha} \cup \partial \mathfrak{S}_{r,p,1}^{(i),\alpha}$

(iii) the maps $v_{r,p}^{(i),\alpha} : \mathfrak{S}_{r,p}^{(i),\alpha} \to U$ of class C^1 ;

(iv) the functions $W_{r,p}^{(i),\alpha} : \mathfrak{S}_{r,p}^{(i),\alpha} : \to R$ of class C^1, where $\alpha = 1, 2, \ldots n_{i,r}$, $p = 1, 2, \ldots m_{i,r}$.

The routine $(\mathcal{C}_q^{(\mathrm{I})}, g_q^{(\mathrm{I})})$

We shall describe the operations to obtain the numbers $m_{\mathrm{I},q}$, $n_{\mathrm{I},q}$ and the elements $\mathfrak{S}_{q,p,1}^{(\mathrm{I}),\alpha}$, $\mathfrak{S}_{q,p,2}^{(\mathrm{I}),\alpha}$, $v_{q,p}^{(\mathrm{I}),\alpha}$, $W_{q,p}^{(\mathrm{I})\alpha}$ for $p = 1, 2, \ldots m_{\mathrm{I},q}$ and $\alpha = 1, 2, \ldots n_{\mathrm{I},q}$.

II-1. We take $n_{\mathrm{I},q}$ = the number of the connected components of the manifold $\mathcal{C}_q^{(\mathrm{I})}$ and we denote $\mathcal{C}_q^{(\mathrm{I}),\alpha}$, $\alpha = 1, 2, \ldots, n_{\mathrm{I},q}$ these connected components. We take by definition $g_q^{(\mathrm{I}),\alpha} = g_q^{(\mathrm{I})} \mid \mathcal{C}_q^{(\mathrm{I}),\alpha}$

In order to avoid still more complicated notations we shall consider that the connected differential manifold $\mathcal{C}_q^{(\mathrm{I}),\alpha}$ admits a global parametric representation :

$$x = \chi_q^{(\mathrm{I}),\alpha}(s^1, s^2, \ldots, s^{k_\mathrm{I}}) \tag{3.2}$$

If not, we have to repete the construction for every coordinate neighborhood.

We denote $h_q^{(\mathrm{I}),\alpha}(s^1, s^2, \ldots, s^{k_\mathrm{I}})$ the local representative of the function $g_q^{(\mathrm{I}),\alpha}$ with respect to the parametrisation (3.2).

We take $\alpha = 1$ and we pass to :

The subroutine $(\mathcal{C}_q^{(\mathrm{I}),\alpha}, h_q^{(\mathrm{I}),\alpha})$

II-2. For every set U_j^*, $j = 1, 2, \ldots, m$, we define the following set : $\mathcal{C}_{q,j}^{*(\mathrm{I}),\alpha}$ = the set of all points $(x, \lambda) \in \bar{U}_j^*$ which satisfy the following three conditions :

1. $$x = \chi_q^{(\mathrm{I}),\alpha}(s^1, s^2, \ldots, s^{k_\mathrm{I}}) \in \mathcal{C}_q^{(\mathrm{I}),\alpha}$$

2. $\lambda = \lambda_q^{(\mathrm{I}),\alpha}(s^1, s^2, \ldots, s^{k_\mathrm{I}})$ is given by the system :

$$\begin{cases} \lambda \cdot \dfrac{\partial \chi_q^{(\mathrm{I}),\alpha}}{\partial s^\beta}(s^1, s^2, \ldots, s^{k_\mathrm{I}}) = \dfrac{\partial h_q^{(\mathrm{I}),\alpha}}{\partial s^\beta}(s^1, s^2, \ldots, s^{k_\mathrm{I}}) \; \beta = 1, 2, \ldots, k_\mathrm{I} \\ \tilde{H}_j(\chi_q^{(\mathrm{I}),\alpha}(s^1, s^2, \ldots, s^{k_\mathrm{I}}), \lambda) = 0. \end{cases} \tag{3.3}$$

3. (The condition of transversal intersection.) For every $(s^1, s^2, \ldots, s^{k_\mathrm{I}})$ the set $\bar{u}_q^{(\mathrm{I}),\alpha}$ of all the points u in the set $\bar{u}(\chi_q^{(\mathrm{I}),\alpha}(s^1, s^2, \ldots, s^{k_\mathrm{I}}), \lambda_q^{(\mathrm{I}),\alpha}(s^1, s^2, \ldots, s^{k_\mathrm{I}}))$ for which the matrix with columns $\dfrac{\partial \chi_q^{(\mathrm{I}),\alpha}}{\partial s^j}(s^1, \ldots, s^{k_\mathrm{I}})$, $j = 1, \ldots, k_\mathrm{I}$, and $\mathcal{J}(\chi_q^{(\mathrm{I}),\alpha}(s^1, \ldots, s^{k_\mathrm{I}}), u)$ has the maximum rank $k_\mathrm{I} + 1$, is nonempty.

If the boundary of the control space U is piecewise smooth then this condition assure also the application of the implicite functions theorem to give $\lambda_1, \lambda_2, \ldots \lambda_n$ from the system (3.3).

If all the sets $\mathcal{C}^{*(\mathrm{I}),\alpha}_{q,j}$, $j = 1, 2, \ldots, m$ are empty we pass to the operation II-8. If there exists a set $\mathcal{C}^{*(\mathrm{I}),\alpha}_{q,j} \neq \Phi$ we pass to the next operation.

II-3. Let $(x^{(\mathrm{I}),\alpha}_{q,j}(\cdot\,; x, \lambda), \lambda^{(\mathrm{I}),\alpha}_{q,j}(\cdot\,; x, \lambda))$ the solution defined for $t \leqslant 0$ of the system :

$$(3.4)_{(j)} \qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\partial \tilde{H}_j}{\partial \lambda}(x, \lambda) \\[2ex] \dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = -\dfrac{\partial \tilde{H}_j}{\partial x}(x, \lambda) \end{cases}$$

with the initial conditions :

$$\begin{cases} x^{(\mathrm{I}),\alpha}_{q,j}(0; x, \lambda) = x \\ \lambda^{(\mathrm{I}),\alpha}_{q,j}(0; x, \lambda) = \lambda \end{cases}$$

for every point $(x, \lambda) \in \mathcal{C}^{*(\mathrm{I}),\alpha}_{q,j}$.

For every such point (x, λ), we define $\tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda) \leqslant 0$ as follows :

(i) if $(x^{(\mathrm{I}),\alpha}_{q,j}(t\,; x, \lambda), \lambda^{(\mathrm{I}),\alpha}_{q,j}(t\,; x, \lambda)) \in \bar{U}^*_j$ for all

$$t \leqslant 0 \text{ then } \tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda) = -\infty\,;$$

(ii) if for every $t < 0$ there exists $t' < 0$, $t' > t$ such that

$$(x^{(\mathrm{I}),\alpha}_{q,j}(t'\,; x, \lambda), \lambda^{(\mathrm{I}),\alpha}_{q,j}(t'\,; x, \lambda)) \notin \bar{U}^*_j \qquad \text{then} \qquad \tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda) = 0\,;$$

(iii) if neither (i) nor (ii) occur, then $\tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda)$ is the negative number which satisfies the conditions :

$$(x^{(\mathrm{I}),\alpha}_{q,j}(t\,; x, \lambda), \lambda^{(\mathrm{I}),\alpha}_{q,j}(t\,; x, \lambda)) \in \bar{U}^*_j \text{ for } \tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda) \leqslant t \leqslant 0 \text{ and}$$
$$(x^{(\mathrm{I}),\alpha}_{q,j}(t\,; \lambda), \lambda^{(\mathrm{I}),\alpha}_{q,j}(t\,; x, \lambda)) \notin \bar{U}^*_j \text{ for } t < \tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda).$$

We define now the sets (1) :

$$(3.5) \qquad \mathfrak{S}^{(\mathrm{I}),\alpha}_{q,j,1} = \{\, x^{(\mathrm{I}),\alpha}_{q,j}(t\,; x, \lambda) \mid \tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda) < t < 0, (x, \lambda) \in \mathcal{C}^{*(\mathrm{I}),\alpha}_{q,j} \}$$

(3.6)

$$\mathfrak{S}^{(\mathrm{I}),\alpha}_{q,j,2} = \{\, x^{(\mathrm{I}),\alpha}_{q,j}(\tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda)\,; x, \lambda) \mid (x, \lambda) \in \mathcal{C}^{*(\mathrm{I}),\alpha}_{q,j} - \infty < \tau^{(\mathrm{I}),\alpha}_{q,j}(x, \lambda) < 0 \}$$

for every $j = 1, 2, \ldots m$.

If all the sets $\mathfrak{S}^{(\mathrm{I}),\alpha}_{q,j,1}$, $\mathfrak{S}^{(\mathrm{I}),\alpha}_{q,j,2}$, $j = 1, 2, \ldots, m$ are empty we pass to the operation II-8 ; otherwise we pass to the next operation.

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(1) With the natural convention that a set is empty if its definition is meaningless.

c) **II-4.** We delate the sets $\mathfrak{S}_{q,j,1}^{(\mathrm{I}),\alpha}$, $\mathfrak{S}_{q,j,2}^{(\mathrm{I}),\alpha}$ such that $\mathfrak{S}_{q,j,1}^{(\mathrm{I}),\alpha}$ is contained in $\mathcal{C}_{q,j}^{(\mathrm{I}),\alpha}$ or $\mathfrak{S}_{q,j,1}^{(\mathrm{I}),\alpha}$ is not a differentiable manifold (possibly with boundary) of dimension $k_{\mathrm{I}}+1$ or if $\mathfrak{S}_{q,j,2}^{(\mathrm{I}),\alpha}$ is not empty or a differentiable manifold of dimension k_{I}.

If there not exist some sets $\mathfrak{S}_{q,j,1}^{(\mathrm{I}),\alpha}$, $\mathfrak{S}_{q,j,2}^{(\mathrm{I}),\alpha}$ that satisfy the condition (*c*) we pass to the operation II-8 ; otherwise we pass to the next operation.

d) **II-5.** For every $x \in \mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha} = \mathfrak{S}_{q,j,1}^{(\mathrm{I}),\alpha} \cup \mathfrak{S}_{q,j,2}^{(\mathrm{I}),\alpha}$ we consider the point $(x_0, \lambda_0) \in \mathcal{C}_{q,j}^{*(\mathrm{I}),\alpha}$ and $t_1 < 0$ such that :

$$x = x_{q,j}^{(\mathrm{I}),\alpha}(t_1 ; x_0, \lambda_0). \tag{3.7}$$

Condition (*d*). For every $x \in \mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha}$ we may take a point $v_{q,j}^{(\mathrm{I}),\alpha}(x)$ in the union of all the sets $\bar{u}(x, \lambda_{q,j}^{(\mathrm{I}),\alpha}(t_1 ; x_0, \lambda_0))$ where t_1, x_0, λ_0 satisfies the condition (3.7) such that the following conditions hold :

(i) the map $v_{q,j}^{(\mathrm{I}),\alpha} : \mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha} \to U$ so defined is of class C^1 ;

(ii) there exists an open neighborhood $\tilde{\mathfrak{S}}_{q,j}^{(\mathrm{I}),\alpha}$ of the closure $\overline{\mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha}}$ and a C^1-extension $\tilde{v}_{q,j}^{(\mathrm{I}),\alpha}$ to $\tilde{\mathfrak{S}}_{q,j}^{(\mathrm{I}),\alpha}$ of the map $v_{q,j}^{(\mathrm{I}),\alpha}$.

Again we retain only the sets $\mathfrak{S}_{q,j,1}^{(\mathrm{I}),\alpha}$, $\mathfrak{S}_{q,j,2}^{(\mathrm{I}),\alpha}$ and the corresponding maps $v_{q,j}^{(\mathrm{I}),\alpha}$ for which this condition is verified. If there exists such a set we pass to the next operation ; otherwise we pass to the operation II-8.

We notice that for a set $\mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha}$ we may obtain two or more maps $v_{q,j}^{(\mathrm{I}),\alpha}$ and hence we may obtain more than one optimal synthesis. In what follows we work with one of these maps.

e) **II-6.** Let $x \in \mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha}$ and $t_1 < 0$, x_0, λ_0, with the condition (3.7). Then we define the map $\varphi_{q,j}^{(\mathrm{I}),\alpha}(\cdot\,; x) : [0, -t_1] \to \mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha}$ by :

$$\varphi_{q,j}^{(\mathrm{I}),\alpha}(t ; x) = x_{q,j}^{(\mathrm{I}),\alpha}(t + t_1 ; x_0, \lambda_0) \text{ for } t \in [0, -t_1]. \tag{3.8}$$

Condition (*e*). We verify that $\varphi_{q,j}^{(\mathrm{I}),\alpha}(\cdot\,; x)$ defined in this way is a solution of the differential system :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, v_{q,j}^{(\mathrm{I}),\alpha}(x)) \tag{3.9}$$

with the initial condition :

$$\varphi_{q,j}^{(\mathrm{I}),\alpha}(0 ; x) = x \tag{3.10}$$

We note that in this case we have :

$$\varphi_{q,j}^{(\mathrm{I}),\alpha}(-t_1 ; x) = x_0. \tag{3.11}$$

REMARK 3.1

We notice that this condition is verified if we have :

$$\frac{\partial \tilde{H}_j}{\partial \lambda}(x, \lambda) = f(x, v_{q,j}^{(\mathrm{I}),\alpha}(x)),\ x \in \tilde{\mathfrak{S}}_{q,j}^{(\mathrm{I}),\alpha} \tag{3.12}$$

and (3.12) is satisfied if the boundary of U is piecewise smooth.

We retain only the sets $\mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha}$ and the maps $v_{q,j}^{(\mathrm{I}),\alpha}$ for that the condition (e) is verified. If such elements do not exist we pass to the operation II-8 ; otherwise we pass to the next operation.

f) **II-7.** We define the map $W_{q,j}^{(\mathrm{I}),\alpha} : \mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha} \to R$ by :

$$W_{q,j}^{(\mathrm{I}),\alpha}(x) = g_{q,j}^{(\mathrm{I}),\alpha}(\varphi_{q,j}^{(\mathrm{I}),\alpha}(-t_1)) + \int_0^{-t_1} f^0(\varphi_{q,j}^{(\mathrm{I}),\alpha}(t\,;x), v_{q,j}^{(\mathrm{I}),\alpha}(\varphi_{q,j}^{(\mathrm{I}),\alpha}(t\,;x)))\mathrm{d}t \tag{3.13}$$

Condition (f). We verify that the map $W_{q,j}^{(\mathrm{I}),\alpha}$ is of class C^1.

We retain the sets $\mathfrak{S}_{q,j}^{(\mathrm{I}),\alpha}$ (and hence the maps $v_{q,j}^{(\mathrm{I}),\alpha}$, $W_{q,j}^{(\mathrm{I}),\alpha}$) for that the condition (f) is verified and pass to the next operation.

We notice that at this moment the subroutine $(\mathcal{C}_q^{(\mathrm{I}),\alpha}, g_q^{(\mathrm{I}),\alpha})$ is finished. Now we repete this subroutine for all pairs $(\mathcal{C}_q^{(\mathrm{I}),\alpha}, g_q^{(\mathrm{I}),\alpha})$ for $\alpha = 1, 2, \ldots n_{\mathrm{I},q}$.

II-8. If $\alpha < n_{\mathrm{I},q}$ then we take $\alpha + 1$ instead of α and pass to the subroutine $(\mathcal{C}_q^{(\mathrm{I}),\alpha}, g_q^{(\mathrm{I}),\alpha})$ (to the operation II-2).

If $\alpha = n_{\mathrm{I},q}$ (hence we have considered all connected components $\mathcal{C}_q^{(\mathrm{I}),\alpha}$ of the manifold $\mathcal{C}_\alpha^{(\mathrm{I})}$) we pass to the next operation.

II-9. We continue the routine $(\mathcal{C}_q^{(\mathrm{I})}, g_q^{(\mathrm{I})})$. For all indeces i, α, r, p, γ and I, β, q, j, δ for that $\mathfrak{S}_{r,p,\gamma}^{(i),\alpha} \cap \mathfrak{S}_{q,j,\delta}^{(\mathrm{I}),\beta} \neq \varphi$ we define the sets :

$$N_{(r,p,\gamma),(q,j,\delta)}^{(i),\alpha(\mathrm{I}),\beta} = \{\, x \in \mathfrak{S}_{r,p,\gamma}^{(i),\alpha} \cap \mathfrak{S}_{q,j,\delta}^{(\mathrm{I}),\beta} \mid W_{r,p}^{(i),\alpha}(x) = W_{q,j}^{(\mathrm{I}),\beta}(x) \,\} \tag{3.14}$$

$$\mathfrak{S}_{(r,p,\gamma),(q,j,\delta)}^{(i),\alpha,(\mathrm{I}),\beta} = (\mathfrak{S}_{r,p,\gamma}^{(i),\alpha} \setminus \mathfrak{S}_{q,j,\delta}^{(\mathrm{I}),\beta}) \cup \{\, x \in \mathfrak{S}_{r,p,\gamma}^{(i),\alpha} \cap \mathfrak{S}_{q,j,\delta}^{(\mathrm{I}),\beta} \mid W_{r,p}^{(i),\alpha}(x) < W_{q,j}^{(\mathrm{I}),\beta}(x) \,\} \tag{3.15}$$

$$\mathfrak{S}_{(q,j,\delta),(r,p,\gamma)}^{(\mathrm{I}),\beta,(i),\alpha} = (\mathfrak{S}_{q,j,\delta}^{(\mathrm{I}),\beta} \setminus \mathfrak{S}_{r,p,\gamma}^{(i),\alpha}) \cup \{\, x \in \mathfrak{S}_{r,p,\gamma}^{(i),\alpha} \cap \mathfrak{S}_{q,j,\delta}^{(\mathrm{I}),\beta} \mid W_{r,p}^{(i),\alpha}(x) > W_{q,j}^{(\mathrm{I}),\beta}(x) \,\} \tag{3.16}$$

If all the above intersections are empty then we pass to the operation II-11; otherwise we continue.

II-10. We write the sets $\mathfrak{S}^{(i),\alpha,(\mathrm{I}),\beta}_{(r,p,\gamma),(q,j,\delta)}$ above determined in the form $\mathfrak{S}^{(i),\alpha}_{r,p,\gamma}$ by changing the integers m_i, $m_{i,r}$, $n_{i,r}$, for $i \leqslant \mathrm{I}$, $r \leqslant q$, and the routine $(\mathcal{C}^{(\mathrm{I})}_q, g^{(\mathrm{I})}_q)$ is finished. We pass to the next operation.

II-11. We define the set :

$$\mathcal{C}^{(\mathrm{I})}_{q+1} = \bigcup_{j=1}^{m_{\mathrm{I},q}} \bigcup_{\alpha=1}^{n_{\mathrm{I},q}} (\mathfrak{S}^{(\mathrm{I}),\alpha}_{q,j,2} \cup \partial\mathfrak{S}^{(\mathrm{I}),\alpha}_{q,j,1}) \tag{3.17}$$

If $\mathcal{C}^{(\mathrm{I})}_{q+1} \neq \Phi$ we define the map $g^{(\mathrm{I})}_{q+1} : \mathcal{C}^{(\mathrm{I})}_{q+1} \to R$ by :

(3.18) $\quad g^{(\mathrm{I})}_{q+1}(x) = W^{(\mathrm{I})}_{q,j}(x)$ for $x \in \mathfrak{S}^{(\mathrm{I}),\alpha}_{q,j}$. We put $q+1$ instead of q and pass to the routine $(\mathcal{C}^{(\mathrm{I})}_q, g^{(\mathrm{I})}_q)$ (to the operation II-1).

If $\mathcal{C}^{(\mathrm{I})}_{q+1} = \Phi$ and $k_{\mathrm{I}} + 1 < n$ we take $m_{\mathrm{I}} = q$, $k_{\mathrm{I}+1} = k_{\mathrm{I}} + 1$ and we define the set :

$$\mathcal{C}^{(\mathrm{I}+1)}_0 = \bigcup_{r=0}^{q} \bigcup_{p=1}^{m_{\mathrm{I},r}} \bigcup_{\alpha=1}^{n_{\mathrm{I},r}} (\mathrm{int}\,(\mathfrak{S}^{(\mathrm{I}),\alpha}_{r,p,1})) \quad \text{and the map :} \tag{3.19}$$

$$g^{(\mathrm{I}+1)}_0(x) = W^{(\mathrm{I}),\alpha}_{r,p}(x) \text{ if } x \in \mathrm{int}\,(\mathfrak{S}^{(\mathrm{I}),\alpha}_{r,p,1}) \tag{3.20}$$

We take $\mathrm{I}+1$ instead of I, 0 instead of q and pass to the routine $(\mathcal{C}^{(\mathrm{I})}_q, g^{(\mathrm{I})}_q)$.

If $\mathcal{C}^{(\mathrm{I})}_{q+1} = \Phi$ and $k_{\mathrm{I}} + 1 = n$ we pass to the step III.

The step III

g) **III-1.** We define the cells of the optimal synthesis :

(i) the k_i-dimensional cells of type I are the connected components of the sets $\mathfrak{S}^{(i),\alpha}_{r,p,1} \cap \mathcal{C}^{(i)}_r$ and $\partial\mathfrak{S}^{(i),\alpha}_{r,p,1}$;

(ii) the k_i-dimensional cells of type II are the sets $\mathfrak{S}^{(i),\alpha}_{r,p,2}$.

Condition (*g*). We retain only the cells that verify the conditions B-(ii) and B-(iii) from the definition of the admissible synthesis.

If such cells do not exist we stop : the algorithm does not work for our problem ; otherwise we pass to the next operation.

h) **III-2.** We define the sets $N, P^k, P^{k+1}, \ldots P^{n-1}, P^n$ of the admissible synthesis :

(i) N is the union of all the sets in the form $N^{(i),\alpha,(\mathrm{I}),\beta}_{(r,p,\gamma)(q,j,\delta)}$ defined in the operation II-9 ;

(ii) P^i is the union of all the cells of dimension less or equal to i, $i = k, k+1, \ldots n$, and we define the set $G' = \mathrm{int}\,(N \cup P^n)$ which is the new phase space of our problem in which the optimal synthesis exists.

We notice that in this operation the set P^n is not the same as in the definition of the admissible synthesis.

Condition *(h)*. We verify that the set N is a piecewise smooth set of dimension less than n and P^i is a piecewise smooth set of dimension i ($i = k, k+1, \dots n$). If it is not the case we stop : the algorithm does not apply for our problem ; otherwise we continue.

i) **III-3.** We verify the fact that every solution of the differential system :

$$\frac{dx}{dt} = f(x, v(x)) \tag{3.21}$$

reaches transversally and in a finite time the terminal manifold $\mathcal{T}$ and intersects only a finite number of cells.

We may retain only an open subset $G'' \subset G'$ such that the points in G'' have this property. If such a set G'' does not exists we stop : the algorithm does not work for our problem ; otherwise we continue.

j) **III-4.** We verify that the function $W : G'' \to R$ defined by the functions $W_{r,p}^{(i),\alpha}$ on every cell is a continuous one.

We may retain only an open subset G''' of G'' for which this condition is satisfied. If such a set G''' does not exists we stop : the algorithm does not work for our problem ; otherwise we continue.

III-5. The sets $N, P^k, P^{k+1}, \dots P^n$ and the map $v : G''' \to U$ represent an *optimal synthesis.* STOP.

REMARK 3.2

In certain cases we may apply the algorithm even if some of the numbers m_i, $m_{i,r}$, $n_{i,r}$ are not finite.

Indeed, if the fact that such a number is infinite is caused by the fact that G is non-bounded, then we may restrict our considerations to a bounded open subdomain $G' \subset G$ and we deal with finite numbers m_i, $m_{i,r}$, $n_{i,r}$.

In other cases there are some general formulae or some recurrence relations that allow to work with infinite numbers m_i, $m_{i,r}$.

To justify the statement of operation III-5 of the algorithm (that is the fact that the sets $N, P^k, .. P^{n-1}$ P^n anf the map $v : G''' \to U$ represent an optimal synthesis) we observe that every condition of the definition 2.1 of the admissible synthesis is implied by a corresponding condition in the algorithm. Hence we have obtained an admissible synthesis.

Moreover, from the operations I-1 and II-5 we deduce that the maps $v : G''' \to U$ and $\mathcal{H} : G''' \times U \times R \to R^n$ verify the condition :

$$\mathcal{H}(\varphi_x(t), u, \eta_{\lambda(x)}(t)) \geqslant \mathcal{H}(\varphi_{(x)}(t), v(\varphi_{(x)}(t), \eta_{\lambda(x)}(t)) = 0$$

for every $u \in U$, $t \in [0, t_F]$ where, $(\varphi_x(\cdot), \eta_{\lambda(x)}(\cdot))$ is the solution of the Hamiltonian

system (3.4) with the initial condition $(x, \lambda(x))$ and which satisfies the relations (3.3). From the theorem 2.2 in section 2 it follows that the obtained synthesis is optimal. Therefore we may state :

Theorem 3.1

If all the operations of the algorithm may be accomplished (that is we arrive to the operation III-5) then an optimal synthesis is obtained.

REMARK

Conversely, if there exists an optimal synthesis and if the conditions (*a*) and (*b*) are verified then the algorithm is working.

A proof of this statement will be given separately.

4. EXAMPLES

Example I : the forced pendulum ([9])

The control system is the following :

$$\begin{cases} \dfrac{d\theta}{dt} = z \\[2ex] \dfrac{dz}{dt} = -\sin\theta - \alpha z + u, \text{ where } |u| \leqslant 1, \alpha > 2, \end{cases}$$

The terminal manifold is $\mathcal{C} = \{ (2\pi n, 0) \mid n = 0, \pm 1, \pm 2, \dots \}$. The functional to minimize is defined by $g \equiv 0, f^0 \equiv 1$.

The step I

a) **I-1.** We define the map $\mathcal{H}(\theta, z, u, \lambda_1, \lambda_2) = 1 + \lambda_1 z + \lambda_2 (u - \sin\theta + \alpha z)$ and we find : $\min\limits_{|u| \leqslant 1} \mathcal{H}(\theta, z, u, \lambda_1, \lambda_2) = \bar{H}(\theta, z, \lambda_1, \lambda_2)$ where :

$$\bar{H}(\theta, z, \lambda_1, \lambda_2) = \begin{cases} 1 + \lambda_1 z + \lambda_2(1 - \sin\theta - \alpha z) & \text{for } \lambda_2 \leqslant 0 \\ 1 + \lambda_1 z - \lambda_2(1 + \sin\theta + \alpha z) & \text{for } \lambda_2 > 0 \end{cases}$$

b) **I-2.** We take

$$U_1^* = \{ (\theta, z, \lambda_1, \lambda_2) \mid \lambda_2 < 0 \}, U_2^* = \{ (\theta, z, \lambda_1, \lambda_2) \mid \lambda_2 > 0 \}$$

and we observe that the function $\bar{H}_j = \bar{H} \mid_{U_j^*}, j = 1, 2$, are of class C^1, hence the condition (*b*) is verified.

The step II

We take by definition $\mathcal{C}_0^{(0)} = \mathcal{C}$, $g_0^{(0)} \equiv g \equiv 0$, $k_0 = \dim(\mathcal{C}_0^{(0)}) = 0$ and we pass to

The routine $(\mathcal{C}_0^{(0)}, g_0^{(0)})$

II-1. The connected components of the terminal manifold $\mathcal{C}_0^{(0)}$ are the points $\mathcal{C}_0^{(0),n} = \{(2\pi n, 0)\}$, $n = \theta, \pm 1, \pm 2, \ldots$, which are 0-dimensional manifolds with the parametrisations :

$$(4.1) \qquad \begin{cases} \theta = \chi_0^{1,(0),n}(s) = 2\pi n \\ z = \chi_0^{2,(0),n}(s) = 0 \qquad \text{for} \qquad s = 0, n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

The local representative of the function $g_0^{(0)}$ corresponding to this parametrisation is :

$$(4.2) \qquad h_0^{(0),n}(s) = 0 \qquad \text{for} \qquad s = 0.$$

We note that we are in the situation from the remark 3.2 since the number $n_{0,0}$ of the connected components of $\mathcal{C}_0^{(0)}$ is infinite, but in this case $\mathcal{C}_0^{(0),n}$ are regularly disposed on the θ-axis. This regularity allows us to obtain the results for $\mathcal{C}_0^{(0)}$ translating the results for $\mathcal{C}_0^{(0),0}$ along the θ-axis. Therefore we shall effect the subroutine $(\mathcal{C}_0^{(0),0}, h_0^{(0),0})$ and translate its results with $2\pi n$ along the θ-axis and obtain the results of the subroutine $(\mathcal{C}_0^{(0),n}, h_0^{(0),n})$. Hence we take $\alpha = 0$ and pass to

The subroutine $(\mathcal{C}_0^{(0),0}, h_0^{(0),0})$

II-2. We define the sets $\mathcal{C}_{0,j}^{*(0),0}$, $j = 1, 2$:

For $j = 1$ the system (3.3) becomes :

$\tilde{H}_1(0, 0, \lambda_1^0, \lambda_2^0) = 1 + \lambda_2^0 = 0$ and hence $\lambda_2^0 = -1$. Therefore

$$\mathcal{C}_{0,1}^{*(0),0} = \{(0, 0, \lambda_1^0, -1) \mid \lambda_1^0 \in R\}$$

For $j = 2$ we have :

$\tilde{H}_2(0, 0, \lambda_1^0, \lambda_2^0) = 1 - \lambda_2^0 = 0$ and hence $\mathcal{C}_{0,2}^{*(0),0} = \{(0, 0, \lambda_1^0, 1) \mid \lambda_1^0 \in R\}$

II-3. We integrate the differential systems :

$$(4.3)_{(j)} \qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial \tilde{H}_j}{\partial \lambda}(x, \lambda) \\ \dfrac{d\lambda}{dt} = -\dfrac{\partial \tilde{H}_j}{\partial x}(x, \lambda) \end{cases}$$

where $x = (\theta, z)$ and $\lambda = (\lambda_1, \lambda_2)$, with the initial conditions from the sets $\mathcal{C}^{*(0),0}_{0,j}$.

For $j = 1$ we have :

$$(4.4) \qquad \begin{cases} \dfrac{d\theta}{dt} = z \\[2ex] \dfrac{dz}{dt} = -\alpha z - \sin\theta + 1 \\[2ex] \dfrac{d\lambda_1}{dt} = \lambda_2 \cos\theta \\[2ex] \dfrac{d\lambda_2}{dt} = -\lambda_1 + \alpha\lambda_2 \end{cases}$$

and the initial conditions : $\theta(0) = 0, z(0) = 0, \lambda_1(0) = \lambda_1^0 \in R, \lambda_2(0) = -1$. The first two equations do not depend on λ_1, λ_2 and in the right hand side verify the conditions for existence and uniqueness of the solutions on the whole plane (θ, z) ([7], [12]).

It follows that there exists a solution $(\theta^{(0),0}_{0,1}(\cdot), z^{(0),0}_{0,1}(\cdot))$ of the system :

$$(4.5) \qquad \begin{cases} \dfrac{d\theta}{dt} = z \\[2ex] \dfrac{dz}{dt} = -\sin\theta - \alpha z + 1 \end{cases}$$

which satisfies the conditions : $\theta^{(0),0}_{0,1}(0) = 0$, $z^{(0),0}_{0,1}(0) = 0$ and which is defined for all $t \in R$. It is easy to show that this solution is monotonically decreasing to $-\infty$ for $t \to -\infty$ and that it does not intersect the θ-axis for any point $t < 0$ (fig. 1).

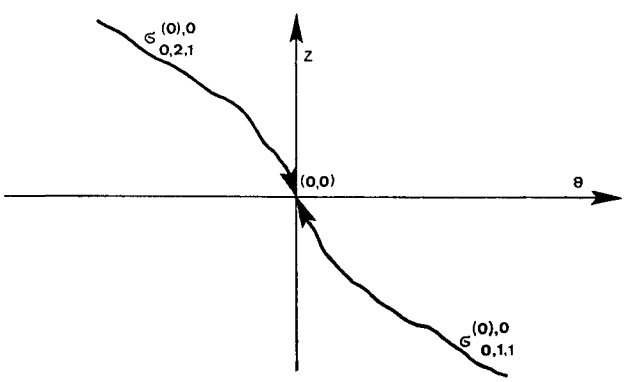

Figure 1

The other two equations of the system (4.4) form the linear differential system :

$$(4.6)\qquad \begin{cases} \dfrac{d\lambda_1}{dt} = \lambda_2 \cos \theta_{0,1}^{(0),0}(t) \\[2ex] \dfrac{d\lambda_2}{dt} = -\lambda_1 + \alpha\lambda_2 \end{cases}$$

with the initial conditions : $\lambda_1(0) = \lambda_1^0 \in R \quad , \quad \lambda_2(0) = -1.$

To define the sets $\mathfrak{S}_{0,1,\gamma}^{(0),0}$, $\gamma = 1, 2$, we need only the second component $\lambda_2(t)$ of the solution of this system. Hence we consider the second order differentiial equation :

$$\frac{d^2\lambda_2}{dt^2} - \alpha \frac{d\lambda_2}{dt} + \lambda_2 \cos \theta_{0,1}^{(0),0}(t) = 0$$

with the initial conditions : $\lambda_2(0) = -1$, $\dfrac{d\lambda_2}{dt}(0) = -\lambda_1^0 - \alpha$ which is eaquivalent with the system (4.6).

To find the set of all $t < 0$ for which the solution

$$(\theta_{0,1}^{(0),0}(t), z_{0,1}^{(0),0}(t), \lambda_1(t\,;\lambda_1^0), \lambda_2(t\,;\lambda_1^0))$$

belong to the set $\bar{U}_1^* = \{ (\theta, z, \lambda_1, \lambda_2) \mid \lambda_2 < 0 \}$ we use the change ([9]) :

$$(4.8)\qquad \lambda_2(t) = e^{\frac{\alpha t}{2}} \psi(t)$$

and obtain the differential equations :

$$(4.9)\qquad \frac{d^2\psi}{dt^2} + \left(\cos\theta_{0,1}^{(0),0}(t) - \frac{\alpha_2}{4}\right)\psi = 0$$

Since we have $\alpha > 2$ it follows that $\cos \theta_{0,1}^{(0),0}(t) - \dfrac{\alpha^2}{4} < 0$ and if we apply a comparison theorem ([7] chap. VIII) we deduce that any solution of the equation (4.9) changes the sign at most once. From (4.8) it follows that any solution $\lambda_2(t\,;\lambda_1^0)$ changes the sign also at most once. Now it is easy to show that there exists a $\lambda_1^0 \in R$ such that $\lambda_2(t'\,;\lambda_1^0) < 0$ for any $t' < 0$, that is we have :

$$\mathfrak{S}_{0,1,1}^{(0),0} = \{ (\theta_{0,1}^{(0),0}(t), z_{0,1}^{(0),0}(t)) \mid t < 0 \}, \ \mathfrak{S}_{0,1,2}^{(0),0} = \Phi.$$

For $j = 2$ we must integrate the differential system :

$$(4.10)\qquad \begin{cases} \dfrac{d\theta}{dt} = z \\[2ex] \dfrac{dz}{dt} = -\sin\theta - \alpha z - 1 \\[2ex] \dfrac{d\lambda_1}{dt} = \lambda_2 \cos\theta \\[2ex] \dfrac{d\lambda_2}{dt} = -\lambda_1 + \alpha\lambda_2 \end{cases}$$

with the initial conditions : $\theta(0) = 0$, $z(0) = 0$, $\lambda_1(0) = \lambda_1^0$, $\lambda_2(0) = 1$. In the same manner as above we obtain :

$\mathfrak{S}_{0,2,1}^{(0),0} = \{ (\theta_{0,2}^{(0),0}(t), z_{0,2}^{(0),0}(t)) \mid t < 0 \}$, $\mathfrak{S}_{0,2,2}^{(0),0} = \Phi$ where $(\theta_{0,2}^{(0),0}(t), z_{0,2}^{(0),0}(t))$ is the solution of the differential system :

$$(4.11)\qquad \begin{cases} \dfrac{d\theta}{dt} = z \\[2ex] \dfrac{dz}{dt} = -\sin\theta - \alpha z - 1 \end{cases}$$

which satisfies the initial conditions $\theta_{0,2}^{(0),0}(0) = 0$, $z_{0,2}^{(0),0}(0) = 0$

c) **II-4.** It is clear that $\mathfrak{S}_{0,j,1}^{(0),0}, j = 1, 2$ are $k_0 + 1 = 1$-dimensional manifolds without boundary (hence $\partial\mathfrak{S}_{0,j,1}^{(0),0} = \Phi$, $j = 1, 2$) and the condition (*c*) is verified.

d) **II-5.** Since for every point $(\theta, z, \lambda_1, \lambda_2) \in \bar{U}_1^*$ we have $\bar{u}(\theta, z, \lambda_1, \lambda_2) = 1$ and for every $(\theta, z, \lambda_1, \lambda_2) \in \bar{U}_2^*$ we have $\bar{u}(\theta, z, \lambda_1, \lambda_2) = -1$, we define the functions : $v_{0,j}^{(0),0} : \mathfrak{S}_{0,j,1}^{(0),0} \to [-1, 1], j = 1, 2$, as follows :

$$v_{0,1}^{(0),0}(\theta, z) = 1 \quad , \quad v_{0,2}^{(0),0}(\theta, z) = -1$$

and hence the condition (*d*) is verified.

e) **II-6.** Let us consider the point $(\theta, z) \in \mathfrak{S}_{0,j,1}^{(0),0}$ such that :

$$(4.12)_{(j)}\qquad \theta = \theta_{0,j}^{(0),0}(t_1^j) \quad , \quad z = z_{0,j}^{(0),0}(t_1^j) \quad , \quad t_1^j < 0 (j = 1, 2)$$

We define now the function $\varphi_{0,j}^{(0),0}(\cdot\,; (\theta, z)) : [0, -t_1^j] \to \mathfrak{S}_{0,j,1}^{(0),0}$:

$$(4.13)_{(j)}\qquad \varphi_{0,j}^{(0),0}(t\,; (\theta, z)) = (\theta_{0,j}^{(0),0}(t + t_1), z_{0,j}^{(0),0}(t + t_1)) \text{ for } t \in [0, -t_1]$$

Since the condition from the remark 3.1 is satisfied the condition (*e*) is also satisfied.

f) **II-7.** We define the maps $W_{0,j}^{(0),0}, j = 1, 2 : W_{0,j}^{(0),0}(\theta, z) = -t_1^j$ where $t_1^j < 0$ is given by $(4.12)_{(j)}$.

For every $z \neq 0$ the first relation $(4.12)_{(j)}$ defines $t_1^j < 0$ as an implicite function of θ and hence $W_{0,j}^{(0),0}(j = 1, 2)$ are C^1-functions on $\mathfrak{S}_{0,j,1}^{(0),0}$. The condition (*f*) is verified.

II-8. Instead of change α by $\alpha + 1$ (or by $\alpha - 1$) and pass to the same subroutine with the new data we use the remark from the operation II-1 and deduce that the results of any subroutine $(\mathcal{C}_0^{(0),n}, h_0^{(0),n})(n = \pm 1, \pm 2, \ldots)$ may be obtained by translating corresponding results of the subroutine $(\mathcal{C}_0^{(0),0}, h_0^{(0),0})$ with $2\pi n$ along the θ-axis.

Hence we obtain :

(i) the solutions $(\theta_{0,j}^{(0),n}(\cdot), z_{0,j}^{(0),n}(\cdot))$ of the differential system (4.5) for $j = 1$, and of the system (4.11) if $j = 2$, with the initial conditions

$$\theta_{0,j}^{(0),n}(0) = 2\pi n, z_{0,j}^{(0),n}(0) = 0, j = 1, 2 ;$$

(ii) the sets $\mathfrak{S}_{0,j,1}^{(0),n} = \{ (\theta_{0,j}^{(0),n}(t), z_{0,j}^{(0),n}(t)) \mid t < 0 \}$, $\mathfrak{S}_{0,j,2}^{(0),n} = \Phi$ and $\mathfrak{S}_{0,j,1}^{(0),n}$ are differentiable manifolds of dimension $k_0 + 1 = 1$;

(iii) the maps $v_{0,j}^{(0),n} : \mathfrak{S}_{0,j,1}^{(0),n} \to [-1, 1], j = 1, 2$, defined by :

$$v_{0,1}^{(0),n} \equiv 1 \quad , \quad v_{0,2}^{(0),n} \equiv -1 ;$$

(iv) the functions $W_{0,j}^{(0),n} : \mathfrak{S}_{0,j}^{(0),n} \to R, j = 1, 2$, given by :

$W_{0,j}^{(0),n}(\theta, z) = -t_1^{j,n}$ where $t_1^{j,n}$ satisfies the relations :

$\theta = \theta_{0,j}^{(0),n}(t_1^{j,n}) \quad , \quad z = z_{0,j}^{(0),n}(t_1^{j,n})$ for $j = 1, 2$ respectively.

We are now in the situation $\alpha = n_{0,0}$ (we have considered all the connected components of the terminal manifold $\mathcal{C}_0^{(0)}$) and we pass to the next operation.

II-9. For every two sets we have $\mathfrak{S}_{r,p,\gamma}^{(i),\alpha} \cap \mathfrak{S}_{q,j,\delta}^{(l),\beta} = \Phi$ and hence we must pass to the operation II-11.

II-11. We have $\mathcal{C}_1^{(0)} = \bigcup_{n=-\infty}^{+\infty} \bigcup_{j=2}^{2} (\mathfrak{S}_{0,j,2}^{(0),n} \cup \partial\mathfrak{S}_{0,j,2}^{(0),n}) = \Phi$ and $k_0 + 1 < 2 =$ and hence we must define $\mathcal{C}_0^{(1)} = \bigcup_{n=\infty}^{+\infty} \bigcup_{j=1}^{2} \mathfrak{S}_{0,j,1}^{(0),n}$ and the function

$$g_0^{(1)} : \mathcal{C}_0^{(1)} \to R : g_0^{(1)}(\theta, z) = W_{0,j}^{(0),n}(\theta, z) \text{ if } (\theta, z) \in \mathfrak{S}_{0,j,1}^{(0),n}.$$

We pass now to

The routine $(\mathcal{C}_0^{(1)}, g_0^{(1)})$

II-1. The new terminal manifold $\mathcal{C}_0^{(1)}$ has also infinitely many connected components $\mathcal{C}_0^{(1),(n,j)} = \mathfrak{S}_{0,j,1}^{(0),n}, n = 0, \pm 1, \pm 2, \ldots j = 1, 2$. We denote

$g_0^{(1),(n,j)} = g_0^{(1)} \mid \mathcal{C}_0^{(1),(n,j)}$ and we take the parametrisations :

$$(4.16)_{(n,j)} \qquad \begin{cases} \theta = \chi_0^{1,(1),(n,j)}(s) = \theta_{0,j}^{(0),n}(s) \\ z = \chi_0^{2,(1),(n,j)}(s) = z_{0,j}^{(0),n}(s),\ s < 0 \end{cases}$$

The local representative of the function $g_0^{(1),(n,j)}$ is :

$h_0^{(1),(n,j)}(s) = -s$ for $s < 0$. We take $\alpha = (0, 1)$ and we pass to.

The subroutine $(\mathcal{C}_{0,j}^{(1),(0,1)}, h_0^{(1),(0,1)})$

II-2. To define the sets $\mathcal{C}_{0,j}^{*(1),(0,1)}, j = 1, 2$ we must solve the linear algebraic system :

$$(4.17)_{(j)} \qquad \begin{cases} \lambda_1^0 \dfrac{d\theta_{0,1}^{(0),0}}{ds}(s) + \lambda_2^0 \dfrac{dz_{0,1}^{(0),0}}{ds}(s) = \dfrac{dh_0^{(0),(0,1)}}{ds}(s) \\ \tilde{H}_j(\theta_{0,1}^{(0),0}(s), z_{0,1}^{(0),0}(s), \lambda_1^0, \lambda_2^0) = 0 \end{cases}$$

For $j = 1$ we have :

$$\begin{cases} \lambda_1^0 z_{0,1}^{(0),0}(s) + \lambda_2^0[-\sin\theta_{0,1}^{(0),0}(s) - \alpha z_{0,1}^{(0),0}(s) + 1] = -1 \\ 1 + \lambda_1^0 z_{0,1}^{(0),0}(s) + \lambda_2^0[-\sin\theta_{0,1}^{(0),0}(s) - \alpha z_{0,1}^{(0),0}(s) + 1] = 0 \end{cases}$$

where the condition of transversal intersection is not verified (the rank of the matrix of this system with respect to λ_1^0, λ_2^0 is equal to $1 < k_1 + 1 = 2$).

For $j = 2$ the system (4.17) becomes :

$$\begin{cases} \lambda_1^0 z_{0,1}^{(0),0}(s) + \lambda_2^0(-\sin\theta_{0,1}^{(0),0}(s) - \alpha z_{0,1}^{(0),0}(s) + 1) = -1 \\ 1 + \lambda_1^0 z_{0,1}^{(0),0}(s) + \lambda_2^0(-\sin\theta_{0,1}^{(0),0}(s) - \alpha z_{0,1}^{(0),0}(s) - 1) = 0 \end{cases}$$

and hence $\lambda_2^0(s) = 0$, $\lambda_1^0(s) = -\dfrac{1}{z_{0,1}^{(0),0}(s)}$. It follows that :

$$\mathcal{C}_{0,2}^{*(1),(0,1)} = \left\{ \left(\theta_{0,1}^{(0),0}(s), z_{0,1}^{(0),0}(s), -\frac{1}{z_{0,1}^{(0),0}(s)}, 0 \right) \middle| \ s < 0 \right\}$$

II-3. We must integrate the differential system (4.10) with the initial conditions from the set $\mathcal{C}_{0,2}^{*(1),(0,1)}$. Then for every $s < 0$ we obtain the solution $(\theta_{0,2}^{(1),(0,1)}(\cdot\,; s), z_{0,2}^{(1),(0,1)}(\cdot\,; s))$ of the system (4.11) defined for all $t \in R$ and which satisfies the conditions :

$$(4.18)_{(0,1)} \qquad \begin{cases} \theta_{0,2}^{(1),(0,1)}(0\,; s) = \theta_{0,1}^{(0),0}(s) \\ z_{0,2}^{(1),(0,1)}(0\,; s) = z_{0,1}^{(0),0}(s) \end{cases}$$

If we integrate now the other two equations of the system (4.10) with the initial conditions $\lambda_1(0) = -\frac{1}{z_{0,1}^{(0),0}(s)}$, $\lambda_2(0) = 0$ then we obtain the solution $(\lambda_1(\cdot\,;s), \lambda_2(\cdot\,;s))$ for which we have $\lambda_2(t\,;s) > 0$ for all $t < 0$, $s < 0$ (we apply again the comparison theorem from [7]).

Therefore we have :

$$\mathfrak{S}_{0,2,1}^{(1),(0,1)} = \{ (\theta_{0,2}^{(1),(0,1)}(t\,;s), z_{0,2}^{(1),(0,1)}(t\,;s)) \mid t < 0, s < 0 \} \; \mathfrak{S}_{0,2,2}^{(1),(0,1)} = \Phi$$

c) **II-4.** Since the set $\mathfrak{S}_{0,2,1}^{(1),(0,1)}$ is a 2-dimensional manifold without boundary the condition (*c*) is verified.

d) **II-5.** For every $(\theta, z) \in \mathfrak{S}_{0,2,1}^{(1),(0,1)}$ we have $v_{0,2}^{(1),(0,1)}(\theta, z) = -1$ and the condition (*d*) is verified.

e) **II-6.** Let us consider the point $(\theta, z) \in \mathfrak{S}_{0,2,1}^{(1),(0,1)}$ and $t_1, s_1 < 0$ such that

$$\theta = \theta_{0,2}^{(1),(0,1)}(t_1\,;s_1), \; z = z_{0,2}^{(1),(0,1)}(t_1\,;s_1) \tag{4.19}$$

We define now :

$$\varphi_{0,2}^{(1),(0,1)}(t\,;(\theta, \tau)) = (\theta_{0,2}^{(1),(0,1)}(t + t_1\,;s_1), z_{0,2}^{(1),(0,1)}(t + t_1\,;s_1))$$

for $t \in [0, -t_1]$. As in the preceeding routine the condition (*e*) is verified.

f) **II-7.** Fort every $(\theta, z) \in \mathfrak{S}_{0,2,1}^{(1),(0,1)}$ we define :

$$W_{0,2}^{(1),(0,1)}(\theta, z) = h_0^{(1),(0,1)}(s_1) + \int_0^{-t_1} dt = -s_1 - t_1 \tag{4.20}$$

where $s_1 < 0$, $t_1 < 0$ are given by (4.19). From the implicite functions theorem it follows that $W_{0,2}^{(1),(0,1)}$ is a C^1-function on $\mathfrak{S}_{0,2,1}^{(1),(0,1)}$.

II-8. We take $\alpha = (0, 2)$ and pass to

The subroutine $(\mathcal{C}_0^{(1),(0,2)}, h_0^{(1),(0,2)})$

Since $\mathcal{C}_0^{(1),(0,2)}$ and $\mathcal{C}_0^{(1),(0,1)}$ are symmetrics, the results of this subroutine may be writen directly from the results of the preceeding subroutine changing the system (4.10) with the system (4.4).

As in the preceeding routine we apply the remark 3.2 and deduce that the results of any subroutine $(\mathcal{C}_0^{(1),(n,j)}, h_0^{(1),(n,j)})$, $j = 1, 2$, $n = \pm 1, \pm 2, \ldots$ may be written from the corresponding results of the subroutines $(\mathcal{C}_0^{(1),(0,j)}, h_0^{(1),(0,j)})$, $j = 1, 2$ translating with $2\pi n$ along the θ-axis (fig. 2).

We have considered all the connected components of the manifold $\mathcal{C}_0^{(1)}$ and hence we may pass to the next operation to continue the routine $(\mathcal{C}_0^{(1)}, g_0^{(1)})$.

II-9. We observe that for every $n = 0, \pm 1, \pm 2, \ldots$ the following intersections are nonempty (fig. 2) :

(i) $$\mathfrak{S}^{(1),(n,2)}_{0,1,1} \cap \mathfrak{S}^{(1),(n+k,2)}_{0,1,1} \neq \Phi,\ k = 1, 2, \ldots$$

(ii) $$\mathfrak{S}^{(1),(n,1)}_{0,2,1} \cap \mathfrak{S}^{(1),(n+k,1)}_{0,2,1} \neq \Phi,\ k = -1, -2, \ldots$$

(iii) $$\mathfrak{S}^{(1),(n,2)}_{0,1,1} \cap \mathfrak{S}^{(1),(n+k,1)}_{0,2,1} \neq \Phi,\ k = 1, 2, \ldots$$

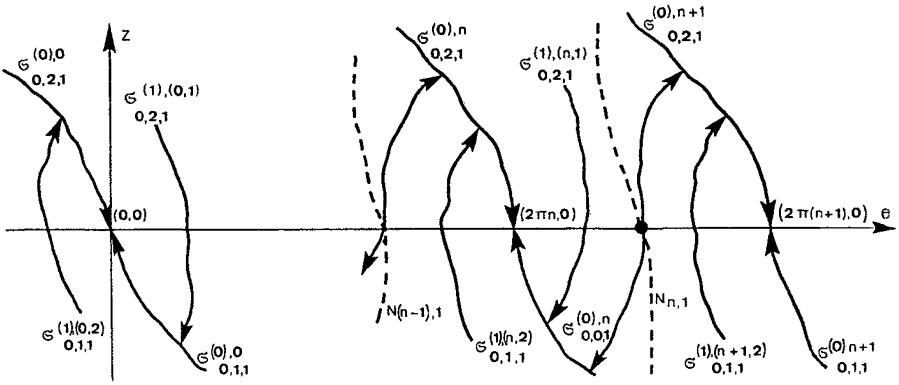

Figure 2

Therefore we must define the new sets $\mathfrak{S}$ and the sets of type N. It is easy to show that for every $n = 0, \pm 1, \pm 2, \ldots$ we have

$$\mathfrak{S}^{(1),(n+k,2)}_{0,1,1} \supset \mathfrak{S}^{(1),(n,2)}_{0,1,1} \text{ and } W^{(1),n+k,2)}_{0,1}(\theta, z) > W^{(1),(n,2)}_{0,1}(\theta, z)$$

for all $(\theta, z) \in \mathfrak{S}^{(1),(n,2)}_{0,1,1}$ and $k = 1, 2, \ldots$

It follows that $N^{(1),(n,2),(1),(n+k,2)}_{(0,1,1),(0,1,1)} = \Phi$, $\mathfrak{S}^{(1),(n+k,2),(1),(n,2)}_{(0,1,1),(0,1,1)} = \mathfrak{S}^{(1),(n+k,2)}_{0,1,1} \setminus \mathfrak{S}^{(1),(k,2)}_{0,1,1}$ and $\mathfrak{S}^{(1),(n,2),(1),(n+k,2)}_{(0,1,1),(0,1,1)} = \mathfrak{S}^{(1),(n,2)}_{0,1,1}$ for every $k = 1, 2, \ldots$

Similarly, for the intersections (ii) we have :

$$N^{(1),(n,1),(1),(n+k,2)}_{(0,2,1),(0,2,1)} = \Phi \quad \mathfrak{S}^{(1),(n+k,1),(1),(n,1)}_{(0,2,1),(0,2,1)} = \mathfrak{S}^{(1),(n+k,1)}_{0,2,1} \setminus \mathfrak{S}^{(1),(n,2)}_{0,2,1}$$

$$\mathfrak{S}^{(1),(n,1),(2),(n+k,1)}_{(0,2,1),(0,2,1)} = \mathfrak{S}^{(1),(n,1)}_{0,2,1},\ k = -1, -2, \ldots$$

However, the sets

$$N_{n,k} = N^{(1),(n,2),(1),(n+k,1)}_{(0,1,1)(0,2,1)},\ \mathfrak{S}^{(1),(n,2),(1),(n+k,1)}_{(0,1,1),(0,2,1)},\ \mathfrak{S}^{(1),(n+k,1),(1),(n,2)}_{(0,2,1),(0,1,1)}$$

from the intersections (iii) are nonempty for $k = 1$ and empty for $k = 2, 3 \ldots$

II-10. In this case it is not necessary to change the indeces of the new sets $\mathfrak{S}$ obtained in the operation II-9. We may change only the definitions of the sets $\mathfrak{S}^{(1),(n,2)}_{0,1,1}$, $\mathfrak{S}^{(1),(n,1)}_{0,2,1}$ that is we take :

$$\mathfrak{S}^{(1),(n,2)}_{0,1,1} = \mathfrak{S}^{(1),(n,2),(1),(n+1,1)}_{(0,1,1)(0,2,1)} \quad , \quad \mathfrak{S}^{(1),(n+1,1)}_{0,2,1} = \mathfrak{S}^{(1),(n+2,1),(1),(n,2)}_{(0,2,1),(0,1,1)}$$

II-11. We have $\mathcal{T}_1^{(1)} = \bigcup_{n=-\infty}^{+\infty} \bigcup_{(i,j)=(1,2),(2,1)} (\mathfrak{S}_{0,j,2}^{(1),(n,j)} \cup \partial\mathfrak{S}_{0,j,1}^{(1),(n,i)}) = \Phi$ and $k_1 + 1 = 2 = n$ and hence we must pass to the step III.

The step III

g) **III-1.** We define the cells of the optimal synthesis :

(1) the 0-dimensional cells (of type II) are :

$$\mathcal{C}_n^{(0)} = \{ (2\pi n, 0) \}, n = 0, \pm 1, \pm 2 \ldots$$

(the connected components of the terminal manifold $\mathcal{T}_0^{(0)}$)

(2) the 1-dimensional cells of type I are: $\mathcal{C}_{n,j}^{(1),1} = \mathfrak{S}_{0,j,1}^{(0),n}$ for $j = 1, 2$ and $n = 0, \pm 1, \pm 2, \ldots$;

(3) there not exist 1-dimensional cells of type II ;

(4) the 2-dimensional cells (of type I) are the following :

$$\mathcal{C}_{n,1}^{(2)} = \mathfrak{S}_{0,1,1}^{(1),(n,2)}, \mathcal{C}_{n,2}^{(2)} = \mathfrak{S}_{0,2,1}^{(1),(n,2)}$$

for $n = 0, \pm 1, \pm 2, \ldots$

It is obvious that all the above defined cells verify the conditions B-(ii) and B-(iii) of the definition of the admissible synthesis.

h) **III-2.** We define the sets N, P^0, P^1, P^2 :

$$N = \bigcup_{n=-\infty}^{+\infty} N_{n,1} \quad , \quad P^0 = \bigcup_{n=-\infty}^{+\infty} \mathcal{C}_n^{(0)} \quad , \quad P^1 = P^0 \cup \left(\bigcup_{n=-\infty}^{+\infty} \bigcup_{j=1}^{2} \mathcal{C}_{n,j}^{(1),1} \right)$$

$$P^2 = P^1 \cup \left(\bigcup_{n=-\infty}^{+\infty} \bigcup_{j=1}^{2} \mathcal{C}_{n,j}^{(2)} \right)$$

We note that $P^2 \cup N = R^2$ and hence $G' = G = R^2$ is the phase space of our control problem.

Since every smooth surface of an Euclidian space is α piecewise smooth set ([4]) it follows that the sets N, P^0, P^1, P^2 are piecewise smooth sets and the condition (*h*) is verified.

i) **III-3.** It is clear that every solution of the system :

$$\begin{cases} \dfrac{d\theta}{dt} = z \\ \\ \dfrac{dz}{dt} = -\sin\theta - \alpha z + v(\theta, z) \end{cases}$$

where the map $v : R^2 \to [-1, 1]$ is given by :

$$v(\theta, z) = \begin{cases} +1 \text{ for } (\theta, z) \in \bigcup_{n=-\infty}^{+\infty} (C_n^{(0)} \cup C_{n,1}^{(1),1} \cup C_{n,1}^{(2)}) \\ -1 \text{ for } (\theta, z) \in \bigcup_{n=-\infty}^{+\infty} (C_{n,2}^{(n),1} \cup C_{n,2}^{(2)}) \end{cases}$$

reaches the terminal manifold $\mathcal{T} = \{ (2\pi n, 0) \mid n = 0, \pm 1, \pm 2, \dots \}$ transversally, in a finite time and intersects only a finite number of cells (at most two cells).

j) **III-4.** It is easy to show using the definitions of the functions $W_{0,j}^{(0),n}$, $W_{0,j}^{(1),(n,j)}$ that the function $W : R^2 \to R$ which represent the value of the synthesis v defined above is continuous on R^2.

III-5. The sets N, P^0, P^1, P^2 and the map v above defined represent an admissible synthesis. STOP.

Example II ([6]).

We apply the algorithm to the time optimal control problem studied by Boltyanskii in [6] as an interesting example in which the phase space that the optimal synthesis exists does not coincides with the domain of controllability.

The control problem is the following :

$$(4.21) \quad \begin{cases} \dfrac{dx^1}{dt} = x^2 \\ \dfrac{dx^2}{dt} = \dfrac{1}{2} u \cdot e^{(x^2)^2} \quad , \quad u \in U = [-1, 1] \quad , \quad f^0 \equiv 1, g \equiv 0 \end{cases}$$

and the terminal manifold is the point $\mathcal{T} = \{ (0, 0) \}$.

The step I

a) **I-1.** We define the map $\mathcal{H} : R^2 \times U \times R^2 \to R$:

$$\mathcal{H}(x^1, x^2, u, \lambda_1, \lambda_2) = 1 + \lambda_1 x^2 + \lambda_2 \cdot \frac{1}{2} u \cdot e^{(x^2)^2}$$

and we find :

$$\min_{|u| \leqslant 1} \mathcal{H}(x^1, x^2, u, \lambda_1, \lambda_2) = \bar{H}(x^1, x^2, \lambda_1, \lambda_2) \text{ where}$$

$$(4.22) \quad \bar{H}(x^1, x^2, \lambda_1, \lambda_2) = \begin{cases} 1 + \lambda_1 x^2 + \lambda_2 \dfrac{1}{2} e^{(x^2)^2} & \text{for} \quad \lambda_2 \leqslant 0 \\ 1 + \lambda_1 x^2 - \lambda_2 \dfrac{1}{2} e^{(x^2)^2} & \text{for} \quad \lambda_2 > 0 \end{cases}$$

b) **I-2.** If we denote

$$U_1^* = \{ (x_1^1, x^2, \lambda_1, \lambda_2) \in R^4 \mid \lambda_2 < 0 \},$$

$$U_2^* = \{ (x^1, x^2, \lambda_1, \lambda_2) \in R^4 \mid \lambda_2 > 0 \}, \bar{H}_j = \bar{H} \mid U_j^*, j = 1, 2,$$

then it is clear that $\bar{H}_j, j = 1, 2$ are C^1-functions and hence the condition (*b*) is verified.

The step II

We denote $\mathcal{C}_0^{(0)} = \mathcal{C}, g_0^{(0)} \equiv g \equiv 0, k_0 = \dim (\mathcal{C}_0^{(0)}) = 0$ and we pass to

The routine $(\mathcal{C}_0^{(0)}, g_0^{(0)})$

II-1. The terminal manifold $\mathcal{C}_0^{(0)} = \mathcal{C} = \{ (0, 0) \}$ has only one connected component $\mathcal{C}_0^{(0)} = \{ (0, 0) \}$ (hence $n_{0,0} = 1$) which is a 0-dimensional manifold with the parametrisation :

$$\begin{cases} x^1 = \chi_0^{1,(0),1}(s) = 0 \\ x^2 = \chi_0^{2,(0),1}(s) = 0 \qquad \text{for} \qquad s = 0. \end{cases}$$

Corresponding to this parametrisation we have :

$$h_0^{(0),1}(s) = 0 \qquad \text{for} \qquad s = 0.$$

We take $\alpha = 1$ and pass to

The subroutine $(\mathcal{C}_0^{(0),1}, h_0^{(0),1})$

II-2. To define the set $\mathcal{C}_{0,1}^{*(0),1}$, we must solve the equation

$$\tilde{H}_1(0, 0, \lambda_1^0, \lambda_2^0) = 1 + \lambda_2^0 = 0$$

and we obtain :

$$\mathcal{C}_{0,1}^{*(0),1} = \{ (0, 0, \lambda_1^0, -1) \mid \lambda_1^0 \in R \}$$

In the same way we obtain :

$$\mathcal{C}_{0,1}^{*(0),1} = \{ (0, 0, \lambda_1^0, 1) \mid \lambda_1^0 \in R \}$$

II-3. For $j = 1$ the Hamiltonian system $(3.5)_{(j)}$ is :

$$(4.23)\qquad \begin{cases} \dfrac{dx^1}{dt} = x^2 \\[2mm] \dfrac{dx^2}{dt} = \dfrac{1}{2}\, e^{(x^2)^2} \\[2mm] \dfrac{d\lambda_1}{dt} = 0 \\[2mm] \dfrac{d\lambda_1}{dt} = -\lambda_1 - x^2\, e^{(x2)^2}\, \lambda_2 \end{cases}$$

with the initial conditions $x^1(0) = 0,\ x^2(0) = 0, \lambda_1(0) = \lambda_1{}^0 \in R, \lambda_2(0) = -1$. We integrate the first two equations of this system and we obtain the solution $(x_{0,1}^{1,(0),1}(\cdot) x_{0,1}^{2,(0),1}(\cdot))$ where $x_{0,1}^{2,(0),1}(t)$ is given by

$$(4.24)\qquad t = 2\int_0^{x^2} e^{-s^2}\, ds \quad \text{and} \quad x_{0,1}^{1,(0),1}(t) \quad \text{by :}$$

$$(4.25)\qquad x_{0,1}^{1,(0),1}(t) = \int_0^t x_{0,1}^{2,(0),1}(s)\, ds$$

We note that the curve $(x_{0,1}^{1,(0),1}(\cdot), x_{0,1}^{2,(0),2}(\cdot))$ may be written in the implicite form :

$$(4.26)\qquad x^1 = 1 - e^{-(x^2)^2}$$

From the third equation of the system (4.23) we obtain: $\lambda_1(t\,;\lambda_1^0) = \lambda_1^0$ for $t \in R$ and from the last equation :

$$(4.27)\qquad \lambda_2(t\,;\lambda_1^0) = -(1 + 2\lambda_1^0 x_{0,1}^{2,(0),1}(t))\, e^{-(x^{2,(0),1}(t))^2}$$

We must define now $\tau_{0,1}^{(0),1}(\lambda_1^0) < 0$ such that for $t \in [\tau_{0,1}^{(0),1}(\lambda_1^0), 0]$ we have

$$(x_{0,1}^{1,(0),1}(t), x_{0,1}^{2,(0),1}(t), \lambda_1^0, \lambda_2^0(t\,;\lambda_1^0)) \in \bar{U}_1^*$$

From (4.23) it follows that $x_{0,1}^{2,(0),1}$ is an increasing function and hence for every $\lambda_1^0 \in R$ there exists at most one $t_1 \in R$ such that $\lambda_2(t_1\,;\lambda_1^0) = 0$, namely if $\lambda_1^0 \geqslant 0$ then we have $\lambda_2(t\,;\lambda_1^0) < 0$ for all $t \in R$ but for $\lambda_1^0 < 0$, $\lambda_2(t\,;\lambda_1^0)$ chages the sign at

$$t_1 = 2\int_0^{-\frac{1}{2\lambda_1^0}} e^{-s^2} ds \geqslant 0$$

It follows that

$$\tau_{0,1}^{(0),1}(\lambda_1^0) = \begin{cases} -\infty & \text{if } \lambda_1^0 \geqslant 0 \\ 0 & \text{if } \lambda_1^0 < 0 \end{cases}$$

and hence

$$\mathfrak{S}^{(0),1}_{0,1,1} = \{(x^{1,(0),1}_{0,1}(t), x^{2,(0),1}_{0,1}(t)) \mid t < 0\}, \mathfrak{S}^{(0),1}_{0,1,2} = \Phi$$

Similarly, for $j = 2$, we obtain :

$$\mathfrak{S}^{(0),1}_{0,2,1} = \{(x^{1,(0),1}_{0,2}(t), x^{2,(0),1}_{0,2}(t)) \mid t < 0\}, \mathfrak{S}^{(0),1}_{0,2,2} = \Phi$$

where $x^{2,(0),1}_{0,2}(t)$ is given by the relation :

$$t = -2\int_0^{x^2} e^{-s^2} ds \tag{4.28}$$

and $x^{1,(0),1}_{0,2}(t)$ by :

$$x^{1,(0),1}_{0,2}(t) = \int_0^t x^{2,(0),1}_{0,2}(s)ds \tag{4.29}$$

As for $j = 1$, the curve $(x^{1,(0),1}_{0,2}(\cdot), x^{2,(0),1}_{0,2}(\cdot))$ may be written in the implicite form :

$$x^1 = e^{-(x^2)^2} - 1 \tag{4.30}$$

c) **II-4.** From (4.26) and (4.30) it follows that $\mathfrak{S}^{(0),1}_{0,j,1}, j = 1, 2$ are $k_0 + 1 = 1$-dimensional manifolds without boundary.

d) **II-5.** We define :

$v^{(0),1}_{0,1}(x^1, x^2) = 1$ for $(x^1, x^2) \in \mathfrak{S}^{(0),1}_{0,1,1}$,

$v^{(0),1}_{0,2}(x^1, x^2) = -1$ for $(x^1, x^2) \in \mathfrak{S}^{(0),1}_{0,2,1}$

and we see that $v^{(0),1}_{0,j} : \mathfrak{S}^{(0),1}_{0,j,1} \to \mathrm{U}, j = 1, 2$ are C^1-functions which may be extended to C^1-functions on some open neighborhoods of the closures $\overline{\mathfrak{S}^{(0),1}_{0,1,1}}$, $\overline{\mathfrak{S}^{(0),1}_{0,2,1}}$ respectively.

e) **II-6.** Let us consider the point $(x^1, x^2) \in \mathfrak{S}^{(0),1}_{0,j,1}$ and $t^j_1 < 0$ such that :

$$x^i = x^{i,(0),1}_{0,j}(t^j_1), i = 1, 2, \text{ for every } j = 1, 2. \tag{4.31$_{(j)}$}$$

Then, for every $j = 1, 2$, the map $\varphi^{(0),1}_{0,j}(\cdot\,; (x^1, x^2)) : [0_1 - t^j_1] \to \mathfrak{S}^{(0),1}_{0,j,1}$, given by

$$\varphi^{(0),1}_{0,j}(t\,; (x^1, x^2)) = (x^{1,(0),1}_{0,j}(t + t^j_1), x^{2,(0),1}_{0,j}(t + t^j_1))$$

is the solution of the differential system :

$$\begin{cases} \dfrac{dx^1}{dt} = x^2 \\[2ex] \dfrac{dx^2}{dt} = \dfrac{1}{2} v^{(0),1}_{0,j}(x^1, x^2)\, e^{(x^2)^2} \end{cases}$$

with the initial condition $\varphi^{(0),1}_{0,j}(0\,; (x^1, x^2)) = (x^1, x^2), j = 1, 2,$

We note that the condition from the Remark 3.1 is satisfied and hence the condition (*e*) is automatically satisfied.

f) **II-7.** We define the maps $W_{0,j}^{(0),1} : \mathfrak{S}_{0,j}^{(0),1} \to R, j = 1, 2$, by :

$$W_{0,j}^{(0),1}(x^1, x^2) = \int_0^{-t_1} \mathrm{d}t = -t_1^j \tag{4.32}$$

where $t_1^j < 0$ is given by $(4.31)_{(j)}$. Therefore $W_{0,j}^{(0),1} j = 1, 2$ are C^1-functions and the condition (*f*) is verified.

The subroutine $(\mathcal{C}_0^{(0)}, h_0^{(0),1})$ is finished and we continue the routine $(\mathcal{C}_0^{(0)}, g_0^{(0)})$.

II-8. We have $\alpha = 1 = n_{0,0}$, and hence we must pass to the next operation to continue the same routine.

II-9. We have $\mathfrak{S}_{0,1,1}^{(0),1} \cap \mathfrak{S}_{0,2,1}^{(0),1} = \Phi$, hence we pass to II-11.

II-11. We have

$$\mathcal{C}_1^{(0)} = \bigcup_{j=1}^{2} (\mathfrak{S}_{0,j,2}^{(0),1} \cup \partial\mathfrak{S}_{0,j,1}^{(0),1}) = \Phi$$

and $k_0 + 1 = 1 < 2 = n$.

Therefore we define :

$$\mathcal{C}_0^{(1)} = \bigcup_{j=1}^{2} (\mathrm{int}\ (\mathfrak{S}_{0,j,1}^{(0),1})) = \mathfrak{S}_{0,1,1}^{(0),1} \cup \mathfrak{S}_{0,2,1}^{(0),1} \text{ and } g_0^{(1)} : \mathfrak{S}_0^{(1)} \to R \text{ using (3.17)}, \tag{3.18}$$

We pass now to the new routine :

The routine $(\mathcal{C}_0^{(1)}, g_0^{(1)})$

II-1. We have $n_{1,0} = 2$ because $\mathcal{C}_0^{(1)}$ has two connected components $\mathcal{C}_0^{(1,1)} = \mathcal{C}_{0,1,1}^{(0)1}$, $\mathcal{C}_0^{(1),2} = \mathcal{C}_{0,2,1}^{(0),1}$ with the parametrisations :

$$\mathcal{C}_0^{(1),1} : \begin{cases} x^1 = \chi_0^{1,(1),1}(s) = 1 - \mathrm{e}^{-s^2} \\ x^2 = \chi_0^{2,(1),1}(s) = s, \quad s < 0 \end{cases} \tag{4.33}$$

$$\mathcal{C}_0^{(1),2} : \begin{cases} x^1 = \chi_0^{1,(1),2}(s) = \mathrm{e}^{-s^2} - 1 \\ x^2 = \chi_0^{2,(1),2} = s, \quad s > 0 \end{cases} \tag{4.34}$$

The local representatives of the functions $g_0^{(1),\alpha} = g_0^{(1)} \mid \mathcal{C}_0^{(1),\alpha}$, $\alpha = 1, 2$, are :

$$h_0^{(1),1}(s) = -2\int_0^s e^{-t^2}dt, \quad s < 0 \tag{4.35}$$

$$h_0^{(1),2}(s) = 2\int_0^s e^{-t^2}dt, \quad s > 0 \tag{4.36}$$

We take $\alpha = 1$ and pass to.

The subroutine $(\mathcal{C}_0^{(1),1}, h_0^{(1),1})$

II-2. For $j = 1$ the system (3.4) has the following form :

$$\begin{cases} \lambda_1^0 \dfrac{d\chi_0^{1,(1),1}}{ds}(s) + \lambda_2^0 \cdot \dfrac{d\,\chi_0^{2,(1),1}}{ds}(s) = \dfrac{dh_0^{(1),1}}{ds}(s) \\ 1 + \lambda_1^0 \cdot \chi_0^{2,(1),1}(s) + 2\lambda_2^0\, e^{(x^{2(1)1}(s))^2} = 0 \end{cases}$$

and from (4.33), (4.35) we obtain :

$$\begin{cases} \lambda_1^0 \cdot s + 2 \cdot \lambda_2^0\, e^{s^2} = -1 \\ 1 + \lambda_1^0 s + 2\lambda_2^0\, e^{-s^2} = 0 \end{cases}$$

Since the rank of the matrix of this system with respect to λ_1^0, λ_2^0 is equal to 1 for every $s < 0$, the condition of transversal intersection is not satisfied.

For $j = 2$ the system to be considered for the definition of the set $\mathcal{C}_{0,2}^{*(1),1}$ is the following :

$$\begin{cases} \lambda_1^0 \cdot 2s \cdot e^{-s^2} + \lambda_2^0 = -2\, e^{-s^2} \\ 1 + \lambda_1^0 \cdot s - \dfrac{1}{2}\lambda_2^0\, e^{s^2} = 0 \end{cases}$$

Therefore we have :

$$\mathcal{C}_{0,2}^{*(1),1} = \left\{ \left(1 - e^{-s^2}, s, -\frac{1}{s}, 0\right) \mid s < 0 \right\}$$

II-3. We integrate the system :

$$(4.37)\qquad \begin{cases} \dfrac{dx^1}{dt} = x^2 \\[2ex] \dfrac{dx^2}{dt} = -\dfrac{1}{2}\, e^{(x^2)^2} \\[2ex] \dfrac{d\lambda_1}{dt} = 0 \\[2ex] \dfrac{d\lambda_2}{dt} = -\lambda_1 - x^2\, e^{(x^2)^2} \cdot \lambda_2 \end{cases}$$

with initial conditions from the set $\mathcal{C}^{*(1),1}_{0,2}$ and obtain the solution $(x^{1,(1),1}_{0,2}(\cdot\,;s), x^{2,(1),1}_{0,2}(\cdot\,;s), -\dfrac{1}{s}, \lambda_2(\cdot\,;s))$ where $x^{2,(1),1}_{0,2}(t;s)$ is given by :

$$(4.38)\qquad t = -2\int_s^{x^2} e^{-t^2}\, dt,$$

$x^{1(1),1}_{0,2}(t\,;s)$ by

$$(4.39)\qquad x^{1,(1),1}_{0,2}(t\,;s) = \int_0^t x^{2,(1),1}_{0,2}(\tau\,;s)\, d\tau$$

and $\lambda_2(t;s)$ by :

$$(4.40)\qquad \lambda_2(t\,;s) = \left(-2 + \frac{2}{s}\, x^{2,(1),1}_{0,2}(t\,;s)\right) e^{-\,(x^{2,(1),1}_{0,2}(t;s))^2}$$

We notice that the curve $(x^{1,(1),1}_{0,2}(\cdot\,;s), x^{2,(1),1}_{0,2}(\cdot\,;s))$ may be written as follows :

$$(4.41)\qquad x^1 = e^{-(x^2)^2} - 2\, e^{-s^2} + 1$$

For every $s < 0$ we have $x^{2,(1),1}_{0,2}(t\,;s) > s$ for $t < 0$ and hence $\lambda_2(t;s) > 0$ for every $t < 0$.

Therefore we have :

$$\mathfrak{S}^{(1),1}_{0,2,1} = \{\, (x^{1,(1),1}_{0,2}(t\,;s), x^{2,(1),1}_{0,2}(t\,;s)) \mid t < 0, s < 0 \,\}\ \mathfrak{S}^{(1),1}_{0,2,2} = \Phi$$

We notice that according to (4.41) the set $\mathfrak{S}^{(1),1}_{0,2,1}$ may be written as follows :

$$\mathfrak{S}^{(1),1}_{0,2,1} = \{\, (e^{-t^2} - 2\, e^{-s^2} + 1, s) \mid s < 0,\ t > s \,\}$$

c) **II-4.** It is clear that $\mathfrak{S}^{(1),1}_{0,2,1}$ is a 2-dimensional differentiable manifold without boundary that looks as in fig. 3 ([6]).

d) **II-5.** For every $(x^1, x^2) \in \mathfrak{S}^{(1),1}_{0,2,1}$ we take $v^{(1),1}_{0,2}(x^1, x^2) = 1$ and we verify the condition (*d*).

e) **II-6.** Since the condition from the remark 3.1 is verified, the condition (*e*) is also verified. Moreover, since the function f^0 (equal to 1) does not contain neither x^1 nor x^2 we need not the function $\varphi^{(1),1}_{0,2}$ for passing to the operation.

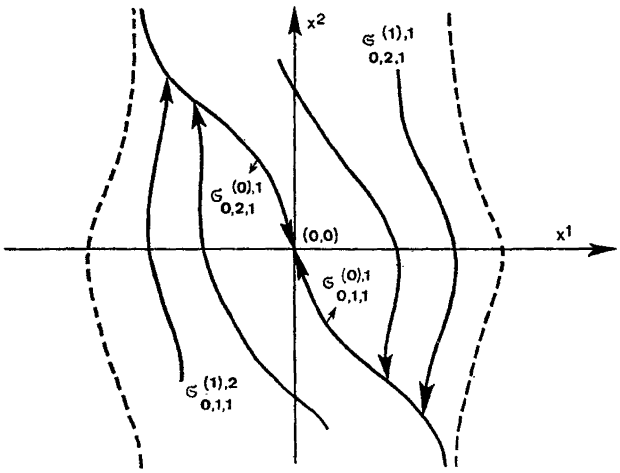

Figure 3

f) **II-7.** According to (3.13) for every $(x^1, x^2) \in \mathfrak{S}^{(1),1}_{0,2,1}$ and $t_1, s_1 < 0$ such that :

(4.42) $x^1 = x^{1,(1),1}_{0,2}(t_1 ; s_1)$, $x^2 = x^{2,(1),1}_{0,2}(t_1 ; s_1)$ we have:

$$W^{(1),1}_{0,2}(x^1, x^2) = h^{(1),1}_{0,2}(s_1) + \int_0^{-t_1} dt = -2 \int_0^{s_1} e^{-t^2}\, dt - t_1$$

and using (4.38) and (4.39) we obtain :

$$W^{(1),1}_{0,2}(x^1, x^2) = -2 \int_0^{s_1} e^{-t^2}\, dt + 2 \int_{s_1}^{x^2} e^{-t^2}\, dt \tag{4.43}$$

where s_1 is given by :

$$s_1 = -\sqrt{\ln \frac{2}{1 + e^{-(x^2)^2} - x^1}} \tag{4.44}$$

Now it is obvious that $W^{(1),1}_{0,2}$ is a C^1-function.

II-8. We have $\alpha = 1 < n_{1,1} = 2$ and hence we take $\alpha = 1 + 1 = 2$ and pass to :

The subroutine $(\mathcal{C}_0^{(1),2}, h_0^{(1),2})$

By symetric reasons the results of this subroutine may be written immediately from the corresponding results of the subroutine $(\mathcal{C}_0^{(1),1}, h_0^{(1),1})$.

II-8. We have $\alpha = 2 = n_{1,1}$ and we pass to the next operation.

II-9. All the intersections $\mathfrak{S}_{q,j,\delta}^{(l),\beta} \cap \mathfrak{S}_{r,p,\gamma}^{(i),\alpha}$ are empty and hence we pass directly to the operation II-11.

II-11. We have

$$\mathcal{C}_1^{(1)} = \bigcup_{j=1}^{2} \bigcup_{\alpha=1}^{2} (\mathfrak{S}_{0,j,2}^{(1),\alpha} \cup \partial\mathfrak{S}_{0,j,1}^{(1),\alpha}) = \Phi$$

and $k_1 + 1 = 2 = n$. Therefore we must pass to the step III.

The step III

g) **III-1.** We define the cells of the admissible synthesis :

(1) the 0-dimensional cells (of type II) are : $\mathcal{C}_1^{(0)} = \mathcal{C}_0^{(0)} = \{ (0, 0) \}$

(2) the 1-dimensional cells of type I are : $\mathcal{C}_{1,2}^{(1),1} = \mathfrak{S}_{0,2,1}^{(0),1}$, $\mathcal{C}_{2,1}^{(1),1} = \mathfrak{S}_{0,1,1}^{(0),2}$

(3) there not exist 1-dimensional cells of type II;

(4) the 2-dimensional cells (of type I) are : $\mathcal{C}_{1,2}^{(2)} = \mathfrak{S}_{0,2,1}^{(1),1}$, $\mathcal{C}_{2,1}^{(2)} = \mathfrak{S}_{0,1,1}^{(1),2}$

(The meaning of the upper and lower indeces of the cells is the following : the left upper index shows the dimension of the cell; the right upper one the type of the cell (we note that we need not show the type of the 0-dimensional or 2-dimensional cells) ; the left lower index is a order number and the right lower one shows the « type » of the trajectories that the cell contains (in our problem we have trajectories of two sorts : the solutions of the system (4.23) and the solutions of the system (4.37).

It is clear that the above defined cells verify the conditions B-(ii) and B-(iii) from the definition 2.3.

h) **III-2.** We define the sets :

$$N = \Phi,\ P^0 = \mathcal{C}_1^{(0)},\ P^1 = P^0 \cup \mathcal{C}_{1,2}^{(1),1} \cup \mathcal{C}_{2,1}^{(1),1},\ P^2 = P^1 \cup \mathcal{C}_{1,2}^{(2)} \cup \mathcal{C}_{2,1}^{(2)} = G.$$

As in the preceeding example the sets N, P^0, P^1, P^2 are piecewise smooth sets and the condition (*h*) is verified.

i) **III-3.** If we define :

$$v(x^1, x^2) = \begin{cases} 1 & \text{for} \quad (x^1, x^2) \in \mathcal{C}_1^{(0)} \cup \mathcal{C}_{2,1}^{(1),1} \cup \mathcal{C}_{2,1}^{(2)} \\ -1 & \text{for} \quad (x^1, x^2) \in \mathcal{C}_{1,2}^{(1),1} \cup \mathcal{C}_{1,2}^{(2)} \end{cases}$$

then from the above results it follows that every solution of the system :

$$\begin{cases} \dfrac{dx^1}{dt} = x^2 \\ \dfrac{dx^2}{dt} = \dfrac{1}{2} v(x^1, x^2)\, e^{(x^2)^2} \end{cases}$$

reaches $\mathfrak{T}$ transversally, in a finite time and intersects only a finite number of cells (at most two cells).

j) **III-4.** From the definition of the functions $W_{r,p}^{(i),\alpha}$ it follows that the functions $W : G \to R$ that coincides on every set $\mathfrak{S}_{r,p}^{(i),\alpha}$ with $W_{r,p}^{(i),\alpha}$ is continuous and the condition (*j*) is verified.

III-5. The sets P^0, P^1, P^2 and the map v represent an optimal synthesis for the control problem (4.21). STOP.

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