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EQUITABLE COLORATIONS OF GRAPHS (*)

par D. de WERRA (1)

Abstract. — *An edge coloration of a graph is a coloration of its edges in such a way that no two edges of the same colour are adjacent. We generalize this concept by introducing the notion of equitable coloration, i.e., coloration of the edges of a graph such that if $f_i(x)$ denotes the number of edges with colour i which are adjacent to vertex x, we have $|f_i(x) - f_j(x)| \leqslant 1$ for every vertex x and every pair of colours i, j. Equitable colorations are also defined for hypergraphs,*

Finally some results on edge colorations are generalized to the case of equitable colorations.

1. Coloration of Hypergraphs

A *hypergraph* $H = (X, U)$ consists of a finite set X of vertices $x_1, ..., x_n$ and a family U of nonempty edges $U_j (j = 1, ..., m)$ satisfying $\bigcup_{j=1}^{m} U_j = X$.

A hypergraph H is *unimodular* if its edge incidence matrix A ($a_{ij} = 1$ if $x_i \varepsilon U_j$ or 0 otherwise) is totally unimodular. The *subhypergraph* of $H = (X, U)$ spanned by a subset $Y \subset X$ is the hypergraph $H(Y) = (Y, U(Y))$ where $U(Y) = \{ U_j \cap Y | U_j \cap Y \neq \emptyset \}$. An *equitable k-coloration* E of $H = (X, U)$ is a partition of X into k subsets $F_1, ..., F_k$ such that for every edge U_j

$$\left| |U_j \cap F_p| - |U_j \cap F_q| \right| \leqslant 1 \qquad \forall\, p, q \,\varepsilon \{1, ..., k\}$$

The result of Camion [1] and Ghouila-Houri [2] about totally unimodular matrices may be formulated in terms of hypergraphs as follows [3] [4] :

Lemma : A hypergraph H is unimodular if and only if all its subhypergraphs have an equitable bicoloration.

We have the following :

Theorem 1 : A unimodulaı hypergraph H has an equitable k-coloration for any k.

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Proof : Given a coloration E of the vertices of H with k colours (E is not necessarily an equitable k-coloration), for each edge U_j we define a vector $E(j) = (f^j_1, f^j_2, \ldots, f^j_k)$ where f^j_p is the number of vertices of U_j which have colour p. Let $F_p \subset X$ be the subset of vertices which have colour p. For every edge let $e(j) = \max\limits_{p,q} (f^j_p - f^j_q) \geqslant 0$; let $e^* = \max\limits_j e(j)$. If $e^* < 2$, E is an equitable k-coloration of H. Otherwise, let U_j be an edge such that $e(j) = e^* = f^j_p - f^j_q$. We consider the subgraph H' spanned by $F_p \cup F_q$. It follows from the lemma that H' has an equitable 2-coloration E'; we colour its vertices with 2 colours p and q in such a way that $|f^j_p - f^j_q| \leqslant 1$ for every U_j. The values f^j_r are unchanged for $r \neq p, q$ and for every U_j. Thus at least one value $e(j)$ is such that the number of pairs p, q with $|f^j_p - f^j_q| \leqslant e(j) - 1$ has increased by at least one unit and the other $e(j)$ have not increased. This procedure can be repeated until $e^* < 2$. We get thus an equitable k-coloration of H. End of proof.

A *transversal* of a hypergraph $H = (X, U)$ is a subset of vertices T such that $T \cap U_j \neq \emptyset$ for $j = 1, \ldots, m$. The following corollary is a slight generalization of a theorem in Berge [3].

Corollary 1 : Let $H = (X, U)$ be a unimodular hypergraph and $k = \min\limits_j |U_j|$ the minimal cardinality of its edges. The set X of vertices of H may be partitioned into k transversals.

Proof : Consider an equitable k-coloration of H where $k = \min\limits_j |U_j|$; such a k-coloration exists from theorem 1. Clearly in each edge there will be at least one vertex of each colour. Hence the subsets $F_1, \ldots, F_k$ defined by the k-coloration are transversals.

Following Berge [3], we call *strong chromatic number* of $H = (X, U)$ the smallest integer k such that there exists a partition of X into subsets $F_1, \ldots, F_k$ with $|F_i \cap U_j| \leqslant 1 \quad \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, m \end{matrix}$. The next corollary is due to Berge [5].

Corollary 2 : The strong chromatic number of a unimodular hypergraph is equal to the maximal cardinality of its edges.

Proof : Let $k = \max |U_j|$ and consider an equitable k-coloration of H; let F_i be the set of vertices with colour i for $i = 1, \ldots, k$. Obviously

$$|F_i \cap U_j| \leqslant 1 \qquad \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, m \end{matrix}$$

We can also apply theorem 1 to graphs; an equitable k-coloration of a graph is then a coloration of its edges with k colours such that for each vertex x, we have :

$$|f_p(x) - f_q(x)| \leqslant 1 \ \forall \ p, q \in \{1, \ldots, k\}$$

where $f_p(x)$ denotes the number of edges with colour p which are adjacent to x.

Corollary 3 : A bipartite graph $G = (X, U)$ has an equitable k-coloration for any k.

Proof : This result is obtained by applying theorem 1 to the hypergraph H obtained as follows : its vertices are the edges of G and its edges are the sets of edges which are adjacent to the same vertex of G. H is unimodular since its edge incidence matrix is the transposed matrix of the edge incidence matrix of G.

When applied to the case of graphs, corollary 1 becomes the theorem of Gupta [3] : If $G = (X, U)$ is a bipartite graph with minimum degree k, then there exists a partition of U into k spanning subsets of edges $H_1, ..., H_k$. (H_i is a spanning subset if the edges in H_i meet all vertices of G.)

Moreover corollary 2 gives the well-known result : the chromatic index of a bipartite graph is equal to the maximum degree of the vertices in G (the chromatic index of G is by definition the smallest k such that the edges of G may be partitioned into k subsets of nonadjacent edges).

2. P-bounded colorations

We will now generalize some results about edge colorations. A *p-bounded* k-coloration E of *a* graph G is a partition of its edges into k nonempty subsets $F_1, ..., F_k$ such that for any vertex x : $|f_j(x) - f_i(x)| \leqslant p$ for $i, j = 1, ..., k$ where $f_j(x)$ is the number of edges of F_j which are adjacent to x. An equitable k-coloration is thus a 1-bounded k-coloration. Let $E = \{ F_1, ..., F_k \}$ be a p-bounded k-coloration and $f_1 \geqslant ... \geqslant f_k$ the cardinalities of $F_1, ..., F_k$ respectively.

Theorem 2 : If the sequence $(f_1, ..., f_k)$ corresponds to a p-bounded k-coloration of G, then any sequence $f'_1, ..., f'_k$ with :

a) $$f'_1 \geqslant ... \geqslant f'_k$$

b) $$\sum_{i=1}^{l} f'_i \leqslant \sum_{i=1}^{l} f_i \qquad l = 1, ..., k-1$$

c) $$\sum_{i=1}^{k} f'_i = \sum_{i=1}^{k} f_i$$

corresponds to a p-bounded k-coloration of G.

Proof : A) We first prove that any couple of subsets F_i, F_j in E with $f_i - f_j = K \geqslant 2$ may be replaced by two subsets F'_i, F'_j with $f'_i - f'_j = K - 2$. $E_{ij} = (F_i, F_j)$ is a p-bounded bicoloration of $G_{ij} = (X, F_i \cup F_j)$; we consider any edge u in G_{ij} and construct an alternating path P containing u (i.e., the edges of which belong alternately to F_i and F_j); we extend the path P as far as possible; we obtain thus either an alternating circuit (with even length) or an alternating open path. We remove P from G_{ij} and repeat the same construction with another edge u, until all edges in G_{ij} are removed.

Since $f_i - f_j = K \geqslant 2$, there is at least one alternating path P in which the first edge and last edge belong to F_i; we interchange the edges of $P \cap F_i$ and $P \cap F_j$.

Let x and y be the endpoints of P. Since P terminates at x with an edge in F_i we have $f_i(x) \geqslant f_j(x) + 1$; by interchanging the edges of P we get

$$f_j(x) \leqslant f_i'(x) = f_i(x) - 1 \leqslant f_i(x)$$

$$f_j(x) \leqslant f_j'(x) = f_j(x) + 1 \leqslant f_i(x)$$

The same inequalities hold for y. Furhtermore, for all vertices $z \neq x, y$ we have $f_i'(z) = f_i(z)$ and $f_j'(z) = f_j(z)$. So we obtain a new p-bounded bicoloration (F_i', F_j') with $f_i' - f_j' = K - 2$.

B) By successive applications of the above described procedure we can obtain p-bounded k-colorations corresponding to any sequence $(f_1', \ldots, f_k')$ satisfying *a*), *b*) and *c*). This ends the proof.

Theorem 2 is a generalization of a result which appears in Folkman and Fulkerson [6]. (Their theorem corresponds to the case where $p = 1$ and k is at least equal to the chromatic index of G.) We raise now and answer the following question : given a graph G, what is the smallest value p such that G has a p-bounded k-coloration for any k? From corollary 3, we know that if G is bipartite, then the minimum value of p is $p = 1$. If G is not bipartite, it is not the case : a triangle has for instance no equitable 2-coloration. (Clearly for any k not less than the chromatic index of G, there is *a* 1-bounded k-coloration of G.)

Théorèm 3 : Let G be any graph; for any k, G has *a* 2-bounded k-coloration.

Proof : The theorem is true for a graph G with one edge. Suppose that it is true for graphs with at most $m - 1$ edges; we will show that it is also true for graphs with m edges. Let G be a graph with m edges; let us remove from G an edge u joining vertices x and y. By our induction hypothesis, $G' = G - u$ has a 2-bounded k-coloration for any k. Given some integer k, let $F_1, \ldots, F_k$ be the subsets of edges defined by such a k-coloration of G'. There exist 2 integers $a, b \geqslant 0$ such that

$$a \leqslant f_i(x) \leqslant a + 2 \qquad \text{for } i = 1, \ldots, k$$

$$b \leqslant f_i(y) \leqslant b + 2 \qquad \text{for } i = 1, \ldots, k$$

We can assume that there is at least one colour, say q, such that $f_q(x) = a$ (otherwise a is replaced by $a + 1$); similarly there is one colour r such that $f_r(y) = b$. We have to examine the following cases :

A) There is a colour s with $f_s(x) < a + 2$ and $f_s(y) < b + 2$. Then u may be introduced into F_s and $F_1, ..., F_k$ is a 2-bounded k-coloration of G.

B) For every colour s with $f_s(x) < a + 2$ we have $f_s(y) = b + 2$ and for every colour t with $f_t(y) < b + 2$ we have $f_t(x) = a + 2$. Let us consider colours q and r; we have $q \neq r$ (otherwise we are in case A).

We determine an alternating chain C starting at x with an r-edge (i.e., an edge in F_r) and whose edges are alternately r-edges and q-edges. We extend chain C as far as possible. Then 2 cases may occur :

B1) The last vertex in C is y; so the last edge in C is a q-edge (because if we arrive at y with an r-edge we can introduce one more q-edge into C since $f_q(y) = b + 2 > f_r(y) = b$). By interchanging the q-edges and the r-edges in C we obtain a 2-bounded k-coloration of G' with $f_q(x) = f_r(x) = a + 1$ and $f_q(y) = f_r(y) = b + 1$. So u may be introduced into F_q (or F_r) and $F_1, ..., F_k$ is a 2-bounded k-coloration of G.

B2) The last vertex in C is $z \neq y$. Again by interchanging the q-edges and the r-edges in C we obtain a 2-bounded k-coloration of G' with $f_r(x) = a + 1$, $f_r(y) = b$ (if C ends for instance with a q-edge we have $f_r(z) + 2 \geqslant f_q(z) > f_r(z)$ and after having interchanged the r-edges and the q-edges, we still have $|f_r(z) - f_q(z)| \leqslant 2$).

We can now introduce edge u into F_r and we still obtain a 2-bounded k-coloration of G.

We have examined all possible cases and the proof is completed.

We now define an *odd cycle* as a connected graph containing an odd number of edges and such that all degrees are even.

Theorem 4 : A connected graph G has an equitable bicoloration if and only if it is not an odd cycle.

Proof : A) Suppose G is an odd cycle; for any equitable bicoloration $\{F_1, F_2\}$ we must have $f_1(x) = f_2(x)$ at each vertex x. Hence, F_1 and F_2 have the same cardinality; but this is not possible since G contains an odd number of edges.

B) Conversely if G is not an odd cycle, then from Euler's theorem, the edges of G may be partitioned into a unique even cycle (if all degrees are even) or into one or more chains joining 2 vertices with odd degrees. By coloring the edges in each chain (or in the unique cycle if all degrees are even) alternately with colours 1 and 2 we obtain an equitable bicoloration of G.

Necessary and sufficient conditions for a graph G to have an equitable k-coloration ($k > 2$) are much more difficult to obtain (this would in fact solve the four color problem). However we can formulate :

Proposition : If in a connected graph G all degrees are multiples of k and if the number of edges is not a multiple of k, then G has no equitable k-coloration.

Proof as in theorem 4, A.

However even if all degrees and the number of edges in a connected graph G are multiples of k, G may not have an equitable k-coloration for $k > 2$.

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