Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique

J. H. Bramble

V. Thomée

Interior maximum norm estimates for some simple finite element methods

Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique, tome 8, n° 2 (1974), p. 5-18.

<http://www.numdam.org/item?id=M2AN_1974__8_2_5_0>

R.A.I.R.O.
(8e année, août 1974, R-2, p. 5 à 18)

INTERIOR MAXIMUM NORM ESTIMATES FOR SOME SIMPLE FINITE ELEMENT METHODS (*)

by J. H. BRAMBLE (1), V. THOMÉE (2)

Summary. — *The approximate solution of a simple constant coefficient second order Dirichlet problem in a plane domain* Ω *by means of Galerkin's method, using continuous, piecewise linear functions on a triangulation of* Ω *is considered. It is proved that if the triangulation is regular in the interior of* Ω *in a certain sense, and if h is the length of the longest edge of a triangle, then the error in the interior of* Ω *is bounded by* $Ch^2|\log h|$ *if the exact solution u is twice continuously differentiable and by* Ch^2 *if the second derivatives are Hölder continuous. Similar results are obtained for continuous, piecewise bilinear functions on a division of the interior of* Ω *into rectangles.*

1. INTRODUCTION

We shall consider the approximate solution of the Dirichlet problem

$$Lu \equiv -\sum_{j,k=1}^{2} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on} \qquad \partial\Omega, \tag{1.2}$$

using a simple finite element method based on triangulating the domain Ω. Here (a_{jk}) is a positive definite constant matrix. For this purpose, for h a small positive parameter, let $\Omega_h \subset \Omega$ be a polygonal domain approximating Ω. Assume that $\bar{\Omega}_h$ is the union of closed triangles having disjoint interiors, such that no vertex of any triangle lies on the interior of an edge of another triangle and such that, uniformly in h, the edges of the triangles have length bounded above and below by constant multiples of h and all the angles are bounded below. We assume throughout that the boundary $\partial\Omega$ is sufficiently regular and the approximation of Ω by Ω_h is sufficiently close for the estimates to be

---

(*) Supported in part by the National Science Foundation.
(1) Mathematics Department, Cornell University, Ithaca.
(2) Department of Mathematics, Chalmers University of Technology, Göteborg.

quoted below the hold. Since we are aiming for interior estimates we shall not be very precise about these assumptions; in fact, we will consider the estimates (1.3), (1.4) and (1.6) below as our assumptions in this respect. In particular these will be satisfied if Ω is convex and smooth and the boundary vertices of $\overline{\Omega}_h$ lie on $\partial\Omega$.

Let V_h be the finite dimensional linear space of functions which are continuous in the whole plane, linear in each triangle and vanish outside Ω_h. Let $\{ P_j \}_{j=1}^{N_h}$ be the interior mesh-points (vertices) of the triangulation. Then a basis of V_h is formed by the elements $\omega_j \in V_h$ for which $\omega_j(P_l) = \delta_{jl}$ and the representation of $v \in V_h$ with respect to this basis takes the simple form

$$v(x) = \sum_{j=1}^{N_h} v(P_j)\omega_j(x).$$

For a given continuous function u on Ω which vanishes on $\partial\Omega$ we define the interpolant $\tilde{u} \in V_h$ by

$$\tilde{u}(x) = \sum_{j=1}^{N_h} u(P_j)\omega_j(x).$$

This function satisfies, as is well-known, under suitable assumptions

$$\|\tilde{u} - u\|_\Omega \leqslant Ch^2 \|u\|_{\Omega,2} , \tag{1.3}$$

$$|\tilde{u} - u|_\Omega \leqslant Ch^2 |u|_{\Omega,2} . \tag{1.4}$$

Here and below we denote by $\| \cdot \|_{\Omega,k}$ (with k omitted when zero) the norm in $W_2^k(\Omega)$ and by $|\cdot|_{\Omega,k}$ the norm in $C^k(\Omega)$. This latter norm will also be used for k non-integral so that, for instance, $|u|_{\Omega,2+\varepsilon}$ is finite when $D^\alpha u \in \mathrm{Lip}^\varepsilon(\Omega)$ for $|\alpha| = 2$ $(0 < \varepsilon < 1)$. In (1.3), (1.4) and in what follows C denotes a positive constant independent of h and the functions involved, but not necessarily the same at different occurances.

Introducing the bilinear form

$$A(v, w) = \int_\Omega \sum_{j,k=1}^{2} a_{jk} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_k} \, dx,$$

the finite element problem may be formulated as follows : Find $v \in V_h$ such that

$$A(v, w) = (f, w) \equiv \int_\Omega f(x)w(x) \, dx, \text{ for all } w \in V_h .$$

An equivalent formulation is : Find $\{ v(P_j) \}_1^{N_h}$ such that

$$\sum_{j=1}^{N_h} v(P_j)A(\omega_j, \omega_l) = (f, \omega_l) \quad , \quad l = 1, \ldots, N_h . \tag{1.5}$$

It is well-known that this problem admits a unique solution $v \in V_h$, and it has been proved (cf. [2], [3]) that under suitable assumptions, if u is the exact solution of (1.1), (1.2), then v is an approximation of u of the same order as the interpolant in the sense that

$$\|v - u\|_\Omega \leqslant Ch^2 \|u\|_{\Omega,2} . \tag{1.6}$$

For the maximum norm of the error it was also proved that

$$|v - u|_\Omega \leqslant Ch \|u\|_{\Omega,2} .$$

However, although this latter estimate is optimal in the sense that its order cannot be improved without changing the norm on the right hand side, one could hope to replace h by h^2 under more stringent regularity assumptions. We shall in fact be able to prove such second order estimates in the interior of the domain Ω when the triangulation is regular in the interior of Ω in the sense that there are three different directions in the plane such that for any Ω^0 with $\Omega^0 \subset\subset \Omega$ (i.e. $\bar{\Omega}^0 \subset \Omega$) and for h sufficiently small, the triangles intersecting Ω^0 are defined by equidistant lines parallel to the given directions.

The following is our main result.

Theorem 1. Assume that the triangulation is regular in the interior. Then for $\Omega^0 \subset\subset \Omega$ there is a constant C and for any $\varepsilon > 0$ a constant C_ε such that

$$|v - u|_{\Omega^0} \leqslant \begin{cases} Ch^2 \log \frac{1}{h} |u|_{\Omega,2} , \\ C_\varepsilon h^2 |u|_{\Omega,2+\varepsilon} . \end{cases}$$

The proof will depend on interpreting (1.5) as a finite difference equation. For this purpose we collect some results from finite difference theory in Section 2 and complete the proof of Theorem 1 in Section 3. In Section 4 we discuss briefly the case of regular quadrilateral elements in the interior of Ω. A weaker form of Theorem 1 was given in [5].

2. ELLIPTIC FINITE DIFFERENCE OPERATORS

For real-valued functions defined on the square mesh hZ^2 we denote $u_\alpha = u(\alpha h)$, and introduce the translation operator T^β defined by $T^\beta u_\alpha = u_{\alpha+\beta}$. We shall consider second order finite difference operators of the form

$$L_h = h^{-2} \sum_\beta b_\beta T^\beta,$$

where only a finite number of the constant real coefficients b_β are non-zero. It is well-known (cf. [6]) that such a finite difference operator L_h is consistent with the differential operator $L = \sum_{|\gamma|=2} a_\gamma D^\gamma$ with $D^\gamma = (\partial/\partial x_1)^{\gamma_1}(\partial/\partial x_2)^{\gamma_2}$ if and only if it can be written

$$L_h = \sum_{\beta,\gamma} c_{\beta\gamma} T^\beta \partial^\gamma \quad \text{with} \quad \sum_\beta c_{\beta\gamma} = a_\gamma \ , \tag{2.1}$$

where $\partial^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}$ and $\partial_l = h^{-1}(T^{e_l} - I)$ for $l = 1,2$, with $e_1 = (1,0)$ and $e_2 = (0,1)$.

For mesh-functions we define, with k a non-negative integer, the following norms

$$\|v\|_{h,\Omega,k} = \sum_{|\gamma| \leqslant k} \|\partial^\gamma v\|_{h,\Omega} \quad \text{with} \quad \|v\|_{h,\Omega} = \left(h^2 \sum_{\alpha h \in \Omega} v^2\right)^{1/2},$$

$$|v|_{h,\Omega,k} = \sum_{|\gamma| \leqslant k} |\partial^\gamma v|_{h,\Omega} \quad \text{with} \quad |v|_{h,\Omega} = \max_{\alpha h \in \Omega} |v_\alpha|,$$

and for $0 < \varepsilon < 1$,

$$|v|_{h,\Omega,\varepsilon} = |v|_{h,\Omega} + \max_{\substack{\alpha \neq \beta \\ \alpha h, \beta h \in \Omega}} \frac{|v_\alpha - v_\beta|}{|(\alpha - \beta)h|^\varepsilon}.$$

For mesh-functions with finite support we also use the inner product

$$(v, w)_h = h^2 \sum_\alpha v_\alpha w_\alpha \ .$$

The adjoint L_h^* of L_h is defined by

$$L_h^* u = h^{-2} \sum_\beta b_\beta T^{-\beta} u \ ,$$

and satisfies for v, w with finite support,

$$(L_h v, w)_h = (v, L_h^* w)_h \ .$$

The operator L_h is said to be elliptic if with $\langle \beta, \theta \rangle = \beta_1\theta_1 + \beta_2\theta_2$,

$$p(\theta) = \sum_\beta b_\beta e^{i\langle \beta,\theta \rangle} \neq 0 \quad \text{for} \quad 0 \neq \theta \in Q = \{\theta \, ; |\theta_j| \leqslant \pi, j = 1, 2\}.$$

For such operators we shall need the fundamental solution given by the following lemma.

Lemma 2.1. Let L_h be elliptic. Then there is a mesh-function g_α defined on hZ^2 such that

$$L_h g_\alpha = h^{-2} \delta_{\alpha,0} \ ,$$

and for any γ there is a constant C such that

$$|\partial^\gamma g_\alpha| \leqslant C(h|\alpha| + h)^{-|\gamma|}.$$

Proof. See [4] where g_α is given in the form

$$g_\alpha = (2\pi)^{-2} \int_Q \frac{e^{i\langle\alpha,\theta\rangle} - 1}{p(\theta)} \, d\theta.$$

We shall now use the fundamental solution to derive an estimate for the maximum norm of a mesh-function v in the interior of Ω in terms of $L_h v$ and the discrete L_2-norm of v.

Lemma 2.2. Let $\Omega^1 \subset\subset \Omega^2 \subset\subset \Omega$ and let $\chi \in C_0^\infty(\Omega^2)$ with $\chi \equiv 1$ in a neighborhood of Ω^1. Then there is a constant C such that for all mesh-functions v on hZ^2,

$$|v|_{h,\Omega^1} \leqslant C \{ \max_{\alpha h \in \Omega^1} |(L_h v, \chi T^{-\alpha} g)_h|_{h,\Omega^1} + \|v\|_{h,\Omega^2} \},$$

where g is the fundamental solution from Lemma 2.1 for the elliptic operator L_h^*.

Proof. It follows from Lemma 2.1 that there is a constant C such that for $\alpha h \in \Omega^1$,

$$|L_h^*(\chi T^{-\alpha} g)_\beta - h^{-2} \delta_{\alpha,\beta}| \leqslant C.$$

Consequently,

$$(2.2) \qquad |(v, L_h^*(\chi T^{-\alpha} g))_h - v_\alpha| \leqslant Ch^2 \sum_{\beta h \in \Omega^2} |v_\beta| \leqslant C \|v\|_{h,\Omega^2} .$$

Since χ has compact support in Ω^2 we have

$$(2.3) \qquad (v, L_h^*(\chi T^{-\alpha} g))_h = (L_h v, \chi T^{-\alpha} g)_h .$$

Together (2.2) and (2.3) imply

$$|v_\alpha| \leqslant C \{ |(L_h v, \chi T^{-\alpha} g)_h| + \|v\|_{h,\Omega^2} \},$$

which proves the lemma.

In the application of this estimate below we shall have occasion to use the following lemma :

Lemma 2.3. Let M_h be a second order finite difference operator of the form (2.1). Then with the notation of Lemma 2.2 there is a constant C and for each $\varepsilon \in (0,1)$ a constant C_ε such that for any mesh-function w on hZ^2,

$$|(M_h w, \chi T^{-\alpha} g)_h|_{h,\Omega^2} \leqslant \begin{cases} C \log \frac{1}{h} |w|_{h,\Omega^2} , \\ C_\varepsilon |w|_{h,\Omega^2,\varepsilon} . \end{cases}$$

Proof. Since $\chi \in C_0^\infty(\Omega^2)$ we have for small h and $\alpha h \in \Omega^1$,

$$|(M_h w, \chi T^{-\alpha} g)_h| = |(w, M_h(\chi T^{-\alpha} g))_h| \leqslant C\, |w|_{h,\Omega^2} h^2 \max_{|\gamma| \leqslant 2} \sum_{\beta h \in \Omega^2} |\partial^\gamma g_{\beta-\alpha}|.$$

By Lemma 2.1 we have for $\alpha h \in \Omega_1$, $|\gamma| \leqslant 2$, with d the diameter of Ω,

$$h^2 \sum_{\beta h \in \Omega^2} |\partial^\gamma g_{\beta-\alpha}| \leqslant C \sum_{h|\beta| \leqslant d} (|\beta| + 1)^{-2} \leqslant C \log \frac{1}{h},$$

which proves the first inequality.

Since M_h annihilates constants, we also find

$$(M_h w, \chi T^{-\alpha} g)_h = (M_h(w - w_\alpha), \chi T^{-\alpha} g)_h = (w - w_\alpha, M_h^*(\chi T^{-\alpha} g))_h \;,$$

so that for $\alpha h \in \Omega^1$,

$$\begin{aligned} |(M_h w, \chi T^{-\alpha} g)_h| &\leqslant C h^2 \sum_{\beta h \in \Omega^2} \frac{|w_\beta - w_\alpha|}{(|\beta - \alpha|\, h + h)^2} \\ &\leqslant C\, |w|_{h,\Omega^2,\varepsilon}\, h^2 \sum_{h|\beta| \leqslant d} (|\beta|\, h + h)^{-(2-\varepsilon)} \leqslant C_\varepsilon\, |w|_{h,\Omega^2,\varepsilon} \;, \end{aligned}$$

which proves the second inequality.

3. PROOF OF THEOREM 1

Let $\Omega^0 \subset\subset \Omega^1 \subset\subset \Omega^2 \subset\subset \Omega$. In considering the regular triangulation in the interior of Ω it is no restriction of generality, since L has arbitrary coefficients, to assume that the three families of straight lines defining the triangles are $x_1 = nh$, $x_2 = nh$, $x_1 + x_2 = nh$ with $n = 0, \pm 1, \ldots$. In this case the mesh-points of Ω^2, for small h, are of the form αh with $\alpha \in Z^2$ and we may denote the corresponding basis functions by ω_α and set $u_\alpha = u(\alpha h)$. The basis of our analysis is then the following representation of the Galerkin equation (1.5) corresponding to the point $P = \alpha h$ as a finite difference equation. Here we use in addition to the forward difference quotients ∂_j also the backward difference quotients,

$$\bar{\partial}_j v_\alpha = h^{-1}(v_\alpha - v_{\alpha - e_j}).$$

Lemma 3.1. For $\alpha h \in \Omega^2$ the Galerkin equation (1.5) may be written

$$L_h v_\alpha = h^{-2}(f, \omega_\alpha), \tag{3.1}$$

where L_h is the elliptic finite difference operator

$$L_h = -\sum_{j,k=1}^{2} a_{jk} \bar{\partial}_j \partial_k \;.$$

Proof. It suffices to show (3.1) for $\alpha = 0$. Let then $P_0 = (0,0)$ and let $\{P_j\}_{j=1}^6$ be its neighbors, $P_1 = (h, 0)$, $P_2 = (0, h)$, $P_3 = (-h, h)$, $P_4 = (-h, 0)$, $P_5 = (0, -h)$, $P_6 = (h, -h)$.

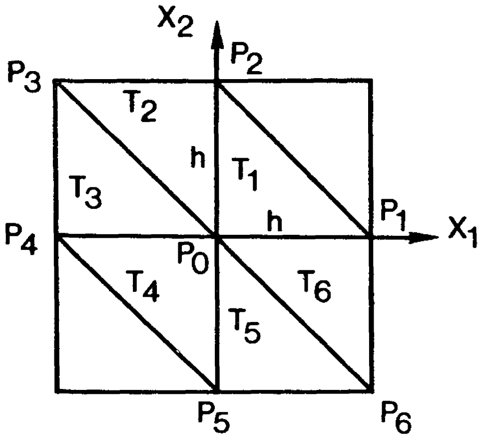

Figure 1

Simple calculations give then for the corresponding basis functions

$$A(\omega_0, \omega_0) = 2(a_{11} + a_{12} + a_{22}),$$

$$A(\omega_1, \omega_0) = A(\omega_4, \omega_0) = -(a_{11} + a_{12}),$$

$$A(\omega_2, \omega_0) = A(\omega_5, \omega_0) = -(a_{12} + a_{22}),$$

$$A(\omega_3, \omega_0) = A(\omega_6, \omega_0) = a_{12}.$$

This proves (3.1). We obtain for the characteristic polynomial of L_h,

$$p(\theta) = -\sum_{j,k=1}^{2} a_{jk}(1 - e^{-i\theta_j})(e^{i\theta_k} - 1)$$

$$= \sum_{j,k=1}^{2} a_{jk}(1 - \cos\theta_j)(1 - \cos\theta_k) + \sum_{j,k=1}^{2} a_{jk} \sin\theta_j \sin\theta_k ,$$

so that since (a_{jk}) is positive definite the operator L_h is an elliptic finite difference operator.

Our purpose is now to rewrite the right hand side of the difference equation (3.1) for $f = Lu$ as a combination of difference operators applied to certain averages of u along the edges of the triangulation. We define

$$U_\alpha^l = h^{-1} \int_0^h u(\alpha h + te_l)\, \mathrm{d}t \quad , \qquad l = 1, 2,$$

$$U_\alpha^3 = h^{-1} \int_0^h u(\alpha_1 h + t, \alpha_2 h + h - t)\, \mathrm{d}t,$$

and obtain (cf. [1]) the following :

Lemma 3.2. With the above notation we have for $\alpha h \in \Omega^2$,

$$(3.2) \qquad h^{-2}(Lu, \omega_\alpha) = - a_{11}[\bar\partial_1 \bar\partial_2 U_\alpha^3 + \partial_1(\bar\partial_1 - \bar\partial_2)U_\alpha^2] - 2a_{12}\bar\partial_1\bar\partial_2 U_\alpha^3 - a_{22}[\bar\partial_1\bar\partial_2 U_\alpha^3 + \partial_2(\bar\partial_2 - \bar\partial_1)U_\alpha^1].$$

Proof. It is clearly sufficient to consider $\alpha = 0$. We obtain then, since ω_0 has its support in $\bigcup_{l=1}^{6} T_l$ (see fig. 1) and has constant gradient in T_l,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}, \omega_0\right) = -\left(\frac{\partial u}{\partial x_k}, \frac{\partial \omega_0}{\partial x_j}\right) = -\sum_{l=1}^{6} \frac{\partial \omega_0}{\partial x_j}\bigg|_{T_l} \int_{T_l} \frac{\partial u}{\partial x_k}\, \mathrm{d}x$$

$$= -\sum_{l=1}^{6} \frac{\partial \omega_0}{\partial x_j}\bigg|_{T_l} \oint_{T_l} \nu_k\, u ds,$$

where $\nu = (\nu_1, \nu_2)$ denotes the exterior normal. For $j, k = 1$ we have hence, using the values of $\frac{\partial \omega_0}{\partial x_1}$ in T_l ,

$$h^{-2}\left(\frac{\partial^2 u}{\partial x_1^2}, \omega_0\right) = h^{-2}[U_0^3 - U_{-e_2}^3 - U_{-e_1}^3 + U_{-e_1-e_2}^3 + U_{e_1-e_2}^2 - U_{-e_2}^2 - U_0^2 + U_{-e_1}^2] = \bar\partial_1\bar\partial_2 U_0^3 + \partial_1(\bar\partial_1 - \bar\partial_2)U_0^2 ,$$

which is the coefficient of $-a_{11}$ in (3.2). The remaining combinations of j and k are treated similarly.

We define now the following discrete averages,

$$u_\alpha^l = \frac{1}{2}(u_\alpha + u_{\alpha+e_l}), \qquad l = 1, 2, \qquad u_\alpha^3 = \frac{1}{2}(u_{\alpha+e_1} + u_{\alpha+e_2}),$$

and find easily, analogously to (3.2),

$$L_h u_\alpha = - a_{11}[\bar\partial_1\bar\partial_2 u_\alpha^3 + \partial_1(\bar\partial_1 - \bar\partial_2)u_\alpha^2] - 2a_{12}\bar\partial_1\bar\partial_2 u_\alpha^3 - a_{22}[\bar\partial_1\bar\partial_2 u_\alpha^3 + \partial_2(\bar\partial_2 - \bar\partial_1)u_\alpha^1] .$$

Hence, setting $w_\alpha^l = u_\alpha^l - U_\alpha^l$, $l = 1, 2, 3$, we find

(3.3)
$$L_h u_\alpha - h^{-2}(Lu, \omega_\alpha) = -\sum_{l=1}^{3} M_h^l w_\alpha^l ,$$

where

$$M_h^1 = a_{22}\partial_2(\bar\partial_2 - \bar\partial_1),\ M_h^2 = a_{11}\partial_1(\bar\partial_1 - \bar\partial_2),\ M_h^3 = (a_{11} + 2a_{12} + a_{22})\bar\partial_1\bar\partial_2 .$$

For the w^l we have the following estimates :

Lemma 3.3 There is a constant C such that for $0 \leqslant \varepsilon < 1$ and $l = 1, 2, 3$,

$$|w^l|_{h,\Omega^2,\varepsilon} \leqslant Ch^2 \, |u|_{\Omega,2+\varepsilon} .$$

Proof. We have for instance

$$\begin{aligned} w_\alpha^1 &= \frac{1}{2}(u(\alpha h) + u(\alpha h + e_1 h)) - h^{-1}\int_0^h u(\alpha h + te_1)\,dt \\ &= \frac{1}{2h}\int_0^h t(h-t)\frac{\partial^2 u}{\partial x_1^2}(\alpha h + te_1)\,dt , \end{aligned}$$

from which the estimate immediately follows for $l = 1$. The cases $1 = 2$, 3 are similar.

We now complete the proof of Theorem 1. Since L_h is elliptic we may apply Lemma 2.2 to the restriction of $u - v$ to the mesh-points of Ω^2. We obtain by Lemma 3.1 and (3.3),

$$L_h(u - v)_\alpha = L_h u_\alpha - L_h v_\alpha = L_h u_\alpha - h^{-2}(Lu, \omega_\alpha) = -\sum_{l=1}^{3} M_h^l w_\alpha^l ,$$

and hence applying Lemmas 2.3 and 3.3,

(3.4)
$$\begin{aligned} |(L_h(u-v), \chi T^{-\alpha} g)_h|_{h,\Omega^1} &\leqslant \sum_{l=1}^{3} |(M_h^l w^l, \chi T^{-\alpha} g)_h|_{h,\Omega^1} \\ &\leqslant \begin{cases} Ch^2 \log\dfrac{1}{h}\,|u|_{\Omega,2} , \\ C_\varepsilon h^2\,|u|_{\Omega,2+\varepsilon} \quad \text{for} \quad 0 < \varepsilon < 1. \end{cases} \end{aligned}$$

We further obtain, using (1.3) and (1.6),

(3.5)
$$\begin{aligned} \|u - v\|_{h,\Omega^2} = \|\tilde u - v\|_{h,\Omega^2} &\leqslant C\,\|\tilde u - v\|_\Omega \leqslant C(\|v - u\|_\Omega + \|\tilde u - u\|_\Omega) \\ &\leqslant Ch^2\,\|u\|_{\Omega,2} \leqslant Ch^2\,|u|_{\Omega,2} . \end{aligned}$$

Together, Lemma 2.2, (3.4) and (3.5) now prove that

$$|\tilde{u} - v|_{\Omega^0} \leqslant |u - v|_{h,\Omega^1} \leqslant \begin{cases} Ch^2 \log \frac{1}{h} |u|_{\Omega,2} , \\ C_\varepsilon h^2 |u|_{\Omega,2+\varepsilon} \quad \text{for} \quad 0 < \varepsilon < 1. \end{cases}$$

Using also (1.4) this completes the proof of Theorem 1.

4. QUADRILATERAL ELEMENTS

In this section we shall consider the case in which the elements in the interior of the domain are squares with sides of length h and the approximating functions are continuous in the union of the squares and bilinear in each square. Such a division of Ω in the interior may then be completed by means of triangles to a polygonal domain $\Omega_h \subset \Omega$ and we shall denote by V_h the finite dimensional linear space of functions which are continuous in the whole plane, bilinear in the squares, linear in the triangles and which vanish outside Ω_h. We may again in the present space V_h define basis functions ω_j with $\omega_j(P_l) = \delta_{jl}$ where $\{ P_l \}_1^{N_h}$ are the interior mesh-points and we assume that the approximation of Ω by Ω_h is such that the interpolant

$$\tilde{u}(x) = \sum_{j=1}^{N_h} u(P_j)\omega_j(x)$$

satisfies (1.3) and (1.4) as before. The finite element problem (1.5) still has a unique solution $v \in V_h$ and the error estimate (1.6) holds.

We shall prove the following analogue of Theorem 1.

Theorem 2. Under the present assumptions and for $\Omega^0 \subset\subset \Omega$ there is a constant C and for any $\varepsilon \in (0, 1)$ a constant C_ε such that

$$|v - u|_{\Omega^0} \leqslant \begin{cases} Ch^2 \log \frac{1}{h} |u|_{\Omega,2} , \\ C_\varepsilon h^2 |u|_{\Omega,2+\varepsilon} . \end{cases}$$

Let $\Omega^0 \subset\subset \Omega^1 \subset\subset \Omega^2 \subset\subset \Omega$ and assume that the mesh-points of Ω^2 for small h are in $\Omega \cap hZ^2$. We now denote the basis function corresponding to the point $h\alpha \in \Omega^2$ by ω_α. This time we may represent the interior Galerkin equations as follows.

Lemma 4.1. For $\alpha h \in \Omega^2$ the Galerkin equation (1.5) may be written

$$L_h v_\alpha = h^{-2}(f, \omega_\alpha), \tag{4.1}$$

where L_h is the elliptic finite difference operator

$$L_h = -a_{11}\frac{1}{6}(T^{e_2} + 4 + T^{-e_2})\partial_1\bar{\partial}_1 - a_{22}\frac{1}{6}(T^{e_1} + 4 + T^{-e_1})\partial_2\bar{\partial}_2$$
$$-\frac{1}{2}a_{12}(\partial_1 + \bar{\partial}_1)(\partial_2 + \bar{\partial}_2).$$

Proof. As above it suffices to consider $\alpha = 0$ and we introduce $P_0 = (0, 0)$ and its neighbors as in fig. 2 :

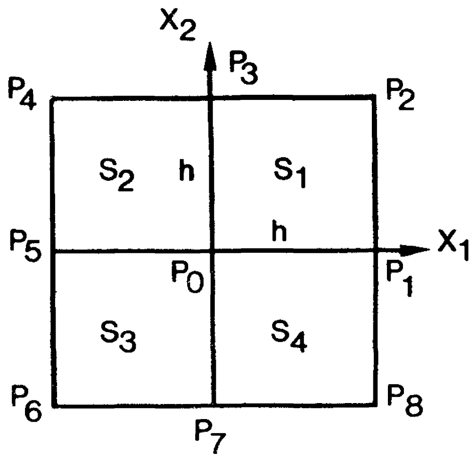

Figure 2

We obtain now

$$A(\omega_0, \omega_0) = \frac{4}{3}(a_{11} + a_{22}),$$

$$A(\omega_1, \omega_0) = A(\omega_5, \omega_0) = -\frac{2}{3}a_{11} + \frac{1}{3}a_{22},$$

$$A(\omega_3, \omega_0) = A(\omega_7, \omega_0) = \frac{1}{3}a_{11} - \frac{2}{3}a_{22},$$

$$A(\omega_2, \omega_0) = A(\omega_6, \omega_0) = -\frac{1}{6}a_{11} - \frac{1}{6}a_{22} - \frac{1}{2}a_{12},$$

$$A(\omega_4, \omega_0) = A(\omega_8, \omega_0) = -\frac{1}{6}a_{11} - \frac{1}{6}a_{22} + \frac{1}{2}a_{12},$$

which proves (4.1). The characteristic polynomial of L_h is now

$$p(\theta) = \frac{1}{3} a_{11}(4 + 2\cos\theta_2)(1 - \cos\theta_1)$$

$$+ \frac{1}{3} a_{22}(4 + 2\cos\theta_1)(1 - \cos\theta_2) + 2a_{12}\sin\theta_1 \sin\theta_2 \ ,$$

or with $s_j = \sin\frac{1}{2}\theta_j$, $c_j = \cos\frac{1}{2}\theta_j$,

(4.2) $$\frac{1}{4} p(\theta) = a_{11}\left(1 - \frac{2}{3}s_2^2\right)s_1^2 + 2a_{12}s_1s_2c_1c_2 + a_{22}\left(1 - \frac{2}{3}s_1^2\right)s_2^2 \ .$$

Since the matrix (a_{jk}) is positive definite we have

$$a_{11}c_2^2s_1^2 + 2a_{12}s_1s_2c_1c_2 + a_{22}c_1^2s_2^2 \geqslant 0,$$

so that

$$\frac{1}{4}p(\theta) \geqslant \frac{1}{3}(a_{11} + a_{22})s_1^2s_2^2 \geqslant 0.$$

Hence for p to vanish we must have $s_1 = 0$ or $s_2 = 0$ and we then see from (4.2) that $s_1 = s_2 = 0$ which proves the ellipticity.

For the purpose of giving an analogue of Lemma 3.2 we define

$$U_\alpha^l = h^{-2}\int_{-h}^{h} u(\alpha h + se_l)(h - |s|)\,ds, \qquad l = 1, 2,$$

$$U_\alpha^3 = h^{-2}\int_0^h\int_0^h u(\alpha h + x)\,dx.$$

We then have the following :

Lemma 4.2. With the above notation we have for $\alpha h \in \Omega^2$,

$$h^{-2}(Lu, \omega_\alpha) = - a_{11}\partial_1\bar\partial_1 U_\alpha^2 - a_{22}\partial_2\bar\partial_2 U_\alpha^1 - 2a_{12}\bar\partial_1\bar\partial_2 U_\alpha^3 \ .$$

Proof. It is again sufficient to consider $\alpha = 0$. Applying the notation of fig. 2 we have

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}, \omega_0\right) = -\left(\frac{\partial u}{\partial x_k}, \frac{\partial \omega_0}{\partial x_j}\right) = -\sum_{l=1}^{4}\int_{S_l}\frac{\partial u}{\partial x_k}\frac{\partial \omega_0}{\partial x_j}\,dx.$$

For $j = k = 1$ we obtain, using Green's formula

$$\left(\frac{\partial^2 u}{\partial x_1^2}, \omega_0\right) = h^{-2}\int_{S_1 \cup S_4}\frac{\partial u}{\partial x_1}(h - |x_2|)\,dx - h^{-2}\int_{S_2 \cup S_3}\frac{\partial u}{\partial x_1}(h - |x_2|)\,dx$$

$$= U_{e_1}^2 - 2U_0^2 + U_{-e_1}^2 = h^2\partial_1\bar\partial_1 U_0^2 \ ,$$

and similarly for $\left(\frac{\partial^2 u}{\partial x_2^2}, \omega_0\right)$. Finally,

$$\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}, \omega_0\right) = h^{-2} \int_{S_1 \cup S_2} \frac{\partial u}{\partial x_1}(h - |x_1|)\, dx - h^{-2} \int_{S_3 \cup S_4} \frac{\partial u}{\partial x_1}(h - |x_1|)\, dx.$$

For the integral over S_1 we obtain

$$h^{-2} \int_{S_1} \frac{\partial u}{\partial x_1}(h - |x_1|)\, dx = h^{-2} \int_{S_1} \left[\frac{\partial}{\partial x_1}((h - x_1)u) + u\right] dx$$

$$= - h^{-1} \int_0^h u(0, x_2)\, dx_2 + U_0^3 .$$

Adding the analogous expressions for the remaining S_l we obtain

$$\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}, \omega_0\right) = h^2 \bar{\partial}_1 \bar{\partial}_2 U_0^3 ,$$

which completes the proof.

Defining this time

$$u_\alpha^l = \frac{1}{6}(u_{\alpha - e_l} + 4u_\alpha + u_{\alpha + e_l}), \qquad l = 1, 2,$$

$$u_\alpha^3 = \frac{1}{4}(u_\alpha + u_{\alpha + e_1} + u_{\alpha + e_2} + u_{\alpha + e_1 + e_2}),$$

and again $w_\alpha^l = u_\alpha^l - U_\alpha^l$, $l = 1, 2, 3,$ we find now

$$L_h u_\alpha - h^{-2}(Lu, \omega_\alpha) = - a_{11} \partial_1 \bar{\partial}_1 w_\alpha^2 - a_{22} \partial_2 \bar{\partial}_2 w_\alpha^1 - 2a_{12} \bar{\partial}_1 \bar{\partial}_2 w_\alpha^3 .$$

For the w_α^l the obvious analogue of Lemma 3.3 holds. In fact, the w_α^l depend linearly on u and vanish for u linear so that the result follows using the integral representation with second derivatives of the remainder in Taylor's formula.

The proof of Theorem 2 can now be completed analogously to Theorem 1.

REFERENCES

[1] J. H. Bramble, *On the convergence of difference approximations for second order uniformly elliptic operators*. Numerical Solution of Field Problems in Continuum Physics. SIAM-AMS Proceedings, Vol. 2, Providence R.I. 1970, 201-209.

[2] J. Nitsche, *Lineare Spline-Funktionen und die Methoden von Ritz für elliptische Randwertprobleme*, Arch. Rational Mech. Anal., *36* (1970), 348-355.

[3] L. A. Oganesjan and P. A. Rukhovets, *Investigation of the convergence rate of variational-difference schemes for elliptic second order equations in a two-dimensional domain with a smooth boundary*. Ž. Vyčisl. Mat. i Mat. Fir. *9* (1969), 1102-1120 (Russian). (Translation : U.S.S.R. Comput. Math. and Math. Phys.).

[4] V. Thomée, *Discrete interior Schauder estimates for elliptic difference operators*. SIAM J. Numer. Anal., *5* (1968), 626-645.

[5] V. Thomée, *Approximate solution of Dirichlet's problem using approximating polygonal domains*. Topics in Numerical Analysis. Edited by J. J. H. Miller. Academic Press 1973, 311-328.

[6] V. Thomée and B. Westergren, *Elliptic difference equations and interior regularity*, Numer. Math. *11* (1968), 196-210.