REVUE FRANÇAISE D'AUTOMATIQUE, INFORMATIQUE, RECHERCHE OPÉRATIONNELLE. ANALYSE NUMÉRIQUE

F. BREZZI
L. D. MARINI

On the numerical solution of plate bending problems by hybrid methods

Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique, tome 9, nº 3 (1975), p. 5-50.

<http://www.numdam.org/item?id=M2AN_1975__9_3_5_0>

R.A.I.R.O.
(9e année, décembre 1975, R-3, p. 5-50)

ON THE NUMERICAL SOLUTION OF PLATE BENDING PROBLEMS BY HYBRID METHODS

par F. BREZZI [1] and L. D. MARINI [2]

Communiqué par P.-A. RAVIART

Summary. — *We study the convergence of the « assumed stresses hybrid method » of Pian and Tong for plate bending problems. We also give the error bound for a large class of approximations.*

INTRODUCTION

Let us consider the « model problem » of an homogeneous isotropic thin plate clamped along the entire boundary and acted by an uniformely distributed load p. It is well known that, if Ω is the portion of the (x_1, x_2) plane occupied by the plate, the transversal displacement $w(x_1, x_2)$ of the plate is solution of the following boundary value problem :

$$\text{(P)} \qquad \begin{cases} \Delta^2 w = p & \text{in} \quad \Omega, \\ w = \dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\partial\Omega$ is the boundary of Ω and n is the normal outward direction to $\partial\Omega$. The problem (P) is a classical one, and has been studied from a theoretical and numerical point of view by many authors for a long time (see e.g. [17], [18], [44], but the literature on this subject is quite large). The use of finite element methods has recently contributed new developments to the numerical approach

(1) Università di Pavia e Laboratorio di Analisi Numerica del C.N.R.
(2) Laboratorio di Analisi Numerica del C.N.R. Palazzo dell' Università, Pavia, Italie

to the problem ; presently many different types of approximations by means of finite element methods are used : conforming, non conforming, equilibrium, hybrid, and mixed (see e.g. [9], [11], [19], [33], [42], [46] and the bibliography of these papers). We shall treat here the « assumed stresses hybrid method » due to Pian and Tong (see e.g. [32]). Although this method has been used for quite a few years, the first proof of convergence has been given only in the last year in [7] and only for some particular choice of discretisations. In the present paper we construct a very large family of discretisations, including the classical ones, depending on three parameters (m = degree of the stress field inside each element, r = degree of the displacements at the interelement boundaries, and s = degree of the normal derivatives of the displacements at the interelement boundaries) and we give sufficient conditions on the value of the parameters in order to have convergence (Theorem 3.7.). A bound for the error is also given for each choice of the parameters. Another result of some interest deals with the problem of the search for a « particular solution » f of the equation

$$f_{ij/ij} = p \qquad \text{in each element,}$$

which is needed in order to apply the method. We show that the practical computation of each term of the discrete problem which contains f can be reduced to the computation of integrals of the known function p, times some suitable known polynomials ; therefore a knowledge of f is not really needed. At the end of the paper we also give some results which have been obtained in the numerical experiments performed by the authors using the Honeywell 6030 of the « Centro di Calcoli Numerici dell'Università di Pavia ».

The scheme of the paper is the following.

In paragraph 1 we give a general idea of the assumed stresses hybrid method of Pian and Tong : for any given decomposition $\mathcal{T}_h$ of Ω into convex polygonal subdomains (for the sake of simplicity Ω is supposed to be a convex polygon), problem (P) is transformed into a saddle point problem (P') in which two spaces appear : a space of stresses $V(\mathcal{T}_h)$ defined independently in each element and a space (of Lagrangian multipliers) of displacements $W(\mathcal{T}_h)$ defined (essentially) with their first derivatives at the interelement boundaries. A theorem of existence and uniqueness of problem (P') is given and its solution is related to the solution of (P).

In paragraph 2 we give, at first, an abstract theorem of convergence (Theorem 2.1) for a general discretisation of (P') by means of finite dimensional spaces $V_h \subset V(\mathcal{T}_h)$ and $W_h \subset W(\mathcal{T}_h)$; the convergence is proved if V_h and W_h satisfy an « abstract » hypothesis, H1. After that we introduce a general family of spaces V_h and W_h; sufficient conditions in order that V_h and W_h, constructed in the indicated manner, satisfy H1 are given in Theorem 3.7. The last part of paragraph 2 and the whole of paragraph 3 deal with the different steps which lead to the proof of Theorem 3.7. Others different

sufficient conditions can be obtained by means of the « intermediate steps », i.e. Theorem 2.2, Theorem 3.1 and Lemma 3.1.

In paragraph 4 we study the behaviour (in $|h|$) of the error bounds given by Theorem 2.1, in the case in which V_h and W_h are constructed in the indicated manner. We underline in particular Lemma 4.1, first proved by L. Tartar, which is a generalisation of the Bramble-Hilbert lemma and which can be useful in many other situations. A new proof to this lemma is also reported.

In paragraph 5 we study the discrete problem from the computational point of view. We indicate a procedure which avoids the difficulties connected with a knowledge of the particular solution $\underline{f}$; we show that the linear system of equations which is obtained is « equivalent » (in the sense that we have the same degrees of freedom, the same topological matrix, and, with our procedure on $\underline{f}$, the difficulties for the computation of the « known vector » are of the same type) to a « displacement method », conforming or non conforming, but in a more general context. We can find in this way the « hybrid analogues » of the classical « displacement » approaches. At the end of paragraph 5 we give finally the results obtained in some numerical experiments; in particular, we find (see also e.g. [31]) that different choices of the discrete stresses V_h do not change the « structure » of the final matrix but they can affect the precision.

1. THE HYBRID APPROACH TO THE PROBLEM

Let us consider the problem

$$(1.1)\qquad \begin{cases} \Delta^2 w = p(x_1, x_2) & \text{in} \quad \Omega \\ w = \dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \Gamma = \partial\Omega \end{cases}$$

where Ω is a convex polygon in the (x_1, x_2) plane, $p(x_1, x_2)$ is an element of $L^2(\Omega)$ and n is the direction of the outward normal to Γ. It is well known that the solution $w(x_1, x_2)$ of (1.1) can be regarded as the displacement along the x_1-axis of an elastic uniform isotropic plate which is clamped along the entire boundary Γ and subjected to an uniformly distributed load $p(x_1, x_2)$. Let us introduce the space of stresses

$$(1.2)\qquad \mathcal{S} = \{ \underline{v} \mid v_{ij} \in L^2(\Omega)(i, j = 1, 2), v_{12} = v_{21} \}$$

with the scalar product

$$(1.3)\qquad [\underline{v}, \underline{u}] = \int_\Omega v_{ij} u_{ij} \, dx$$

where (here and in the following) the convention of summation of repeated indices is used. The norm of an element $\underline{v}$ in $\mathcal{S}$ will be noted by

$$\|\underline{v}\|_0 = [\underline{v}, \underline{v}]^{1/2}. \tag{1.4}$$

We define also the space

$$S = \{ \underline{v} \mid \underline{v} \in \mathcal{S}, v_{ij/ij} \in L^2(\Omega) \} \tag{1.5}$$

where (here and in the following) the classical notation $g_{/i} = \dfrac{\partial g}{\partial x_i}$ is used.

From the principle of minimum complementary energy we have that if w is the solution of (1.1) then the tensor $\underline{\sigma}$ with components given by

$$\sigma_{ij} = w_{/ij}, \tag{1.6}$$

minimizes the functional

$$J(\underline{v}) = \frac{1}{2} \int_\Omega v_{ij} v_{ij} \, dx = \frac{1}{2} \|\underline{v}\|_0^2 \tag{1.7}$$

over the manifold

$$S_p = \{ \underline{v} \mid \underline{v} \in S, v_{ij/ij} = p \text{ in } \Omega \}. \tag{1.8}$$

Let us consider now a decomposition $\mathcal{T}_h$ of Ω into convex subdomains; to the decomposition $\mathcal{T}_h$ we associate the space

$$\mathcal{V} = \{ \underline{v} \mid \underline{v} \in \mathcal{S}, v_{ij/ij} \in L^2(K) \text{ for each } K \in \mathcal{T}_h \} \tag{1.9}$$

with the norm

$$\|\underline{v}\|_{\mathcal{V}}^2 = \|\underline{v}\|_0^2 + \sum_{K \in \mathcal{T}_h} \|v_{ij/ij}\|_{L^2(K)}^2 \tag{1.10}$$

and we define the continuous bilinear form on $\mathcal{V} \times H_0^2(\Omega)$ (1) :

$$b(\underline{v}, \varphi) = \sum_{K \in \mathcal{T}_h} \int_K (v_{ij} \varphi_{/ij} - v_{ij/ij} \varphi) \, dx. \tag{1.11}$$

It is easy to verify that if $\underline{v} \in \mathcal{V}$ then

$$b(\underline{v}, \varphi) = 0 \qquad \forall \varphi \in H_0^2(\Omega) \tag{1.12}$$

iff $\underline{v} \in S$. So, introducing the manifold

$$V_p(\mathcal{T}_h) = \{ \underline{v} \mid \underline{v} \in \mathcal{S}, v_{ij/ij} = p \text{ in each } K \in \mathcal{T}_h \} \tag{1.13}$$

we have that S_p can be presented as

$$S_p = \{ \underline{v} \mid \underline{v} \in V_p(\mathcal{T}_h), b(\underline{v}, \varphi) = 0 \qquad \forall \varphi \in H_0^2(\Omega) \}. \tag{1.14}$$

(1) For the definitions of the spaces $H^m(\Omega)$ and $H_0^m(\Omega)$, see e g [28]

The central idea of the stress hybrid method of Pian and Tong [33] is now to minimize $J(\underline{v})$ over $V_p(\mathcal{T}_h)$ requiring (1.12) to be satisfied by the method of Lagrangian multipliers. We are therefore led to consider a problem of the type

$$(1.15)\qquad \begin{cases} \text{Find } (\underline{\sigma}, \psi) \text{ in } V_p(\mathcal{T}_h) \times H_0^2(\Omega), \text{ saddle point of} \\ \mathfrak{L}(\underline{v}, \varphi) = J(\underline{v}) - b(\underline{v}, \varphi). \end{cases}$$

By Green's formula, it can be shown that $b(\underline{v}, \varphi)$ depends only on the values of $\underline{v}$, φ and of their first derivatives along the interelement boundaries. So, no uniqueness for ψ in the problem (1.15) can be expected. It is reasonable, then, to « restrict » the space of Lagrangian multipliers by considering the space

$$(1.16)\qquad W(\mathcal{T}_h) = \{ \varphi \mid \varphi \in H_0^2(\Omega),\ \Delta^2\varphi = 0 \text{ in each } K \in \mathcal{T}_h \},$$

with the norm

$$(1.17)\qquad \|\varphi\|_W^2 = |\varphi|_{2,\Omega}^2 = \int_\Omega \varphi_{/ij}\varphi_{/ij}\, dx.$$

Our problem becomes now :

$$(1.18)\qquad \begin{cases} \text{Find } (\underline{\sigma}, \psi) \text{ in } V_p(\mathcal{T}_h) \times W(\mathcal{T}_h) \text{ saddle point of :} \\ \mathfrak{L}(\underline{v}, \varphi) = J(\underline{v}) - b(\underline{v}, \varphi). \end{cases}$$

Problem (1.18) can be linearized by considering an element $\underline{f}$ of $V_p(\mathcal{T}_h)$ and introducing the space

$$(1.19)\qquad V(\mathcal{T}_h) = V_p(\mathcal{T}_h) - \{ \underline{f} \} = \{ \underline{v} \mid \underline{v} \in \mathcal{S},\ v_{ij/ij} = 0 \text{ in each } K \in \mathcal{T}_h \}.$$

Since $V(\mathcal{T}_h)$ is a closed subspace of $\mathcal{S}$, it will be equipped with the norm

$$(1.20)\qquad \|\underline{v}\|_V = \|\underline{v}\|_0.$$

It will also be convenient for the following to associate to every φ in $W(\mathcal{T}_h)$ a stress tensor $M(\varphi)$ in $V(\mathcal{T}_h)$ defined by

$$(1.21)\qquad M(\varphi)_{ij} = \varphi_{/ij} \qquad (i, j = 1, 2);$$

we have therefore for all φ in $W(\mathcal{T}_h)$:

$$(1.22)\qquad \|\varphi\|_W = \|M(\varphi)\|_V = \|M(\varphi)\|_0.$$

It can be easily verified that (1.18) is equivalent to the following problem :

$$(1.23)\qquad \begin{cases} \textit{find } (u, \psi) \textit{ in } V(\mathcal{T}_h) \times W(\mathcal{T}_h) \textit{ such that :} \\ [\underline{u}, \underline{v}] + [\underline{f}, \underline{v}] - b(\underline{v}, \psi) = 0 \qquad \forall \underline{v} \in V(\mathcal{T}_h), \\ b(\underline{u}, \varphi) + b(\underline{f}, \varphi) = 0 \qquad \forall \varphi \in W(\mathcal{T}_h), \end{cases}$$

in the sense that $(\underline{\sigma}, \psi)$ is a solution of (1.18) iff $(\underline{u}, \psi) = (\underline{\sigma} - \underline{f}, \psi)$ is a solution of (1.23). We shall give now a theorem that characterizes the relations between the solution of (1.23) and the solution of (1.1).

Theorem 1.1. — *Problem* (1.23) *has a unique solution* $(\underline{u}, \psi)$ *which is related to the solution* w *of* (1.1) *by*

$$(1.24)\qquad \begin{cases} u_{ij} + f_{ij} = w_{/ij} & (i, j = 1, 2), \\ \psi = w \text{ on } \Sigma = \bigcup_{K \in \mathcal{T}_h} \partial K, \\ \psi_{/i} = w_{/i} \text{ on } \Sigma & (i = 1, 2). \end{cases}$$

Proof. — Let $(\underline{u}, \psi)$ be a couple in $V(\mathcal{T}_h) \times W(\mathcal{T}_h)$ satisfying conditions (1.24). We have then for all $\underline{v}$ in $V(\mathcal{T}_h)$:

$$(1.25)\qquad [\underline{u}, \underline{v}] + [\underline{f}, \underline{v}] - b(\underline{v}, \psi) = [\underline{u} + \underline{f}, \underline{v}] - b(\underline{v}, \psi) = \\ = \int_\Omega w_{/ij} v_{ij}\, dx - \int_\Omega v_{ij} \psi_{/ij}\, dx = 0.$$

Moreover, for all φ in $W(\mathcal{T}_h)$,

$$(1.26)\qquad b(\underline{u}, \varphi) + b(\underline{f}, \varphi) = b(\underline{u} + \underline{f}, \varphi) = 0$$

since $\underline{u} + \underline{f}$ belongs to S and $\varphi \in H_0^2(\Omega)$.

Therefore $(\underline{u}, \psi)$ is a solution of (1.23); let now $(\underline{u}^*, \psi^*)$ be another solution of (1.23); with classical arguments we have immediately that $\underline{u}^* = \underline{u}$, and therefore

$$(1.27)\qquad b(\underline{v}, \psi - \psi^*) = 0 \qquad \forall \underline{v} \in V(\mathcal{T}_h).$$

Hence by taking $\underline{v} = M(\psi - \psi^*)$ in (1.27) we have

$$(1.28)\qquad b(M(\psi - \psi^*), \psi - \psi^*) = \int_\Omega (\psi - \psi^*)_{/ij} (\psi - \psi^*)_{/ij}\, dx = 0$$

and then $\psi = \psi^*$. So (1.23) has a unique solution and the proof is complete.

REMARK 1.1. — Existence and uniqueness of the solution of (1.23) follows immediately also from the abstract results of Brezzi [8]. In fact it is sufficient to observe that for all φ in $W(\mathcal{T}_h)$ $(\varphi \neq 0)$:

$$\operatorname*{Sup}_{\underline{v} \in V(\mathcal{T}_h) - \{0\}} \|\underline{v}\|_0^{-1} b(\underline{v}, \varphi) \geqslant \|M(\varphi)\|_0^{-1} b(M(\varphi), \varphi) = \|M(\varphi)\|_0 = \|\varphi\|_W.$$

2. NUMERICAL APPROXIMATION

Let us now consider a sequence $\{ \mathcal{T}_h \}_h$ of decompositions of Ω into convex subdomains, and let, for any decomposition $\mathcal{T}_h$, V_h and W_h be closed subspace of $V(\mathcal{T}_h)$ and $W(\mathcal{T}_h)$ respectively. We consider the approximate problem :

$$(2.1)\qquad \begin{cases} \textit{find } (\underline{u}_h, \psi_h) \textit{ in } V_h \times W_h \textit{ such that :} \\ [\underline{u}_h + \underline{f}, \underline{v}_h] - b(\underline{v}_h, \psi_h) = 0 & \forall \underline{v}_h \in V_h, \\ b(\underline{u}_h + \underline{f}, \varphi_h) = 0 & \forall \varphi_h \in W_h. \end{cases}$$

Suppose that V_h and W_h satisfy the following hypothesis :

H1. — *There exists a positive constant* $\gamma > 0$, *independent of the decomposition such that :*

$$\text{(2.2)} \qquad \operatorname*{Sup}_{\underline{v} \in V_h - \{0\}} \|\underline{v}\|_0^{-1} b(\underline{v}, \varphi) \geqslant \gamma \, \|\varphi\|_W \qquad \forall \varphi \in W_h.$$

Then, from the abstract theory of [8], we get the following result.

Theorem 2.1. — *If* H1 *is satisfied,* (2.1) *has a unique solution* $(\underline{u}_h, \psi_h)$; *moreover if* $(\underline{u}, \psi)$ *is the solution of* (1.23) *then :*

$$\text{(2.3)} \qquad \|\underline{u} - \underline{u}_h\|_0 + \|\psi - \psi_h\|_W \leqslant c(\operatorname*{Inf}_{\underline{v} \in V_h} \|\underline{u} - \underline{v}\|_0 + \operatorname*{Inf}_{\varphi \in W_h} \|\psi - \varphi\|_W),$$

where c is a constant independent of the decomposition.

We shall give in the following some general examples of spaces V_h, W_h which satisfy H1. First of all, suppose that the sequence $\{ \mathcal{T}_h \}_h$ verifies the following conditions.

c1) There exists a convex polygon $\hat{K}$ such that for every $\mathcal{T}_h$ and for every K in $\mathcal{T}_h$ we can find a transform F which maps $\hat{K}$ onto K, of the type :

$$\text{(2.4)} \qquad x = F\hat{x} = B\hat{x} + \xi$$

where B is a 2×2 non singular matrix and ξ is a vector in $\mathbb{R}^2$.

c2) There exists two positive constants σ_1, σ_2, independent of $\mathcal{T}_h$ and of K, such that :

$$\text{(2.5)} \qquad \|B\| \leqslant \sigma_1 \, |h|, \; \|B^{-1}\| \leqslant \sigma_2 \, |h|^{-1}$$

where $|h|$ is defined, for each decomposition $\mathcal{T}_h$, as

$$\text{(2.6)} \qquad |h| = \max_{K \in \mathcal{T}_h} \{ \text{diameter of } K \}.$$

Let now $\hat{V}$ be a finite dimensional space of smooth symmetric tensors $\underline{v}$, defined on $\hat{K}$ and self-equilibrating, in the sense that :

$$\text{(2.7)} \qquad v_{ij/ij} = 0 \text{ in } \hat{K};$$

for any given K in $\mathcal{T}_h$ we define the space

$$\text{(2.8)} \qquad V(\hat{V}, K) = \{ \underline{v} \,|\, \exists \hat{\underline{v}} \in \hat{V} ; \; v_{ij} = (\hat{v}_{rs} o F) b_{ir} b_{js} \}$$

where F is the transform of the type (2.4) which maps $\hat{K}$ on K and b_{lm} are the coefficients of the matrix B. We have obviously from (2.7) and (2.8) that, for every $\underline{v}$ in $V(\hat{V}, K)$,

$$\text{(2.9)} \qquad v_{ij/ij} = 0 \text{ on } K.$$

Therefore we can define

$$V(\hat{V}, \mathcal{T}_h) = \{ \underline{v} \,|\, \underline{v} \in \mathcal{S},\ v_{|K} \in V(\hat{V}, K)\ \forall K \in \mathcal{T}_h \} \tag{2.10}$$

and we obtain that $V(\hat{V}, \mathcal{T}_h)$ is a closed subspace of $V(\mathcal{T}_h)$.

We define now, for every decomposition $\mathcal{T}_h$ and for every pair (r, s) of integers such that $r \geqslant 3$, $s \geqslant 1$, the space $W(r, s, \mathcal{T}_h)$ as

(2.11)

$$W(r, s, \mathcal{T}_h) = \left\{ \varphi \mid \varphi \in W(\mathcal{T}_h), \varphi_{|\partial K} \in P_r(\partial K), \frac{\partial \varphi}{\partial n}_{|\partial K} \in P_s(\partial K)\, \forall K \in \mathcal{T}_h \right\},$$

where $P_m(\partial K)$ indicates, for every integer m, the space of functions defined on ∂K which are polynomials of degree $\leqslant m$ on each side (and not necessarly continuous). We observe that the condition « $\varphi \in W(\mathcal{T}_h)$ » implies that the functions φ, $\varphi_{/1}$, $\varphi_{/2}$ are continuous, and this justifies the requirements $r \geqslant 3$, $s \geqslant 1$.

We want now to prove some sufficient conditions on $\hat{V}$, r, s such that the spaces $V(\hat{V}, \mathcal{T}_h)$ and $W(r, s, \mathcal{T}_h)$ verify condition H1. For the sake of simplicity we will examine, as possible choice for $\hat{K}$, only the classical cases $\hat{K}$ = triangle (and then each K will be a triangle) and $\hat{K}$ = unit square (and then each K will be a parallelogram).

First of all, we remark that each transform F of the type (2.4) can be decomposed in a finite number of ways as the product of :

i) a transform of type :

$$\begin{aligned} \bar{x}_1 &= \beta_{11} x_1, \\ \bar{x}_2 &= \beta_{21} x_1 + x_2; \end{aligned} \tag{2.12}$$

ii) a contraction :

$$\begin{aligned} \bar{\bar{x}}_1 &= \rho \bar{x}_1, \\ \bar{\bar{x}}_2 &= \rho x_2; \end{aligned} \tag{2.13}$$

iii) a rigid displacement :

$$(x_1, x_2) = R(\bar{\bar{x}}_1, \bar{\bar{x}}_2). \tag{2.14}$$

Moreover, condition c2) guarantees that there exist four positive constants ρ_1, ρ_2, ρ_3, ρ_4, independents of K and of $\mathcal{T}_h$ such that :

$$\rho_1 |h| \leqslant \rho \leqslant \rho_2 |h|, \tag{2.15}$$

$$\beta_{11} \geqslant \rho_3, \tag{2.16}$$

$$\beta_{11}^2 + \beta_{21}^2 \leqslant \rho_4. \tag{2.17}$$

From the decomposition i), ii), iii) we obtain that each function φ such that $\Delta^2\varphi = 0$ in K is transformed by F into a function

$$\hat{\varphi} = \varphi \circ F \tag{2.18}$$

such that

$$A\hat{\varphi} = 0 \text{ in } \hat{K} \tag{2.19}$$

where A is a fourth order elliptic operator given by

$$A = \left[\left(\frac{\partial}{\partial x_1} - \beta_{21}\frac{\partial}{\partial x_2}\right)^2 + \left(\beta_{11}\frac{\partial}{\partial x_2}\right)^2\right]^2. \tag{2.20}$$

We will now prove some technical lemmas. For this, first of all, we define the operator

$$\vec{\gamma} : H^2(\hat{K}) \to (L^2(\partial\hat{K}))^3 \tag{2.21}$$

by

$$\vec{\gamma}(\varphi) = (\varphi, \varphi_{/1}, \varphi_{/2})_{\partial\hat{K}}, \tag{2.22}$$

and the space $T(\partial\hat{K}) = \vec{\gamma}(H^2(\hat{K}))$ (1). We define also, for each A of type (2.20) the operator $G_A : T(\partial\hat{K}) \to H^2(\hat{K})$ as follows :

$$\psi = G_A\vec{\xi} \Leftrightarrow \begin{cases} A\psi = 0 \text{ in } \hat{K} \\ \vec{\gamma}\psi = \vec{\xi}. \end{cases} \tag{2.23}$$

Finally we consider the space

$$\hat{L} = \{ \underline{v} \mid v_{ij} \in L^2(\hat{K})\ (i, j = 1, 2),\ v_{12} = v_{21} \} \tag{2.24}$$

with the norm

$$\|\underline{v}\|^2_{\hat{L}} = \|\underline{v}\|^2_{\Lambda} = \sum_{i,j=1}^{2} \|v_{ij}\|^2_{L^2(\hat{K})}, \tag{2.25}$$

and the operator $\hat{M}$ from $H^2(\hat{K})$ into $\hat{L}$ defined as

$$\underline{v} = \hat{M}\varphi \Leftrightarrow v_{ij} = \varphi_{/ij}. \tag{2.26}$$

Lemma 2.1. — *Let* $\{ A_\lambda \}_{\lambda\in I}$ *be a family of operators of type :*

$$A_\lambda = \left(\left(\frac{\partial}{\partial x_1} - \beta_{21}(\lambda)\frac{\partial}{\partial x_2}\right)^2 + \left(\beta_{11}(\lambda)\frac{\partial}{\partial x_2}\right)^2\right)^2, \tag{2.27}$$

---

(1) $T(\partial\hat{K})$ will be equipped with the (natural) norm :

$$\|\vec{\xi}\|_{T(\partial\hat{K})} = \operatorname*{Inf}_{\substack{\varphi\in H^2(K) \\ \vec{\gamma}\varphi = \vec{\xi}}} \|\varphi\|_{H^2(K)}.$$

with the conditions

$$\beta_{11}(\lambda) \geqslant \bar{\beta} \qquad \forall \lambda \in I, \tag{2 28}$$

$$\beta_{11}^2 + \beta_{21}^2 \leqslant \bar{\bar{\beta}} \qquad \forall \lambda \in I \tag{2 29}$$

Then there exists a constant α, *independent of* λ, *such that, for all* $\vec{\xi} \in T(\partial \hat{K})$, *we have*

$$\| \hat{M} G_{A_\lambda} \vec{\xi} \|_\Lambda \leqslant \alpha \, \| \hat{M} G_{\Delta^2} \vec{\xi} \|_\Lambda \tag{2 30}$$

Proof — Let, for all $\vec{\xi}$ in $T(\partial \hat{K})$,

$$\zeta = G_{\Delta^2} \vec{\xi}, \qquad \psi = G_{A_\lambda} \vec{\xi}, \qquad \bar{\psi} = \zeta - \psi \tag{2 31}$$

We have immediately that

$$\begin{cases} A_\lambda \bar{\psi} = A_\lambda \zeta, \\ \vec{\gamma} \bar{\psi} = 0 \end{cases} \tag{2 32}$$

Therefore, if $a_\lambda(u, v)$ is the continuous bilinear form associated with the operator A_λ, we have that $\bar{\psi}$ is solution of the problem

$$a_\lambda(\bar{\psi}, \varphi) = a_\lambda(\zeta, \varphi) \qquad \forall \varphi \in H_0^2(\hat{K}) \tag{2 33}$$

Moreover, from (2 28), (2 29), setting

$$|\varphi|_{2,K} = \| \hat{M} \varphi \|_\Lambda, \qquad \varphi \in H^2(\hat{K}), \tag{2 34}$$

we get immediately that there exist two positive constants α_1, α_2, independent of λ, such that

$$a_\lambda(\varphi, \varphi) \geqslant \alpha_1 \, |\varphi|_{2\,\hat{K}}^2 \qquad \forall \varphi \in H_0^2(\hat{K}) \tag{2 35}$$

$$a_\lambda(\varphi_1, \varphi_2) \leqslant \alpha_2 \, |\varphi_1|_{2\,\hat{K}} \, |\varphi_2|_{2,\hat{K}} \qquad \forall \varphi_1, \varphi_2 \in H^2(\hat{K}) \tag{2 36}$$

So from (2 33), (2 35), (2 36) we get

$$|\psi|_{2,\hat{K}} \leqslant \alpha_1^{-1} \alpha_2 \, |\zeta|_{2\,\hat{K}} \tag{2 37}$$

and from (2 31), (2 37)

$$|\bar{\psi}|_{2,\hat{K}} \leqslant (1 + \alpha_1^{-1} \alpha_2) \, |\zeta|_{2\,\hat{K}} \tag{2 38}$$

We have then, for all $\vec{\xi} \in T(\partial \hat{K})$, that

$$\| M G_{A_\lambda} \vec{\xi} \|_\Lambda \leqslant (1 + \alpha_1^{-1} \alpha_2) \, \| M G_{\Delta^2} \vec{\xi} \|_\Lambda \tag{2 39}$$

which proves (2 30) with $\alpha = (1 + \alpha_1^{-1} \alpha_2)$

Lemma 2.2. — *Let* $\{ A_\lambda \}_{\lambda \in \Lambda}$ *be a family of operators which satisfy* (2 27), (2 28), (2 29), *let* $\hat{\mathfrak{W}}$ *be a closed cone belonging to a finite dimensional subspace of* $T(\partial \hat{K})$, *with the property that*

$$\forall \vec{\xi} \in \hat{\mathfrak{W}}, \forall \vec{\xi}_1 \in \mathcal{N}, \qquad \vec{\xi} + \vec{\xi}_1 \in \hat{\mathfrak{W}}, \tag{2 40}$$

where $\mathcal{N}$ is the space of traces of polynomials of degree $\leqslant 1$ in $\hat{K}$, that is :

$$\mathcal{N} = \vec{\gamma}(P_1(\hat{K})) = \{ \vec{\xi} \mid \vec{\xi} \in T(\partial\hat{K}),\ \hat{M}G_{\Delta^2}\vec{\xi} = 0 \}. \tag{2.41}$$

Let finally $\hat{V}$ be a finite dimensional subspace of $\hat{L}$ of smooth self-equilibrating tensors, and suppose that the following condition holds :

$$\text{H1}\,(\hat{V}, \hat{\mathfrak{W}}) \quad \begin{cases} \forall \vec{\xi} \in \hat{\mathfrak{W}}, \\ \textit{if} \displaystyle\int_{\hat{K}} (G_{\Delta^2}\vec{\xi})_{/ij} v_{ij}\, d\hat{x} = 0 \qquad \forall \underline{v} \in \hat{V} \\ \textit{then}\ \|\hat{M}G_{\Delta^2}\vec{\xi}\|_\Lambda = 0. \end{cases}$$

Then there exists a positive constant $\hat{\alpha}$, depending only on $\hat{V}$ and $\hat{\mathfrak{W}}$ such that .

$$\operatorname*{Sup}_{\underline{v} \in \hat{V} - \{0\}} \|\underline{v}\|_\Lambda^{-1} \int_{\hat{K}} (G_{A_\lambda}\vec{\xi})_{/ij} v_{ij}\, d\hat{x} \geqslant \hat{\alpha}\, \|\hat{M}G_{A_\lambda}\vec{\xi}\|_\Lambda, \tag{2.42}$$

for all $\vec{\xi}$ in $\hat{\mathfrak{W}}$.

Proof. — First of all we remark that the value of

$$\int_{\hat{K}} (G_{A_\lambda}\vec{\xi})_{/ij} v_{ij}\, d\hat{x} \tag{2.43}$$

is independent of λ; in fact, for all φ in $H^2(\hat{K})$ and for all smooth self equilibrating tensor $\underline{v}$ in $\hat{L}$ we have Green's formula :

$$\int_{\hat{K}} \varphi_{/ij} v_{ij}\, dx = \int_{\partial\hat{K}} (\varphi_{/i} v_{ij} \hat{n}_j - \varphi v_{ij/j} \hat{n}_i)\, d\hat{l}, \tag{2.44}$$

where $\underline{\hat{n}}$ is the outward normal direction to $\partial\hat{K}$ and $d\hat{l}$ is the elementary part of $\partial\hat{K}$. Therefore the value of (2.43) depends only on $\vec{\xi}$ and $\underline{v}$ and we can set :

$$\beta(\underline{v}, \vec{\xi}) = \int_{\hat{K}} (G_{\Delta^2}\vec{\xi})_{/ij} v_{ij}\, d\hat{x}. \tag{2.45}$$

Hence, from lemma 2.1, it is sufficient to show that there exists a positive constant $\bar{\alpha}$ such that, for all $\vec{\xi}$ in $\hat{\mathfrak{W}}$, we have :

$$\operatorname*{Sup}_{\underline{v} \in \hat{V} - \{0\}} \|\underline{v}\|_\Lambda^{-1} \beta(\underline{v}, \vec{\xi}) \geqslant \bar{\alpha}\, \|\hat{M}G_{\Delta^2}\vec{\xi}\|_\Lambda. \tag{2.46}$$

For this, we remark that, for all $\vec{\xi}$ in $\hat{\mathfrak{W}}$ such that $\|\hat{M}G_{\Delta^2}\vec{\xi}\|_\Lambda \neq 0$, we have from H1 $(\hat{V}, \hat{\mathfrak{W}})$ that :

$$\operatorname*{Sup}_{\underline{v} \in \hat{V} - \{0\}} \|\underline{v}\|_\Lambda^{-1} \beta(\underline{v}, \vec{\xi}) = \alpha(\vec{\xi}) > 0. \tag{2.47}$$

Let us now define the sets

$$\hat{S} = \{ \vec{\xi} \mid \vec{\xi} \in \hat{\mathfrak{W}}, \|\hat{M}G_{\Delta^2}\vec{\xi}\|_\Lambda = 1 \} \tag{2.48}$$

and

$$\hat{D} = \{ \vec{\xi} \mid \vec{\xi} \in \hat{S}, \vec{\xi} \text{ is orthogonal to } \mathcal{N} \}; \tag{2.49}$$

we note that $\hat{D}$ is a compact set, since $\hat{\mathfrak{W}}$ is closed. We also note that, from (2.47) and from the continuity of $\beta(\underline{v}, \vec{\xi})$, $\alpha(\vec{\xi})$ is continuous, and therefore we have :

$$\operatorname*{Inf}_{\vec{\xi} \in \hat{D}} \alpha(\vec{\xi}) \geqslant \bar{\alpha} > 0. \tag{2.50}$$

Let now $\vec{\xi}$ be an element of $\hat{\mathfrak{W}}$ and let

$$\rho = \|\hat{M}G_{\Delta^2}\vec{\xi}\|_\Lambda. \tag{2.51}$$

If $\rho = 0$ then (2.46) holds; if $\rho \neq 0$ we have that

$$\vec{\xi}' = \frac{\vec{\xi}}{\rho} \in \hat{S} \tag{2.52}$$

since $\hat{\mathfrak{W}}$ is a cone. Let now $\vec{\xi}_1$ be the projection of $\vec{\xi}'$ on $\mathcal{N}$; we have that $\vec{\xi}' - \vec{\xi}_1 \in \hat{D}$ from (2.40), and moreover :

$$\begin{aligned} \operatorname*{Sup}_{\underline{v} \in \hat{V} - \{0\}} \|\underline{v}\|_\Lambda^{-1} \beta(\underline{v}, \vec{\xi}) &= \rho \operatorname*{Sup}_{\underline{v} \in \hat{V} - \{0\}} \|\underline{v}\|_\Lambda^{-1} \beta(v, \vec{\xi}') = \rho \\ &\operatorname*{Sup}_{\underline{v} \in \hat{V} - \{0\}} \|\underline{v}\|_\Lambda^{-1} \beta(\underline{v}, \vec{\xi}' - \vec{\xi}_1) \geqslant \rho\bar{\alpha} = \bar{\alpha} \|\hat{M}G_{\Delta^2}\vec{\xi}\|_\Lambda. \end{aligned} \tag{2.53}$$

Therefore (2.46) holds and the proof is complete.

The following theorem gives us a connection between hypothesis H1 $(\hat{V}, \hat{\mathfrak{W}})$ and hypothesis H1.

Theorem 2.2. — *Let $V(\hat{V}, \mathcal{T}_h)$ and $W(r, s, \mathcal{T}_h)$ be constructed as previously stated. For all $\mathcal{T}_h$ and for all K in $\mathcal{T}_h$ we construct the spaces :*

$$W(K, \mathcal{T}_h) = \{ \hat{\varphi} \mid \exists \varphi \in W(r, s, \mathcal{T}_h), \hat{\varphi} = (\varphi_{|K}) \circ F \}, \tag{2.54}$$

$$\mathfrak{W}(K, \mathcal{T}_h) = \vec{\gamma}(W(K, \mathcal{T}_h)), \tag{2.55}$$

and we define :

$$\hat{\mathfrak{W}} = \text{closure in } T(\partial\hat{K}) \text{ of } \bigcup_{\mathcal{T}_h} \bigcup_{K \in \mathcal{T}_h} \mathfrak{W}(K, \mathcal{T}_h). \tag{2.56}$$

Then, if H1 $(\hat{V}, \hat{\mathfrak{W}})$ *holds, the spaces $V(\hat{V}, \mathcal{T}_h)$ and $W(r, s, \mathcal{T}_h)$ verify hypothesis* H1.

Proof. — Given a decomposition $\mathcal{T}_h$, let $\varphi \in W(r, s, \mathcal{T}_h)$; for every K in $\mathcal{T}_h$ we define

$$\hat{\varphi} = (\varphi_{|K}) \circ F; \tag{2.57}$$

we have obviously $\hat{\varphi} \in W(K, \mathcal{T}_h)$; moreover we have

$$A_\lambda \hat{\varphi} = 0 \tag{2.58}$$

for some operator A_λ of the type (2.27)-(2.29). Hence

$$\vec{\gamma}\hat{\varphi} \in \mathfrak{W}(K, \mathcal{T}_h) \subseteq \hat{\mathfrak{W}} \tag{2.59}$$

and

$$\hat{\varphi} = G_{A_\lambda}(\vec{\gamma}\hat{\varphi}). \tag{2.60}$$

Then from lemma 2.2 there exists a $\underline{\hat{v}}$ in $\hat{V}$ such that

$$\left\{ \begin{array}{l} \displaystyle\int_{\hat{K}} \hat{\varphi}_{/ij}\hat{v}_{ij}\, d\hat{x} \geqslant \hat{\alpha}\, |\hat{\varphi}|^2_{2,\hat{K}}, \\ \|\underline{\hat{v}}\|^2_\Lambda \leqslant |\hat{\varphi}|_{2,\hat{K}}. \end{array} \right. \tag{2.61}$$

We define now an element $\underline{v} = \underline{v}(K)$ in $V(\hat{V}, K)$ by

$$v_{ij} = (\hat{v}_{rs} \circ F) b_{ir} b_{js}, \tag{2.62}$$

and we obtain

$$\int_K \varphi_{/ij} v_{ij}\, dx = \int_{\hat{K}} \hat{\varphi}_{/ij}\hat{v}_{ij}\, |\det B|\, d\hat{x} \geqslant \hat{\alpha}\, |\hat{\varphi}|^2_{2,\hat{K}}\, |\det B| \geqslant \geqslant \hat{\alpha}\, |\varphi|^2_{2,K}\, \|B^{-1}\|^{-4}, \tag{2.63}$$

$$\begin{aligned} \int_K v_{ij} v_{ij}\, dx &= \int_{\hat{K}} \hat{v}_{rs}\hat{v}_{lm} b_{ir} b_{js} b_{il} b_{jm}\, |\det B|\, d\hat{x} \leqslant \\ &\leqslant \|\underline{\hat{v}}\|^2_\Lambda\, \|B\|^4\, |\det B| \leqslant |\hat{\varphi}|^2_{2,\hat{K}}\, \|B\|^4\, |\det B| \leqslant \\ &\leqslant |\varphi|^2_{2,K}\, \|B\|^8. \end{aligned} \tag{2.64}$$

Let now $\underline{v}$ be the element of $V(\hat{V}, \mathcal{T}_h)$ defined by :

$$\underline{v}_{|K} = \underline{v}(K) \qquad \forall K \in \mathcal{T}_h; \tag{2.65}$$

we have :

$$b(\underline{v}, \varphi) = \sum_{K \in \mathcal{T}_h} \int_K \varphi_{/ij} v_{ij}\, dx \geqslant \sum_{K \in \mathcal{T}_h} \hat{\alpha}\, |\varphi|^2_{2,K}\, \|B^{-1}\|^{-4} \geqslant \hat{\alpha}\, |\varphi|^2_{2,\Omega} \sigma_2^{-4}\, |h|^4 \tag{2.66}$$

$$\|\underline{v}\|^2_0 = \sum_{K \in \mathcal{T}_h} \int_K v_{ij} v_{ij}\, dx \leqslant \sum_{K \in \mathcal{T}_h} |\varphi|^2_{2,K}\, \|B\|^8 \leqslant |\varphi|^2_{2,\Omega} \sigma_1^8\, |h|^8. \tag{2.67}$$

Hence :

$$\frac{b(\underline{v}, \varphi)}{\|\underline{v}\|_0} \geqslant \hat{\alpha} \sigma_2^{-4} \sigma_1^4\, \|\varphi\|_W, \tag{2.68}$$

and therefore H1 holds with $\gamma = \hat{\alpha} \sigma_2^{-4} \sigma_1^4$.

We characterize now the closed cone $\hat{\mathfrak{W}}$, or, more generally, some closed cone containing $\hat{\mathfrak{W}}$, for different choices of the parameters r, s and of $\hat{K}$.

Proposition 2.1. — *Let $\hat{K}$ be a triangle or the unit square. Then if $s \geqslant r - 1$, $\hat{\mathfrak{W}}$ is contained in the space $T(r, s, K)$ of triplets $(\varphi, \varphi_{/1}, \varphi_{/2})$ which are continuous on ∂K and such that :*

$$\varphi \in P_r(\partial \hat{K}) \tag{2.69}$$

$$\varphi_{/i} \in P_s(\partial \hat{K}) \qquad i = 1, 2. \tag{2.70}$$

The proof is immediate.

Proposition 2.2. — *Let $\hat{K}$ be a triangle and suppose that $s \leqslant r - 1$. Then $\hat{\mathfrak{W}}$ is contained in the cone $TC(r, s, \vartheta)$ of triplets of functions $(\varphi, \varphi_{/1}, \varphi_{/2})$ that are continuous on $\partial \hat{K}$ and such that the following conditions are satisfied :*

i) $\varphi \in P_r(\partial \hat{K})$,

ii) $\varphi_{/i} \in P_{r-1}(\partial \hat{K}) \qquad (i = 1, 2)$,

iii) *there exist three directions $\underline{\tilde{n}}^{(1)}, \underline{\tilde{n}}^{(2)}, \underline{\tilde{n}}^{(3)}$ such that :*

$$\begin{cases} \dfrac{\partial \varphi}{\underline{\tilde{n}}^{(i)}} \text{ is of degree } \leqslant s \text{ on } L_i, \\ |\underline{\tilde{n}}^{(i)} \cdot \underline{\tilde{n}}^{(j)}| \leqslant \vartheta < 1 \quad i \neq j, \end{cases} \tag{2.71}$$

where L_1, L_2, L_3 are the edges of $\hat{K}$, and ϑ depends on the constants σ_1 and σ_2 which appear in (2.5).

The proof is immediate, if one consideres $\underline{\tilde{n}}^{(1)}, \underline{\tilde{n}}^{(2)}, \underline{\tilde{n}}^{(3)}$ as the images by F of the normal directions $\underline{n}^{(1)}, \underline{n}^{(2)}, \underline{n}^{(3)}$ to the edges of K (of course $\tilde{n}^{(i)}$ depend on φ) and observing that, since F transforms parallel directions into parallel directions, we have $|\underline{\tilde{n}}^{(i)} \cdot \underline{\tilde{n}}^{(j)}| < 1$, and then, if K is « not too flat » (condition c2)), we have : $|\underline{\tilde{n}}^{(i)} \cdot \tilde{n}^{(j)}| \leqslant \vartheta < 1$ for $i \neq j$.

Proposition 2.3. — *Let $\hat{K}$ be the unit square and suppose that $s < r - 1$. Then $\hat{\mathfrak{W}}$ is contained in the cone $TCS(r, s, \vartheta)$ of triplets of functions $(\varphi, \varphi_{/1}, \varphi_{/2})$ that are continuous on $\partial \hat{K}$ and satisfy the following conditions :*

i) $\varphi \in P_r(\partial \hat{K})$,

ii) $\varphi_{/i} \in P_{r-1}(\partial \hat{K}) \qquad (i = 1, 2)$,

iii') *there exist two directions $\underline{\tilde{n}}^{(1)}, \underline{\tilde{n}}^{(2)}$ such that :*

$$\begin{aligned} &\frac{\partial \varphi}{\partial \underline{\tilde{n}}^{(1)}} \text{ is of degree } \leqslant s \text{ on the sides } x_1 = \text{const}, \\ &\frac{\partial \varphi}{\partial \underline{\tilde{n}}^{(2)}} \text{ is of degree } \leqslant s \text{ on the sides } x_2 = \text{const}, \\ &|\underline{\tilde{n}}^{(1)} \cdot \underline{\tilde{n}}^{(2)}| \leqslant \vartheta < 1 \end{aligned} \tag{2.72}$$

where ϑ *depends on the constants* σ_1 *and* σ_2 *which appear in* (2.5).

In the following paragraph we shall give some examples of choices of $\hat{V}$, r, s, $\hat{K}$ such that H1 $(\hat{V}, \hat{\mathfrak{W}})$ is satisfied.

3. EXAMPLES

First of all, we introduce the Southwell functions U, V which are associated with a self-equilibrating stress field v by the formulas

$$U_{/2} = v_{11}, \qquad -\frac{1}{2}(U_{/1} + V_{/2}) = v_{12} = v_{21}, \qquad V_{/1} = v_{22}, \tag{3.1}$$

and we consider, for φ and $\underline{v}$ sufficiently smooth, the formula :

$$\beta(\underline{v}, \varphi) = \int_{\hat{K}} \varphi_{/ij} v_{ij} \, \mathrm{d}\hat{x}; \tag{3.2}$$

by substituting (3.1) in (3.2) we have :

$$\beta(\underline{v}, \varphi) = \int_{\hat{K}} (\varphi_{/11} U_{/2} - \varphi_{/12} U_{/1} - \varphi_{/21} V_{/2} + \varphi_{/22} V_{/1}) \, \mathrm{d}\hat{x}. \tag{3.3}$$

On the other hand, it is well known that if f and g are sufficiently smooth functions (say, if f and g belong to $H^2(\hat{K})$) then we have :

$$\int_{\hat{K}} (f_{/1} g_{/2} - f_{/2} g_{/1}) \, \mathrm{d}\hat{x} = \int_{\partial\hat{K}} f \frac{\partial g}{\partial s} \, \mathrm{d}\hat{l} = - \int_{\partial\hat{K}} \frac{\partial f}{\partial s} g \, \mathrm{d}\hat{l} \tag{3.4}$$

where s is the tangent direction to $\partial\hat{K}$.

Therefore from (3.3) and (3.4) we have :

$$\beta(\underline{v}, \varphi) = \int_{\partial\hat{K}} \left(\varphi_{/1} \frac{\partial U}{\partial s} - \varphi_{/2} \frac{\partial V}{\partial s} \right) \mathrm{d}\hat{l}. \tag{3.5}$$

We note now that the couple (U, V) associated with a given stress field $\underline{v}$ is not unique, but if (U, V) and $(\overline{U}, \overline{V})$ are associated to the same $\underline{v}$, then

$$U = \overline{U} + \alpha x_1 + \beta \tag{3.6}$$

$$V = \overline{V} - \alpha x_2 + \gamma \tag{3.7}$$

with α, β, γ constants; therefore

$$\int_{\partial\hat{K}} \left(\varphi_{/1} \frac{\partial U}{\partial s} - \varphi_{/2} \frac{\partial V}{\partial s} \right) \mathrm{d}\hat{l} = \int_{\partial\hat{K}} \left(\varphi_{/1} \frac{\partial \overline{U}}{\partial s} - \varphi_{/2} \frac{\partial \overline{V}}{\partial s} \right) \mathrm{d}\hat{l} \tag{3.8}$$

for each φ in $H^2(\hat{K})$.

We consider now, for each space $\hat{V}$ of smooth symmetrical and self-equilibrating tensors in $\hat{K}$, the space $SF(\hat{V})$ of the traces on $\partial\hat{K}$ of all Southwell functions which are associated to some element $\underline{v}$ in $\hat{V}$. From the previous considerations we obtain immediately the following theorem.

Theorem 3.1. — *Let $\hat{V}$ be a space of smooth symmetric self-equilibrating tensors in $\hat{K}$ and let $\hat{\mathfrak{W}}$ be a closed cone in $T(\partial\hat{K})$. Then* H1 $(\hat{V}, \hat{\mathfrak{W}})$ *is satisfied iff the following condition holds :*

$$C(\hat{V}, \hat{\mathfrak{W}}) \quad \begin{cases} \text{If } \vec{\xi} = (\varphi, \varphi_{/1}, \varphi_{/2}) \in \hat{\mathfrak{W}} \text{ and} \\ \displaystyle\int_{\partial\hat{K}} \varphi_{/i} \frac{\partial Z_i}{\partial s} \, \mathrm{d}\hat{l} = 0 \qquad \forall (Z_1, Z_2) \in SF(\hat{V}), \\ \text{then } \varphi_{/i} \text{ is constant } (i = 1, 2). \end{cases}$$

The proof is immediate since condition

$$\varphi_{/i} = \text{constant} \qquad (i = 1, 2) \tag{3.9}$$

is equivalent to

$$\| \hat{M} G_{\Delta^2}(\varphi, \varphi_{/1}, \varphi_{/2}) \|_{\Lambda} = 0 \tag{3.10}$$

for all $(\varphi, \varphi_{/1}, \varphi_{/2})$ in $T(\partial\hat{K})$.

We will now prove some technical lemmas.

Lemma 3.1. — *Let K be a (convex) polygon with edges $L_1, L_2, \ldots, L_k$. Let, for any integer $m \geqslant 1$,*

$$\tilde{P}_m(\partial K)$$

be the set of all continuous functions defined on ∂K which are polynomials of degree $\leqslant m$ on each side of K. Then if φ is an element of $\tilde{P}_m(\partial K)$ such that

$$\int_{\partial K} \varphi \frac{\partial p}{\partial s} \, \mathrm{d}l = 0 \qquad \forall p \in \tilde{P}_m(\partial K) \tag{3.11}$$

we get the existence of $(k + 1)$ constants $c, c_1, \ldots, c_k$ such that :

$$\varphi = c_i l_i^{(m)} + c \qquad \text{on} \qquad L_i \qquad (i = 1, \ldots, k), \tag{3.12}$$

where $l_i^{(m)}$ is the (normalized) Legendre polynomial of degree m on L_i.

Proof. — First of all we remark that if $p_1, p_2, \ldots, p_k$ are k polynomials of degree $\leqslant m - 1$ defined respectively on $L_1, \ldots, L_k$ and such that

$$\sum_{i=1}^{k} \int_{L_i} p_i \, \mathrm{d}l_i = 0, \tag{3.13}$$

then there exists a $p \in \tilde{P}_m(\partial K)$ such that :

$$\frac{\partial p}{\partial s} = p_i \text{ on } L_i \qquad (i = 1, ..., k), \tag{3.14}$$

which is quite easy to verify. We remark also that, for any given φ in $P_m(\partial K)$ it is possible to find $(k + 1)$ constants $c, c_1, ..., c_k$ and k polynomials $p_1, ..., p_k$ of degree $\leqslant m - 1$, such that :

$$\begin{cases} \varphi = c_i l_i^{(m)} + p_i + c \text{ on } L_i, \qquad (i = 1, ..., k), \\ \sum_{i=1}^{k} \int_{L_i} p_i \, \mathrm{d}l_i = 0. \end{cases} \tag{3.15}$$

Let now p be an element of $\tilde{P}_m(\partial K)$ such that

$$\frac{\partial p}{\partial s} = p_i \text{ on } L_i \qquad (i = 1, ..., k); \tag{3.16}$$

from (3.11) we get

$$\int_{\partial K} \varphi \frac{\partial p}{\partial s} \, \mathrm{d}l = 0, \tag{3.17}$$

and from (3.15), (3.16) .

$$\int_{\partial K} \varphi \frac{\partial p}{\partial s} \, \mathrm{d}l = \sum_{i=1}^{k} \int_{L_i} (c_i l_i^{(m)} + p_i) p_i \, \mathrm{d}l_i = \sum_{i=1}^{k} \int_{L_i} p_i^2 \, \mathrm{d}l_i. \tag{3.18}$$

Therefore, from (3.17) (3.18), we get $p_i = 0$ $(i = 1, ..., k)$ and (3.12) is proved.

Lemma 3.2. — *In the same hypotheses of lemma* 3.1, *if k and m are odd,* φ *is constant.*

The proof follows immediately from lemma 3.1 and from the antisymmetry of the Legendre polynomials of odd degree.

Lemma 3.3. — *Let K be as in lemma* 3.1. *If* φ *is an element of* $\tilde{P}_m(\partial K)$ *such that*

$$\int_{\partial K} \varphi \frac{\partial p}{\partial s} \, \mathrm{d}l = 0 \qquad \forall p \in \tilde{P}_{m+1}(\partial K) \tag{3.19}$$

then φ *is constant.*

Proof. — Let us set

$$\bar{c} = \int_{\partial K} \varphi \, \mathrm{d}l, \tag{3.20}$$

$$p_i = \varphi - \bar{c} \text{ on } L_i \qquad (i = 1, ..., k); \tag{3.21}$$

obviously :

$$\sum_{i=1}^{k} \int_{L_i} p_i \, \mathrm{d}l_i = 0. \tag{3.22}$$

Let now p be an element of $\tilde{P}_{m+1}(\partial K)$ such that

$$\frac{\partial p}{\partial s} = p_i \text{ on } L_i \qquad (i = 1, ..., k); \tag{3.23}$$

we have from (3.19), (3.21), (3.23)

$$\int_{\partial K} \frac{\partial p}{\partial s} \, \mathrm{d}l = \sum_{i=1}^{k} \int_{L_i} (p_i + \bar{c}) p_i \, \mathrm{d}l_i = \sum_{i=1}^{k} \int_{L_i} p_i^2 \, \mathrm{d}l_i = 0. \tag{3.24}$$

So from (3.21), (3.24) we have $\varphi = \bar{c}$.

From lemmas 3.1, 3.2, 3.3 we obtain immediately the following theorems.

Theorem 3.2. — *Let $\hat{K}$ be a triangle or the unit square. If $s = r - 1$, and $SF(\hat{V})$ contains $\tilde{P}_{s+1}(\partial\hat{K}) \times \tilde{P}_{s+1}(\partial\hat{K})$, taking $\hat{\mathfrak{W}} = T(r, s, \hat{K})$, condition $C(\hat{V}, \hat{\mathfrak{W}})$ is satisfied.*

Theorem 3.3. — *Let $\hat{K}$ be a triangle ; if $s = r - 1$, if $SF(\hat{V})$ contains*

$$\tilde{P}_s(\partial\hat{K}) \times \tilde{P}_s(\partial\hat{K})$$

and if s is odd, taking $\hat{\mathfrak{W}} = T(r, s, \hat{K})$, condition $C(\hat{V}, \hat{\mathfrak{W}})$ is satisfied.

We shall now study the case $s < r - 1$. For this, suppose first of all that $(\varphi, \varphi_{/1}, \varphi_{/2})$ is an element of $TC(r, s, \vartheta)$ such that

$$\int_{\partial\hat{K}} \varphi_{/i} \frac{\partial p}{\partial s} \mathrm{d}\hat{l} = 0 \qquad \forall p \in \tilde{P}_{r-1}(\partial\hat{K}) \qquad (i = 1, 2). \tag{3.25}$$

Then if $(r - 1)$ is odd, we immediately get from lemma 3.2 that $\varphi_{/i} =$ constant $(i = 1, 2)$. If $(r - 1)$ is even we observe that from (3.25) and lemma 3.1 we get

$$\varphi_{/1} = c_i l_i^{(r-1)} + c \text{ on } L_i \qquad (i = 1, 2, 3), \tag{3.26}$$

$$\varphi_{/2} = \bar{c}_i l_i^{(r-1)} + \bar{c} \text{ on } L_i \qquad (i = 1, 2, 3). \tag{3.27}$$

We also note that, if $(r - 1)$ is even, the value of $l_i^{(r-1)}$ at the boundaries of L_i must be equal, for each $i = 1, 2, 3$. Therefore if, for the sake of simplicity, $\hat{K}$ is an equilateral triangle, we get

$$c_1 = c_2 = c_3, \tag{3.28}$$

$$\bar{c}_1 = \bar{c}_2 = \bar{c}_3. \tag{3.29}$$

By requiring now that $\frac{\partial \varphi}{\partial \tilde{\underline{n}}^{(i)}}$ is of degree $\leqslant s$ on L_i $(i = 1, 2, 3)$, we have :

$$(3.30)\quad \deg \frac{\partial \varphi}{\partial \tilde{\underline{n}}^{(i)}} = \deg (\tilde{n}_1^{(i)} \varphi_{/1} + \tilde{n}_2^{(i)} \varphi_{/2})$$

$$= \deg (\tilde{n}_1^{(i)}(c_1 l_i^{(r-1)} + c) + \tilde{n}_2^{(i)}(\tilde{c}_1 l_i^{(r-1)} + \tilde{c})) \leqslant s < r - 1 ;$$

therefore

$$(3.31)\qquad \tilde{n}_1^{(i)} c_1 + \tilde{n}_2^{(i)} \bar{c}_1 = 0 \qquad (i = 1, 2, 3)$$

and, from (2.71), conditions (3.31) imply

$$(3.32)\qquad c_1 = \bar{c}_1 = 0,$$

and then $\varphi_{/i}$ is constant $(i = 1, 2)$.

We have proved the following theorem.

Theorem 3.4. – *Let $\hat{K}$ be an (equilateral) triangle ; if $s < r - 1$ and $SF(\hat{V})$ contains $\tilde{P}_{r-1}(\partial \hat{K}) \times \tilde{P}_{r-1}(\partial \hat{K})$, taking $\hat{\mathfrak{W}} = TC(r, s, \vartheta)$, condition $C(\hat{V}, \hat{\mathfrak{W}})$ is satisfied.*

In a similar manner we can also prove the analogous result for $K =$ unit square (the only difference will be, when $(r - 1)$ is odd, that conditions (3.28), (3.29) become : $c_1 = - c_2 = c_3 = - c$ and $c_1 = - c_2 = c_3 = - c$), and the following theorem can be stated.

Theorem 3.5. – *Let K be the unit square ; if $s < r - 1$ and $SF(V)$ contains $P_{r-1}(\partial K) \times P_{r-1}(\partial K)$, taking $\mathfrak{W} = TCS(r, s, \vartheta)$, condition $C(V, \mathfrak{W})$ is satisfied.*

We shall now consider the last case, i.e. $s > r - 1$. Suppose that $(\varphi, \varphi_{/1}, \varphi_{/2})$ is an element of $T(r, s, \hat{K})$ (with $s > r - 1$ and $\hat{K}$, for sake of simplicity, equal to the unit square) such that :

$$(3.33)\qquad \int_{\partial \hat{K}} \varphi_{/i} \frac{\partial p}{\partial s} \, d\hat{l} = 0 \qquad \forall p \in \tilde{P}_s(\partial \hat{K}).$$

We have from lemma 3.1 that :

$$(3.34)\qquad \varphi_{/1} = c_i l_i^{(s)} + c \qquad \text{on} \qquad L_i$$

$$(3.35)\qquad \varphi_{/2} = \bar{c}_i l_i^{(s)} + \bar{c} \qquad \text{on} \qquad L_i$$

Moreover if, for instance, $L_1 = \{ 0 \leqslant x_1 \leqslant 1, x_2 = 0 \}$, c_1 must be equal to zero, since $\varphi_{/1}$ coincides, on L_1, with the derivative of φ which is of degree $\leqslant r - 1 < s$. From the continuity of $\varphi_{/1}$ we have then $c_2 = c_3 = c_4 = 0$, and $\varphi_{/1}$ is constant. In a similar manner we prove that $\varphi_{/2}$ is constant. The same arguments can be used for the case $\hat{K} =$ triangle

$$\{ x_1 \geqslant 0 ; x_2 \geqslant 0 ; x_1 + x_2 \leqslant 1 \}.$$

Therefore we can state the following theorem.

Theorem 3.6. — *If $\hat{K}$ is the unit square or the triangle*

$$\{ x_1 \geqslant 0 ; x_2 \geqslant 0 ; x_1 + x_1 \leqslant 1 \},$$

if $s > r - 1$ and if $SF(\hat{V})$ contains $\tilde{P}_s(\partial\hat{K}) \times \tilde{P}_s(\partial\hat{K})$, taking $\hat{\mathfrak{W}} = T(r, s, \hat{K})$, condition $C(\hat{V}, \hat{\mathfrak{W}})$ is satisfied.

We remark now that if, for all integer $m \geqslant 0$, we set :

(3.36) $P_m(\hat{K}) = \{$ polynomials of degree $\leqslant m$ on $\hat{K} \}$.

(3.37) $Q_m(\hat{K}) = \{$ polynomials of degree $\leqslant m$ in each variable on $\hat{K} \}$,

(3.38) $S(m, P, \hat{K}) = \{ \underline{v} \mid v_{12} = v_{21}, v_{ij} \in P_m(\hat{K}), v_{ij/ij} = 0 \text{ in } \hat{K} \}$,

(3.39) $S(m, Q, \hat{K}) = \{ \underline{v} \mid v_{11} = U_{/2}, v_{12} = v_{21} = -\frac{1}{2}(U_{/1} + V_{/2}),$
$v_{22} = V_{/1}, (U, V) \in \underset{m+1}{Q}(\hat{K}) \}$

then sufficient conditions in order to have that $SF(\hat{V})$ contains

$$\tilde{P}_k(\partial\hat{K}) \times \tilde{P}_k(\partial\hat{K}),$$

are respectively :

(3.40) $\hat{V}$ contains $S(k - 1, P, \hat{K})$ if $\hat{K}$ is a triangle,

(3.41) $\hat{V}$ contains $S(k - 1, Q, \hat{K})$ if $\hat{K}$ is the unit square.

All preceeding results can be summarized in the following theorem.

Theorem 3.7. — *Let $\{ \mathcal{T}_h \}_h$ be a sequence of decompositions which satisfies* C1) *and* C2); *let, for each decomposition $\mathcal{T}_h$, $V_h = V(\hat{V}, \mathcal{T}_h)$ and $W_h = W(r, s, \mathcal{T}_h)$ be defined starting from $\hat{V}$ and r, s (respectively) as in* (2.8), (2.10), (2.11); *the following conditions are sufficient in order that V_h and W_h satisfy hypothesis* H1.

1) *$\hat{K}$ is a triangle, $\hat{V}$ contains $S(m, P, \hat{K})$ and m, r, s verify the following conditions :*

$$\begin{array}{lll} m \geqslant r - 2 & \text{if} & r - 1 > s, \\ m \geqslant r - 2 & \text{if} & r - 1 = s \text{ and } s \text{ is odd}, \\ m \geqslant r - 1 & \text{if} & r - 1 = s \text{ and } s \text{ is even}, \\ m \geqslant s - 1 & \text{if} & r - 1 < s. \end{array}$$

2) *$\hat{K}$ is the unit square, $\hat{V}$ contains $S(m, Q, \hat{K})$ and m, r, s verify the following conditions :*

$$\begin{array}{lll} m \geqslant r - 2 & \text{if} & r - 1 > s, \\ m \geqslant r - 1 & \text{if} & r - 1 = s, \\ m \geqslant s - 1 & \text{if} & r - 1 < s, \end{array}$$

where $S(m, P, \hat{K})$ and $S(m, Q, \hat{K})$ are defined in (3.36)-(3.39).

REMARK. — Using lemma 3.1 it is possible to find many other sufficient conditions such that hypothesis H1 is satisfied. For example if we require that the functions φ of $\hat{\mathfrak{W}} \subseteq T(r, s, \hat{K})$, for $s = r - 1$ and $\hat{K}$ = unit square, are such that « $\varphi_{/12}$ is continuous at each vertex », we can prove that if $SF(\hat{V})$ contains $\tilde{P}_s(\partial\hat{K}) \times \tilde{P}_s(\partial\hat{K})$ and s is odd then $C(\hat{V}, \hat{\mathfrak{W}})$ holds.

4. ERROR BOUNDS

We want now to study, in the cases of the examples treated before, the behaviour in $|h|$ of the error bounds :

$$\operatorname*{Inf}_{\underline{v} \in V_h} \|\underline{u} - \underline{v}\|_0 \tag{4.1}$$

and

$$\operatorname*{Inf}_{\varphi \in W_h} \|\psi - \varphi\|_W. \tag{4.2}$$

We shall prove, as an upper bound for (4.1), the following result.

Theorem 4.1. — *If $V_h = V(\hat{V}, \mathcal{T}_h)$ and $\hat{V}$ contains the space $S(m, P, \hat{K})$, then, if $u_{ij|K} \in H^{m+1}(K)$ $(i, j = 1, 2)$ for all K in $\mathcal{T}_h$, we have :*

$$\operatorname*{Inf}_{\underline{v} \in V_h} \|\underline{u} - \underline{v}\|_0 \leqslant C\,|h|^{m+1}\,\|\underline{u}\|_{m+1,\mathcal{T}_h} \tag{4.3}$$

where C is a constant independent of u and $\mathcal{T}_h$, and $\|\underline{u}\|_{m+1,\mathcal{T}_h}$ is given by

$$\|\underline{u}\|^2_{m+1,\mathcal{T}_h} = \sum_{i,j=1}^{2} \sum_{K \in \mathcal{T}_h} \|u_{ij}\|^2_{H^{m+1}(K)}. \tag{4.4}$$

The proof of theorem 4.1 is based on an abstract lemma (*cf.* [43]) of the Bramble-Hilbert type which will be reported here in the most general form since it can be useful in many other situations. We present here a different proof of the same result.

Lemma 4.1. — *Let E be a Banach space and let E_0, E_1, F three normed linear spaces; let moreover A_0, A_1, L be linear continuous operators from E into E_0, E_1, F respectively. If :*

$$\|g\|_E \simeq \|A_0 g\|_{E_0} + \|A_1 g\|_{E_1}, \tag{4.5}$$

$$Lg = 0 \quad \text{if} \quad A_1 g = 0, \tag{4.6}$$

$$A_0 \text{ is compact}, \tag{4.7}$$

then there exists a constant c such that

$$\|Lg\|_F \leqslant c\,\|A_1 g\|_{E_1} \qquad \forall g \in E. \tag{4.8}$$

Proof. — First of all we remark that from (4.5) and (4.7) we have that

$$P = \{ g \mid g \in E, A_1 g = 0 \} = \ker (A_1) \tag{4.9}$$

is a finite dimensional subspace of E. In fact if g_n is a sequence in P such that

$$g_n \rightharpoonup 0 \text{ in } E \tag{4.10}$$

we have from (4.7) that

$$A_0 g \to 0 \text{ in } E_0 \tag{4.11}$$

and therefore from (4.5)

$$g_n \to 0 \text{ in } E. \tag{4.12}$$

The proof of the lemma will be now given in two steps. We shall first prove that, defining, for all g in E,

$$Q(g) = \operatorname*{Inf}_{p \in P} \|g - p\|_E \tag{4.13}$$

we get that there exists a constant c_1, such that

$$Q(g) \leqslant c_1 \|A_1 g\|_{E_1} \qquad \forall g \in E. \tag{4.14}$$

Secondly we shall prove that there exists a constant c_2 such that :

$$\|Lg\| \leqslant c_2 Q(g) \qquad \forall g \in E. \tag{4.15}$$

In order to prove (4.14) we suppose, by contradiction, that there exists a sequence g_n in E such that

$$\|A_1 g_n\|_{E_1} \to 0, \tag{4.16}$$

$$Q(g_n) = 1. \tag{4.17}$$

Then, since P is finite dimensional, there exists a sequence $\tilde{g}_n = g_n - p_n$ defined by

$$\|\tilde{g}_n\|_E = Q(g_n) = \operatorname*{Inf}_{p \in P} \|g_n - p\|_E = \|g_n - p_n\|_E ; \tag{4.18}$$

we have now

$$\|A_1 \tilde{g}_n\|_{E_1} = \|A_1 g_n\|_{E_1} \to 0, \tag{4.19}$$

$$\|\tilde{g}_n\|_E = 1. \tag{4.20}$$

Therefore, there exists a subsequence $\{ \tilde{g}_k \}$ of $\{ \tilde{g}_n \}$ and an element g^* in E such that :

$$\tilde{g}_k \rightharpoonup g^* \text{ in } E, \tag{4.21}$$

and consequently :

$$A_0 \tilde{g}_k \to A_0 g^* \text{ in } E_0, \tag{4.22}$$

$$A_1 g_k \rightharpoonup A_1 g^* \text{ in } E_1. \tag{4.23}$$

Then from (4.23) and (4.19)

$$A_1 \tilde{g}_k \to A_1 g^* = 0 \text{ in } E_1 \tag{4.24}$$

and from (4.5), (4.22), (4.24) we have

$$g_k \to g^* \text{ in } E \tag{4.25}$$

$$\tilde{g}^* \in P. \tag{4.26}$$

Finally we have from (4.25), (4.26) that

$$\operatorname*{Inf}_{p \in P} \|\tilde{g}_k - p\|_E \leqslant \|\tilde{g}_k - g^*\| \to 0$$

which is in contradiction to (4.17), (4.18); hence (4.14) is proved.

Let us prove now (4.15); we have for all g in E and for all p in P :

$$\|Lg\|_F = \|Lg - Lp\| \leqslant c_2 \|g - p\|_E \tag{4.27}$$

since L is continuous. Taking the « Inf » in (4.26) we get immediately (4.15). In order to apply lemma 4.1 to our case, let us consider for any integer k the space :

$$\hat{E}^{(k)} = \{ \underline{v} \mid v_{12} = v_{21}, v_{ij} \in H^k(\hat{K}) \qquad (i, j = 1, 2) \}. \tag{4.28}$$

Setting now for any integer $m \geqslant 1$

$$\left\{ \begin{array}{l} E = \hat{E}^{(m+1)}, \qquad E_0 = \hat{E}^{(0)}, \qquad E_1 = (L^2(\hat{K}))^7, \qquad F = E, \\ A_0 = I \text{ (identity)}, \\ A_1 \underline{v} = \left(\dfrac{\partial^{m+1}}{\partial x_1^{m+1}} \underline{v}, \dfrac{\partial^{m+1}}{\partial x_2^{m+1}} \underline{v}, v_{ij/ij} \right), \\ \hat{\Pi} = \text{projection over } S(m, P, \hat{K}), \\ L = I - \hat{\Pi}, \end{array} \right. \tag{4.29}$$

we are exactly in the hypotheses of lemma 4.1, since $A_1 \underline{v} = 0$ implies $\underline{v} \in S(m, P, \hat{K})$. Then we get that there exists a constant $\hat{c}$ such that

$$\|\underline{u} - \hat{\Pi}\underline{u}\| \leqslant \hat{c} \sum_{i,j=1}^{2} |u_{ij}|_{m+1,\hat{K}} \tag{4.30}$$

for all $\underline{u}$ in $\hat{E}^{(m+1)}$ such that $u_{ij/ij} = 0$ in $\hat{K}$.

We can now prove theorem 4.1.

Proof of theorem 4.1. — Let $\underline{u} \in V(\mathcal{T}_h)$, and suppose that for all K in $\mathcal{T}_h$ we have :

$$(u_{ij})_{|K} \in H^{m+1}(K) \qquad (i, j = 1, 2). \tag{4.31}$$

Then, for any given K in $\mathcal{T}_h$, consider the tensor $\hat{\underline{u}} = \hat{\underline{u}}(K) \in \hat{E}^{(m+1)}$ such that :

$$(u_{ij})_{|K} = (\hat{u}_{rs} \circ F) b_{ir} b_{js}. \tag{4.32}$$

Of course we have :

$$\hat{u}_{rs/rs} = 0, \tag{4.33}$$

since $\underline{u}$ is self-equilibrating on K. Let now :

$$\hat{\underline{v}} = \hat{\underline{v}}(K) = \hat{\Pi}\,\hat{\underline{u}}; \tag{4.34}$$

from (4.29) we obtain :

$$\|\hat{\underline{u}} - \hat{\underline{v}}\|_\Lambda \leqslant \hat{c} \sum_{i,j=1}^{2} |\hat{u}_{ij}|_{m+1,\hat{K}}. \tag{4.35}$$

Let now $\underline{v}(K)$ be defined by :

$$v_{ij}(K) = (\hat{v}_{rs} \circ F) b_{ir} b_{js}; \tag{4.36}$$

we have :

$$\begin{aligned} \int_K (u_{ij} - v_{ij})(u_{ij} - v_{ij})\,\mathrm{d}x &\leqslant \|\underline{u} - \underline{v}\|_\Lambda^2 \|B\|^4 |\det B| \leqslant \\ &\leqslant \hat{c}^2 \sum_{i,j=1}^{2} |\hat{u}_{ij}|_{m+1,\hat{K}}^2 \|B\|^4 |\det B| \leqslant \\ &\leqslant \hat{c}^2 \sum_{i,j=1}^{2} |u_{ij}|_{m+1,K}^2 \|B\|^4 \|B^{-1}\|^4 \|B\|^{2m+2} \leqslant \\ &\leqslant \hat{c}^2 \sum_{i,j=1}^{2} |u_{ij}|_{m+1,K}^2 \sigma_1^{2n+6} \sigma_2^4 |h|^{2m+2}. \end{aligned} \tag{4.37}$$

Let now $\underline{v}$ be the element of $V(\hat{V}, \mathcal{T}_h)$ defined by :

$$\underline{v}_{|K} = v(K) \qquad \forall K \in \mathcal{T}_h. \tag{4.38}$$

We will have :

$$\begin{aligned} \|\underline{u} - \underline{v}\|_0^2 &= \sum_{K \in \mathcal{T}_h} \int_K (u_{ij} - v_{ij})(u_{ij} - v_{ij})\,\mathrm{d}x \leqslant \\ &\leqslant \sum_{K \in \mathcal{T}_h} c^2 \sum_{i,j=1}^{2} |u_{ij}|_{m+1,K}^2 \sigma_1^{2m+6} \sigma_2^4 |h|^{2m+2} \leqslant \hat{c}^2 \sigma_1^{2m+6} \sigma_2^4 |h|^{2m+2} \|\underline{u}\|_{m+1,\mathcal{T}_h}^2. \end{aligned} \tag{4.39}$$

and (4.3) is proved.

Let us examine, now, the quantity (4.2), that is :

$$\operatorname*{Inf}_{\varphi \in W_h} \|\psi - \varphi\|_W. \tag{4.40}$$

We have the following theorem.

Theorem 4.2. — *If $W_h = W(r, s, \mathcal{T}_h)$ and if $\psi_{|K} \in H^{9+2}(K)$ for all K in $\mathcal{T}_h$, we have*

$$\operatorname*{Inf}_{\varphi \in W_h} \|\psi - \varphi\|_W \leqslant c\,|h|^q \left(\sum_{K \in \mathcal{T}_h} \|\psi\|_{q+2,K}^2 \right)^{1/2}, \tag{4.41}$$

where $q = \min(s, r-1)$ and c is a constant independent of ψ and of $\mathcal{T}_h$.

Proof. — We shall give only a sketch of the proof, which is based essentially on the Bramble-Hilbert lemma. First of all we remark that :

$$\|\psi - \varphi\|_W^2 = \sum_{K \in \mathcal{T}_h} |\psi - \varphi|_{2,K}^2, \tag{4.42}$$

and that

$$|\psi - \varphi|_{2,K}^2 = \operatorname*{Inf}_{\chi - (\psi - \varphi) \in H_0^2(K)} |\chi|_{2,K}^2, \tag{4.43}$$

since ψ and φ are biharmonic in K.

Therefore if we consider the space

$$H(r, s, \mathcal{T}_h) = \left\{ \chi \mid \chi \in H_0^2(\Omega), \chi_{|\partial K} \in P_r(\partial K), \frac{\partial \chi}{\partial n}_{|\partial K} \in P_s(\partial K), \forall K \in \mathcal{T}_h \right\}, \tag{4.44}$$

we obtain :

$$\operatorname*{Inf}_{\varphi \in W(r,s,\mathcal{T}_h)} \|\psi - \varphi\|_W \leqslant \operatorname*{Inf}_{\chi \in H(r,s,\mathcal{T}_h)} |\psi - \chi|_{2,\Omega}. \tag{4.45}$$

Now, if $s = r - 1$, it is known (see e.g. [5], [10], [15], [36]) that :

$$\operatorname*{Inf}_{\chi \in H(r,s\,\mathcal{T}_h)} |\psi - \chi|_{2,\Omega} \leqslant c\,|h|^{r-1} \left(\sum_{K \in \mathcal{T}_h} \|\psi\|_{r+1,K}^2 \right)^{1/2}, \tag{4.46}$$

with c constant independent of ψ and $\mathcal{T}_h$. Then the obvious inclusions :

$$H(r, s, \mathcal{T}_h) \subseteq H(r, r - 1, \mathcal{T}_h) \qquad (s \leqslant r - 1), \tag{4.47}$$

$$H(r, s, \mathcal{T}_h) \subseteq H(s + 1, s, \mathcal{T}_h) \qquad (s \geqslant r - 1), \tag{4.48}$$

conclude the proof in the case $s \geqslant 2$.

Let us now consider the case $s = 1$; since for all $r \geqslant 3$ we have

$$W(r, 1, \mathcal{T}_h) \supseteq W(3, 1, \mathcal{T}_h), \tag{4.49}$$

it will be sufficient to study the case $r = 3$. To this aim, let $\varphi \in W(3, 1, \mathcal{T}_h)$ be the function defined by

$$\begin{cases} (\varphi, \varphi_{/1}, \varphi_{/2}) = (\psi, \psi_{/1}, \psi_2) \\ \text{at each vertex of each } K \in \mathcal{T}_h \end{cases} \tag{4.50}$$

and let $\chi \in H(3, 2, \mathcal{T}_h)$ be the function of Clough-Tocher type (*cfr.* e.g. [10]) which verifies (4.50) and also verifies

$$\frac{\partial \chi}{\partial n} = \frac{\partial \psi}{\partial n} \text{ at the middle point of each side of each } K \in \mathcal{T}_h. \tag{4.51}$$

It is now well known (*cfr.* again [10]) that :

$$|\psi - \chi|_{2,\Omega} \leqslant c\,|h| \left(\sum_{K \in \mathcal{T}_h} \|\psi\|_{3,K}^2 \right)^{1/2} \tag{4.52}$$

with c constant independent of ψ and $\mathcal{T}_h$. Considering now the function $\varphi - \chi$, we obvious have that

$$\varphi - \chi = 0 \quad \text{on} \quad \Sigma = \bigcup_{K \in \mathcal{T}_h} \partial K, \tag{4.53}$$

and for all K in $\mathcal{T}_h$ the function

$$\frac{\partial}{\partial n}(\varphi - \chi)_{|\partial K} \tag{4.54}$$

is a quadratic polynomial on each side which is zero at each corner and which values on the middle point of each side are bounded by some constant (independent of K) times the norm of ψ in $W^{1,\infty}(K)$. Therefore with classical arguments (*cfr.* e.g. [36], chap. VI, pp. 24-25) we get

$$|\varphi - \chi|_{2,\Omega} \leqslant c\,|h| \left(\sum_{K \in \mathcal{T}_h} \|\psi\|^2_{3,K} \right)^{1/2} \tag{4.55}$$

with c constant independent of ψ and $\mathcal{T}_h$. Therefore from (4.52) and (4.55) we get the result for $s = 1$ and the proof is completed.

5. NUMERICAL SOLUTION

We shall now make some remarks on the effective computation of the solution of the « discretized problem » (2.1). It is easy to verify that, with the indicated choice for V_h and W_h, (2.1) reduces to a system of $NV + NW$ linear equations with $NV + NW$ unknowns, where, of course, NV is the dimension of V_h and NW is the dimension of W_h. In particular, let $\underline{v}^{(1)}, \ldots, \underline{v}^{(NV)}$ be a basis in V_h and $\varphi^{(1)}, \ldots, \varphi^{(NW)}$ a basis in W_h; writing ·

$$\underline{u}_h = \sum_{i=1}^{NV} U_i \underline{v}^{(i)}, \tag{5.1}$$

$$\psi_h = \sum_{i=1}^{NW} \Psi_i \varphi^{(i)}, \tag{5.2}$$

problem (2.1) becomes

$$\left\{ \begin{aligned} &\sum_{i=1}^{NV} [\underline{v}^{(i)}, \underline{v}^{(j)}] U_i - \sum_{i=1}^{NW} b(\underline{v}^{(j)}, \varphi^{(i)}) \Psi_i = -[\underline{f}, \underline{v}^{(j)}] \qquad j = 1, \ldots, NV, \\ &\sum_{i=1}^{NV} b(\underline{v}^{(i)}, \varphi^{(j)}) U_i = -b(\underline{f}, \varphi^{(j)}) \qquad j = 1, \ldots, NW, \end{aligned} \right. \tag{5.3}$$

and setting :

$$\mathcal{A} = \{ A_{ij} \}; \qquad A_{ij} = [\underline{v}^{(j)}, \underline{v}^{(i)}] \qquad (i = 1, \ldots, NV;\ j = 1, \ldots, NV), \tag{5.4}$$

$$\mathcal{B} = \{ B_{ij} \}; \qquad B_{ij} = b(\underline{v}^{(j)}, \varphi^{(i)}) \qquad (i = 1, \ldots, NW;\ j = 1, \ldots, NV), \tag{5.5}$$

$$F_i = -[\underline{f}, \underline{v}^{(i)}] \qquad (i = 1, \ldots, NV), \tag{5.6}$$

$$G_i = -b(\underline{f}, \varphi^{(i)}) \qquad (i = 1, \ldots, NW), \tag{5.7}$$

problem (5.3) can be written as :

$$\begin{cases} \mathcal{A}\vec{U} - \mathcal{B}^T\vec{\Psi} = \vec{F}, \\ \mathcal{B}\vec{U} = \vec{G}, \end{cases} \tag{5.8}$$

where $\mathcal{A}$ is a square $NV \times NV$ matrix, symmetric and positive definite, $\mathcal{B}$ a rectangle $NW \times NV$ matrix and $\mathcal{B}^T$ is the transposed of matrix $\mathcal{B}$.

The following part of the paragraph will be divided into four steps : *a*) computation of the coefficients, *b*) computation of the « known vectors » $\vec{F}$ and $\vec{G}$, *c*) solution of the linear system, *d*) numerical results.

a) *Computation of the coefficients*

The practical computation of the coefficients A_{ij} of the matrix $\mathcal{A}$ can be performed without difficulty, since

$$[\underline{v}^{(j)}, \underline{v}^{(i)}] = \sum_{K \in \mathcal{T}_h} \int_K v_{rs}^{(j)} v_{rs}^{(i)} \, dx, \tag{5.9}$$

and $v_{rs}^{(l)}$ are smooth known functions (in general polynomials) in each K. Moreover, since the elements $\underline{v}$ of V_h are independently assumed in each K, we have that, if $V_h = V(\hat{V}, \mathcal{T}_h)$ and if n and $N(h)$ are respectively the dimension of $\hat{V}$ and the number of elements K in $\mathcal{T}_h$, then the dimension NV of V_h is given by :

$$NV = n \, . \, N(h). \tag{5.10}$$

Let now $\hat{\underline{v}}^{(1)}, \ldots, \hat{\underline{v}}^{(n)}$ be a basis in $\hat{V}$ and let, for each K in $\mathcal{T}_h$, $\underline{v}^{(1)}(K), \ldots, \underline{v}^{(n)}(K)$ be defined by :

$$v_{ij}^{(l)}(K) = v_{rs}^{(l)} b_{ir} b_{js} \qquad (l = 1, \ldots, n). \tag{5.11}$$

We get, obviously, that $\underline{v}^{(1)}(K), \ldots, \underline{v}^{(n)}(K)$ is a basis in $V(\hat{V}, K)$; ordering now the elements K in $\mathcal{T}_h$, and calling them, say, $K_1, \ldots, K_{N(h)}$, we can set, for each $j = 1, \ldots, n$ and for each $l = 1, \ldots, N(h)$:

$$\underline{v}^{(n(l-1)+j)} = \begin{cases} \underline{v}^{(j)}(K_l) \text{ on } K_l \\ 0 \text{ on } K_m \text{ for } m \neq l; \end{cases} \tag{5.12}$$

we obtain a basis in $V_h = V(\hat{V}, \mathcal{T}_h)$; it is easy to verify that, with a basis of this type, $\mathcal{A}$ becomes a block-diagonal matrix, each block being a square $n \times n$ matrix which is symmetrical and positive definite.

On the other hand, for the computation of the coefficients of $\mathcal{B}$ the use of Green's formula is necessary, since we have

$$b(\underline{v}^{(j)}, \varphi^{(i)}) = \sum_{K \in \mathcal{T}_h} \int_K v_{lm}^{(j)} \varphi_{/lm}^{(i)} \, dx, \tag{5.13}$$

and if $W_h = W(r, s, \mathcal{T}_h)$ (for some value of $r \geqslant 3$ and $s \geqslant 1$) the value of the $\varphi^{(i)}$ at the interior of each element K may not, in general, be known.

Therefore the coefficients $b(\underline{v}^{(j)}, \varphi^{(i)})$ must be written in the following form :

$$b(\underline{v}^{(j)}, \varphi^{(i)}) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\varphi^{(i)}_{/r} v^{(j)}_{rs} n_s - \varphi^{(i)} v^{(j)}_{rs/s} n_r) \, dl \tag{5.14}$$

where, for each K, (n_1, n_2) is the direction of the outward normal derivative to ∂K. The computation of (5.14) is easy, since $v^{(j)}$ and $\varphi^{(i)}$ are known, with their first derivatives, at the interelement boundaries; moreover, if the basis $\underline{v}^{(1)}, \ldots, \underline{v}^{(NV)}$ is chosen of the form (5.12), for each j there is just one term, in the sum which appears in (5.14), which differs from zero.

b) *Computation of the « known vectors »*

From the computational point of view, the evaluation of terms (5.6) and (5.7) is the greatest difficulty, since in general it is impossible to calculate *a priori* a particular solution $\underline{f}$ of the equation :

$$f_{ij/ij} = p \quad \text{in each} \quad K \in \mathcal{T}_h. \tag{5.15}$$

We shall show in the following a procedure which proved efficient in reducing evaluation of (5.6) and (5.7) to the computation of some integrals of the known function p, multiplied by suitable polynomials, performed on the elements K.

We shall first treat the simplest case, in which Ω is a square with sides parallel to the axes, and all the elements K are also squares of the same type, with sides of lenght h. A more general case, in which all the elements K are triangles of general type, will be treated later on.

Let then K be a square element and let (x'_1, x'_2) and (x''_1, x''_2) be the coordinates of the lower left vertex and, respectively, of the upper right vertex of K, as shown in fig. 1.

In each K we choose now $\underline{f}$ of the type :

$$f_{11} = f_{22} = 0 \text{ in } K, \tag{5.16}$$

$$f_{12} = f_{21} = f = \frac{1}{2} \int_{x'_1}^{x_1} \int_{x'_2}^{x_2} p(\xi_1, \xi_2) \, d\xi \tag{5.17}$$

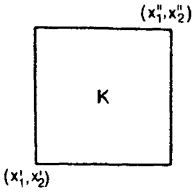

Figure 1

and we remark that, in this case, we get :

$$f_{ij/ij} = p \text{ in } K, \tag{5.18}$$

$$f(x'_1, x_2) = 0 \qquad x'_2 \leqslant x_2 \leqslant x''_2, \tag{5.19}$$

$$f(x_1, x'_2) = 0 \qquad x'_1 \leqslant x_1 \leqslant x''_1. \tag{5.20}$$

We observe that if $\underline{v}^{(i)}$ is an element of a basis of type (5.12) which is not zero in K, then :

$$[\underline{f}, \underline{v}^{(i)}] = \int_K f_{rs} v^{(i)}_{rs} \, dx = 2 \int_K f v^{(i)}_{12} \, dx. \tag{5.21}$$

Setting now, for the sake of simplicity,

$$v = v_{12}^{(i)} \tag{5.22}$$

we define in K the function

$$g(x_1, x_2) = \int_{x_1}^{x''_1} \int_{x_2}^{x''_2} v(\xi_1, \xi_2)\, d\xi.$$

First we observe that, since v is supposed to be a polynomial in K, g is also a polynomial, and its expression can be computed without difficulty. Moreover we have :

$$g_{/12} = v \text{ in } K, \tag{5.24}$$

$$g(x_1, x''_2) = 0 \qquad x'_1 \leqslant x_1 \leqslant x''_1, \tag{5.25}$$

$$g(x''_1, x_2) = 0 \qquad x'_2 \leqslant x_2 \leqslant x''_2. \tag{5.26}$$

Therefore the quantity (5.21), integrating by parts and using (5.18)-(5.20) and (5.24)-(5.26), becomes :

$$\int_K fv\, dx = \int_K fg_{/12}\, dx = \int_K f_{/12} g\, dx - \int_{\partial K} f_{/2} g n_1\, dl + \tag{5.27}$$
$$+ \int_{\partial K} f g_{/1} n_2\, dl = \int_K f_{/12} g\, dx = \frac{1}{2} \int_K pg\, dx,$$

and finally

$$[\underline{f}, \underline{v}^{(i)}] = \int_K pg\, dx; \tag{5.28}$$

Since p is the known function and g is polynomial which can be easily deduced from $\underline{v}^{(i)}$, terms (5.6) can be easily computed using expression (5.28).

Let now deal with terms (5.7); by means of Green's formula and of (5.16), (5.17) we get

$$b(\underline{f}, \varphi^{(i)}) = 2 \sum_{K \in \mathcal{T}_h} \int_{\partial K} (f \varphi_{/2}^{(i)} n_1 - f_{/1} \varphi^{(i)} n_2)\, dl. \tag{5.29}$$

It can be easily verified that if, for instance, $W_h = W(3, 1, \mathcal{T}_h)$ (this procedure can nevertheless be followed also for the general case), then we can construct a basis for W_h in the following way : of all vertices of the elements K, we define as « nodes » those which are internal to Ω, and we associate, with each node P, three functions φ_0, φ_1, φ_2 (of W_h) such that

$$\varphi_r = \frac{\partial \varphi_r}{\partial x_1} = \frac{\partial \varphi_r}{\partial x_2} = 0 \qquad (r = 0, 1, 2) \tag{5.30}$$

at each node different from P, and moreover :

$$\varphi_0(P) = 1, \qquad \frac{\partial \varphi_0}{\partial x_1}(P) = \frac{\partial \varphi_0}{\partial x_2}(P) = 0, \tag{5.31}$$

$$\frac{\partial \varphi_1}{\partial x_1}(P) = 1, \qquad \varphi_1(P) = \frac{\partial \varphi_1}{\partial x_2}(P) = 0, \tag{5.32}$$

$$\frac{\partial \varphi_2}{x_2}(P) = 1, \qquad \varphi_2(P) = \frac{\partial \varphi_2}{x_1}(P) = 0. \tag{5.33}$$

As P describes all nodes, the set of the functions $\varphi_0, \varphi_1, \varphi_2$ verifying (5.30)-(5.33) describes a basis of $W(3, 1, \mathcal{T}_h)$; we shall also say, in other words, that « we have chosen as degrees of freedom in $W(3, 1, \mathcal{T}_h)$ the value of φ and of its first derivatives at each node ».

Let now $\varphi = \varphi^{(i)}$ be an element of such basis, and let $P \equiv (P_1, P_2)$ be the node of $\mathcal{T}_h$ corresponding to φ, in the sense that P is the unique node in which $\varphi = \varphi_{/1} = \varphi_{/2} = 0$ do not hold. The sum which appears in (5.29) is now reduced to the sum of the four integrals over the four sides which have P as a vertex (see fig. 2, where the indicated sides are called $L_{12}, L_{24}, L_{34}, L_{13}$).

If we denote by $f^{(l)}$ the restriction to K_l of f, we get

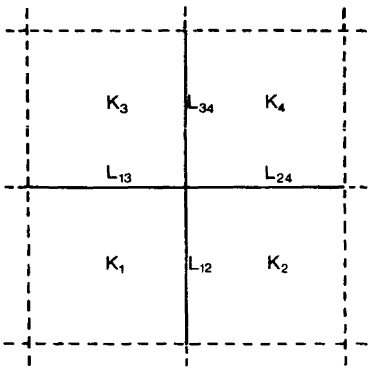

Figure 2

$$\begin{aligned} b(\underline{f}, \varphi) &= \sum_{K \in \mathcal{T}_h} 2 \int_{\partial K} (f\varphi_{/2} n_1 - f_{/1} \varphi n_2)\, \mathrm{d}l = \\ &= 2 \int_{L_{12}} (f^{(1)} - f^{(2)})\varphi_{/2}\, \mathrm{d}l + 2 \int_{L_{24}} (f_{/1}^{(4)} - f_{/1}^{(2)})\varphi\, \mathrm{d}l \\ &\quad + 2 \int_{L_{34}} (f^{(3)} - f^{(4)})\varphi_{/2}\, \mathrm{d}l + 2 \int_{L_{13}} (f_{/1}^{(3)} - f_{/1}^{(1)})\varphi\, \mathrm{d}l, \end{aligned} \tag{5.34}$$

and from (5.19), (5.20) :

$$(5.35)\quad b(\underline{f}, \varphi) = 2\int_{L_{12}} f^{(1)}\varphi_{/2}\,\mathrm{d}l - 2\int_{L_{24}} f^{(2)}_{/1}\varphi\,\mathrm{d}l + 2\int_{L_{34}} f^{(3)}\varphi_{/2}\,\mathrm{d}l - 2\int_{L_{13}} f^{(1)}_{/1}\varphi\,\mathrm{d}l.$$

We shall only show how to calculate the integral :

$$(5.36)\qquad \int_{L_{12}} f^{(1)}\varphi_{/2}\,\mathrm{d}l;$$

the integrals over the other sides can be calculated in a similar manner. Integrating (5.36) by parts we get :

$$\begin{aligned}(5.37)\quad \int_{L_{12}} f^{(1)}\varphi_{/2}\,\mathrm{d}l &= f^{(1)}(P)\varphi(P) - \int_{L_{12}} f^{(1)}_{/2}\varphi(P_1, x_2)\,\mathrm{d}x \\ &= \varphi(P)\frac{1}{2}\int_{K_1} p(\xi_1, \xi_2)\,\mathrm{d}\xi - \frac{1}{2}\int_{K_1} p(\xi_1, \xi_2)\varphi(P_1, \xi_2)\,\mathrm{d}\xi \\ &= \frac{1}{2}\int_{K_1} p(\xi_1, \xi_2)[\varphi(P) - \varphi(P_1, \xi_2)]\,\mathrm{d}\xi.\end{aligned}$$

If we consider now the contribution of the other three integrals appearing in (5.35) we obtain :

$$\begin{aligned}(5.38)\quad b(\underline{f}, \varphi) &= \int_{K_1} p(x_1, x_2)[\varphi(P) - \varphi(P_1, x_2) - \varphi(x_1, P_2)]\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\quad - \int_{K_2} p(x_1, x_2)\varphi(x_1, P_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{K_3} p(x_1, x_2)\varphi(P_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

We remark that, since φ is a polynomial on each L_{ij}, functions $\varphi(P_1, x_2)$ and $\varphi(x_1, P_2)$ will also be polynomials in each K_l, and therefore (5.7) can be easily calculated by (5.38).

We shall examine now the more general case in which the elements K are triangles; we shall show that, even in this case, quantities (5.6) and (5.7) can be calculated by evaluating integrals over some elements K of the function $p(x_1, x_2)$ multiplied by suitable polynomials. Let K a triangle in $\mathcal{T}_h$; we first remark that, by translation and eventually exchanging the x_1-axis with the x_2-axis, we can always reduce to the situation of fig. 3. We denote by $P \equiv (O, O)$, $P' \equiv (x'_1, x'_2)$, $P'' \equiv (x''_1, x''_2)$ the vertices of K and by $L_1 = PP'$ $L_2 = PP''$, $L_3 = P'P''$ the sides of K, whose equations are supposed to be, respectively :

$$(5.39)\qquad x_2 = r_1(x_1) = \alpha_1 x_1, \qquad x_2 = r_2(x_1) = \alpha_2(x_1), \qquad x_3 = r_3(x_1) = \alpha_3 x_1 + \beta$$

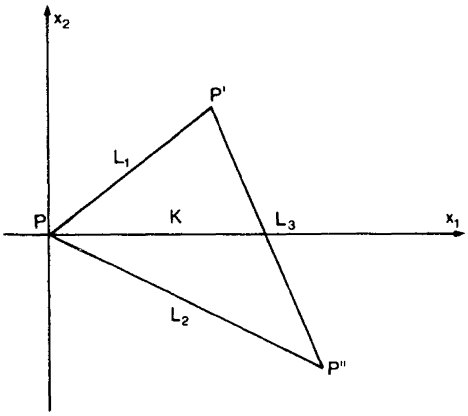

Figure 3

Let us study, first, the quantity (5.6); we suppose that $\underline{v} = \underline{v}^{(i)}$ is an element of the basis (5.12) which is different from zero on K, and we want to calculate the quantity

$$\int_K f_{rs} v_{rs} \, dx. \tag{5.40}$$

If we choose again f of the type

$$f_{11} = f_{22} = 0, \qquad f_{12} = f_{21} = f \tag{5.41}$$

and we set, for the sake of simplicity,

$$v = v_{12} = v_{21},$$

then (5.40) can be written as

$$2 \int_K fv \, dx. \tag{5.42}$$

In order to evaluate (5.42), we calculate first a function $g(x_1, x_2)$ which satisfies the following conditions :

$$g_{/12} = \frac{\partial^2 g}{\partial x_1 \, \partial x_2} = v \text{ in } K, \tag{5.43}$$

$$g = g_{/2} = 0 \text{ on } L_3. \tag{5.44}$$

Such a function can be calculated explicitly by setting

$$g(x_1, x_2) = g^1(x_1, x_2) + g^2(x_2) + g^3(x_1), \tag{5.45}$$

where the functions g^i are such that :

$$g^1_{/12} = \frac{\partial^2 g^1}{\partial x_1 \, \partial x_2} = v \text{ in } K, \tag{5.46}$$

$$g^2_{/2}(x_2) = -\, g^1_{/2}(r_3^{-1}(x_2), x_2), \tag{5.47}$$

$$g^3(x_1) = -\, g^1(x_1, r_3(x_1)) - g^2(r_3(x_1)). \tag{5.48}$$

It can be easily verified that the function defined by (5.45)-(5.48) verifies (5.43), (5.44); we also remark that, since v is a polynomial, the functions g^i can be easily calculated and are also polynomials.

By (5.43) and by Green's formula we get

$$\int_K fv \, \mathrm{d}x = \int_K fg_{/12} \, \mathrm{d}x = \int_K f_{/12} g \, \mathrm{d}x + \int_{\partial K} fg_{/2} \, \mathrm{d}l - \int_{\partial K} f_{/1} g \, \mathrm{d}l. \tag{5.49}$$

We choose f in such a way that :

$$f_{/12} = \frac{1}{2} p \text{ in } K, \tag{5.50}$$

$$f = 0 \text{ on } L_1, \tag{5.51}$$

$$f = 0 \text{ on } L_2. \tag{5.52}$$

An explicit solution of (5.50)-(5.52) can be obtained in the following way. Let $\tilde{p}(x_1, x_2)$ be defined by

$$\tilde{p}(x_1, x_2) = \begin{cases} p(x_1, x_2) & \text{if } (x_1, x_2) \in K, \\ 0 & \text{if } (x_1, x_2) \notin K, \end{cases} \tag{5.53}$$

and let z^1 be defined by,

$$z^1(x_1, x_2) = \frac{1}{2} \int_0^{x_1} \int_{x_2'}^{x_2} \tilde{p}(\xi_1, \xi_2) \, \mathrm{d}\xi ; \tag{5.54}$$

of course $z^1(x_1, x_2)$ satisfies the following conditions :

$$z^1_{/12} = \frac{1}{2} \tilde{p}(x_1, x_2) \tag{5.55}$$

$$z^1(0, x_2) = 0; \qquad z^1(x_1, x_2'') = 0. \tag{5.56}$$

Let now $\chi(x_2)$ be defined by

$$\chi(x_2) = z^1(r_1^{-1}(x_2), x_2); \tag{5.57}$$

we set

$$z^2(x_1, x_2) = \begin{cases} \chi(x_2) & \text{for } x_2 > 0, \\ 0 & \text{for } x_2 \leqslant 0, \end{cases} \tag{5.58}$$

and we have that function f, given by

$$f(x_1, x_2) = z^1(x_1, x_2) - z^2(x_1, x_2), \tag{5.59}$$

i.e.

$$f(x_1, x_2) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^{x_1}\int_{x_2''}^{x_2} \tilde{p}(\xi_1, \xi_2)\, d\xi - \dfrac{1}{2}\int_{x_2''}^{x_2}\int_0^{r_1^{-1}(x_2)} \tilde{p}(\xi_1, \xi_2)\, d\xi & \text{for } x_2 > 0 \\ \dfrac{1}{2}\displaystyle\int_0^{x_1}\int_{x_2''}^{x_2} \tilde{p}(\xi_1, \xi_2)\, d\xi \text{ for } x_2 \leqslant 0 & \end{cases} \tag{5.60}$$

satisfies conditions (5.50)-(5.52). We have, in particular, the following expressions for $f_{/1}$ and $f_{/2}$:

$$f_{/1}(x_1, x_2) = \frac{1}{2}\int_{x_2''}^{x_2} \tilde{p}(x_1, \xi_2)\, d\xi_2, \tag{5.61}$$

$$f_{/2}(x_1, x_2) = \begin{cases} \dfrac{1}{2}\left[\displaystyle\int_0^{x_1} \tilde{p}(\xi_1, x_2)\, d\xi_1 - \int_0^{r_1^{-1}(x_2)} \tilde{p}(\xi_1, x_2)\, d\xi_1 \right. \\ \qquad \left. - \dfrac{1}{\alpha_1}\displaystyle\int_{x''_2}^{x_2} \tilde{p}(r_1^{-1}(x_2), \xi_2)\, d\xi_2\right] \text{ for } x_2 > 0 \\ \dfrac{1}{2}\displaystyle\int_0^{x_1} \tilde{p}(\xi_1, x_2)\, d\xi_1 \text{ for } x_2 \leqslant 0 \end{cases} \tag{5.62}$$

By formulas (5.44), (5.51), (5.52), we have first that (5.49) becomes

$$\int_K fv\, dx = \int_K f_{/12} g\, dx - \int_{L_1} f_{/1} g n_2^{(1)}\, dl - \int_{L_2} f_{/1} g n_2^{(2)}\, dl, \tag{5.63}$$

where $(n_1^{(i)}, n_2^{(i)})$ is the normal outward direction to the side L_i. Substituting (5.50) and (5.61) in (5.63) we get :

$$\int_K fv\, dx = \frac{1}{2}\int_K pg\, dx - \frac{c_1}{2}\int_0^{x_1'}\int_{x_2''}^{r_1(x_1)} \tilde{p}G^{(1)}\, dx - \frac{c_2}{2}\int_0^{x_1'}\int_{x_2''}^{r_2(x_1)} \tilde{p}G^{(2)}\, dx, \tag{5.64}$$

where $G^{(1)}(x_1, x_2) = g(x_1, r_1(x_1))$, $G^{(2)}(x_1, x_2) = g(x_1, r_2(x_1))$ and c_1, c_2 are constants depending on α_1, α_2. We remark now that $\tilde{p}(x_1, x_2)$ is zero for $x_2'' \leqslant x_2 \leqslant r_2(x_1)$; Setting now

$$G(x_1, x_2) = \begin{cases} g(x_1, x_2) - c_1 G^{(1)}(x_1, x_2) & \text{if } 0 \leqslant x_1 \leqslant x_1', \\ g(x_1, x_2) & \text{if } x_1' \leqslant x_1, \end{cases} \tag{5.65}$$

we finally get

$$\int_K fv \, \mathrm{d}x = \frac{1}{2} \int_K pG \, \mathrm{d}x, \tag{5.66}$$

where G is a polynomial in each of the two regions $x_1 \leqslant x_1'$ and $x_1' \leqslant x_1$.

Let us now consider the terms of the type :

$$b(f, \varphi) = \sum_{K \in \mathcal{T}_h} \int_K (f_{ij}\varphi_{/ij} - p\varphi) \, \mathrm{d}x \tag{5.67}$$

where φ is an element of the basis for W_h. It is easy to verify that if, for instance, $W_h = W(3, 1, \mathcal{T}_h)$ and we choose as degrees of freedom the values of the functions and of their first derivatives, then the sum which appears in (5.67) reduces to a few terms. Let us consider just one of them, say a triangle K as in figure 3. By Green's formula and due to the given choice for f, we have :

$$\int_K (f_{ij}\varphi_{/ij} - p\varphi) \, \mathrm{d}x = \int_{\partial K} (f\varphi_{/2}n_1 - f_{/1}\varphi n_2) \, \mathrm{d}l; \tag{5.68}$$

let us first consider the term :

$$\int_{\partial K} f_{/1}\varphi n_2 \, \mathrm{d}l. \tag{5.69}$$

We have

$$\begin{aligned} \int_{\partial K} f_{/1}\varphi n_2 \, \mathrm{d}l &= \sum_{i=1}^{3} \int_{L_i} f_{/1}\varphi n_2^{(i)} \, \mathrm{d}l \\ &= n_2^{(1)}c_1 \int_0^{x_1'} f_{/1}(\xi_1, r_1(\xi_1))\varphi(\xi_1, r_1(\xi_1)) \, \mathrm{d}\xi_1 \\ &\quad + n_2^{(2)}c_2 \int_0^{x_1'} f_{/1}(\xi_1, r_2(\xi_1))\varphi(\xi_1, r_2(\xi_1)) \, \mathrm{d}\xi_1 \\ &\quad + n_2^{(3)}c_3 \int_{x_1'}^{x_1'} f_{/1}(\xi_1, r_3(\xi_1))\varphi(\xi_1, r_3(\xi_1)) \, \mathrm{d}\xi_1, \end{aligned} \tag{5.70}$$

and from (5.61)

$$\begin{aligned} \int_{\partial K} f_{/1}\varphi n_2 &= \frac{n_2^{(1)}c_1}{2} \int_0^{x_1'} \int_{r_2(x_1)}^{r_1(x_1)} p(\xi_1, \xi_2)\Phi_1(\xi_1, \xi_2) \, \mathrm{d}\xi \\ &\quad + \frac{n_2^{(3)}c_3}{2} \int_{x_1'}^{x_1'} \int_{r_2(x_1)}^{r_3(x_1)} p(\xi_1, \xi_2)\Phi_3(\xi_1, \xi_2) \, \mathrm{d}\xi. \end{aligned} \tag{5.71}$$

Since $\Phi_i(x_1, x_2) = \varphi(x_1, r_i(x_1))$ $(i = 1, 3)$ are polynomials, formula (5.71) is of the expected type. We shall deal now with the term

$$\int_{\partial K} f\varphi_{/2} n_1 \, dl \tag{5.72}$$

which reduces, by (5.51), (5.52), to

$$\int_{L_3} f\varphi_{/2} n_1^{(3)} \, dl. \tag{5.73}$$

Let now $Q(x_1)$ be a primitive of the polynomial $\varphi_{/2}(x_1, r_3(x_1))$, which is itself a polynomial and easily computable; changing the variable and integrating by parts we get :

$$\begin{aligned} n_1^{(3)} \int_{L_3} f\varphi_{/2} \, dl &= n_1^{(3)} c_3 \int_{x_1'}^{x_1''} f(\xi_1, r_3(\xi_1))\varphi_{/2}(\xi_1, r_3(\xi_1)) \, d\xi_1 \\ &= n_1^{(3)} c_3 \int_{x_1'}^{x_1''} f_{/1}(\xi_1, r_3(\xi_1)) Q(\xi_1) \, d\xi_1 \\ &\quad - \alpha_3 n_1^{(3)} c_3 \int_{x_1'}^{x_1''} f_{/2}(\xi_1, r_3(\xi_1)) Q(\xi_1) \, d\xi_1. \end{aligned} \tag{5.74}$$

From (5.61), (5.62) we obtain :

$$\begin{aligned} \int_{\partial K} f\varphi_{/2} n_1 \, dl &= -\frac{n_1^{(3)} c_3}{2} \int_{x_1'}^{x_1''} \int_{r_2(x_1)}^{r_3(x_1)} p(\xi_1, \xi_2) Q(\xi_1) \, d\xi \\ &\quad + \frac{n_1^{(3)} c_3}{2} \int_K p(\xi_1, \xi_2) Q(\xi_1) \, d\xi \\ &\quad + \frac{n_1^{(3)} c_3 \alpha_3}{2\alpha_1} \int_{x_1'}^{\bar{x}_1} \int_{r_2(x_1)}^{r_3(x_1)} p(\xi_1, \xi_2) Q(\xi_1) \, d\xi, \end{aligned} \tag{5.75}$$

and finally :

$$\begin{aligned} &\int_K (f_{ij}\varphi_{/ij} - p\varphi) \, dx \\ &= \frac{1}{2} \Bigg\{ (n_1^{(3)} c_3 - n_2^{(1)} c_1) \int_0^{x_1'} \int_{r_2(x_1)}^{r_1(x_1)} p(\xi_1, \xi_2)(Q(\xi_1) + \Phi_1(\xi_1, \xi_2)) \, d\xi \\ &\quad + c_3 \left(\frac{n_1^{(3)} \alpha_3}{\alpha_1} - n_2^{(3)} \right) \int_{x_1'}^{\bar{x}_1} \int_{r_2(x_1)}^{r_3(x_1)} p(\xi_1, \xi_2)(Q(\xi_1) + \Phi_3(\xi_1, \xi_2)) \, d\xi \\ &\quad - n_2^{(3)} c_3 \int_{\bar{x}_1}^{x_1''} \int_{r_2(x_1)}^{r_3(x_1)} p(\xi_1, \xi_2) \Phi_3(\xi_1, \xi_2) \, d\xi \Bigg\} \end{aligned} \tag{5.76}$$

which is again of the desired type.

c) *Solution of the linear system*

We have seen in a) that, if the chosen basis for V_h is of type (5.12), then $\mathcal{A}$ is a block diagonal matrix, each block being an $n \times n$ square matrix, symmetrical and positive definite. Therefore the best manner for solving the system (5.8) is an a priori inversion of $\mathcal{A}$ (cfr. e.g. [32]), which is easily performed and leads, by the substitution

$$\vec{U} = \mathcal{A}^{-1}\mathcal{B}^T\vec{\Psi} + \mathcal{A}^{-1}\vec{F}, \tag{5.77}$$

to the linear system

$$\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^T\vec{\Psi} = \mathcal{B}\mathcal{A}^{-1}\vec{F} + \vec{G} \tag{5.78}$$

in the NW unknowns $\Psi_1, \ldots, \Psi_{NW}$. Let us set :

$$\mathcal{K} = \mathcal{B}\mathcal{A}^{-1}\mathcal{B}^T, \tag{5.79}$$

$$\vec{P} = \mathcal{B}\mathcal{A}^{-1}\vec{F} + \vec{G}. \tag{5.80}$$

Proposition 5.1. – *Hypothesis* H1 *implies that* $\mathcal{K}$ *is symmetrical and positive definite.*

Proof. – From (5.79) we have obviously that $\mathcal{K}$ is symmetrical, since $\mathcal{A}^{-1}$ is symmetrical. Moreover, since $\mathcal{A}^{-1}$ is positive definite, we have for every vector $\vec{\Phi} = \Phi_1, \ldots, \Phi_{NW}$ that :

$$(\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^T\vec{\Phi}, \vec{\Phi}) = (\mathcal{A}^{-1}\mathcal{B}^T\vec{\Phi}, \mathcal{B}^T\vec{\Phi}) \geqslant 0 \tag{5.81}$$

and moreover,

$$(\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^T\vec{\Phi}, \vec{\Phi}) = 0 \Leftrightarrow \mathcal{B}^T\vec{\Phi} = 0. \tag{5.82}$$

Let now φ be the element of W_h defined by

$$\varphi = \sum_{i=1}^{NW} \varphi^{(i)}\Phi_i ; \tag{5.83}$$

if $\mathcal{B}^T\vec{\Phi} = 0$ then, for all $\underline{v}$ in V_h, it follows

$$b(\underline{v}, \varphi) = \sum_{j=1}^{NV} V_j b(\underline{v}^{(j)}, \varphi) = \sum_{j=1}^{NV} V_j(\mathcal{B}^T\vec{\Phi})_j = 0 \tag{5.84}$$

and, from hypothesis H1, (5.84) implies $\varphi = 0$ and therefore $\vec{\Phi} = 0$. So $\mathcal{B}^T\vec{\Phi} = 0$ iff $\vec{\Phi} = 0$ and from (5.81), (5.82) we obtain the result.

The numerical solution of

$$\mathcal{K}\vec{\Psi} = \vec{P} \tag{5.85}$$

can therefore be performed in one of the classical ways for solving linear systems with a symmetrical positive definite matrix.

We shall now make some comments on the « topological structure » of $\mathcal{H}$ (i.e. the position of the non-zero elements). It is easy to verify that, if we choose as degrees of fredom in $W_h = W(r, s, \mathcal{T}_h)$ the values of φ and of some of its partial derivatives at certain points on the sides of the elements K, then the coefficient H_{ij} of $\mathcal{H}$ is different from zero only if $\varphi^{(i)}$ and $\varphi^{(j)}$ have its « corresponding points » on the boundary of the same element.

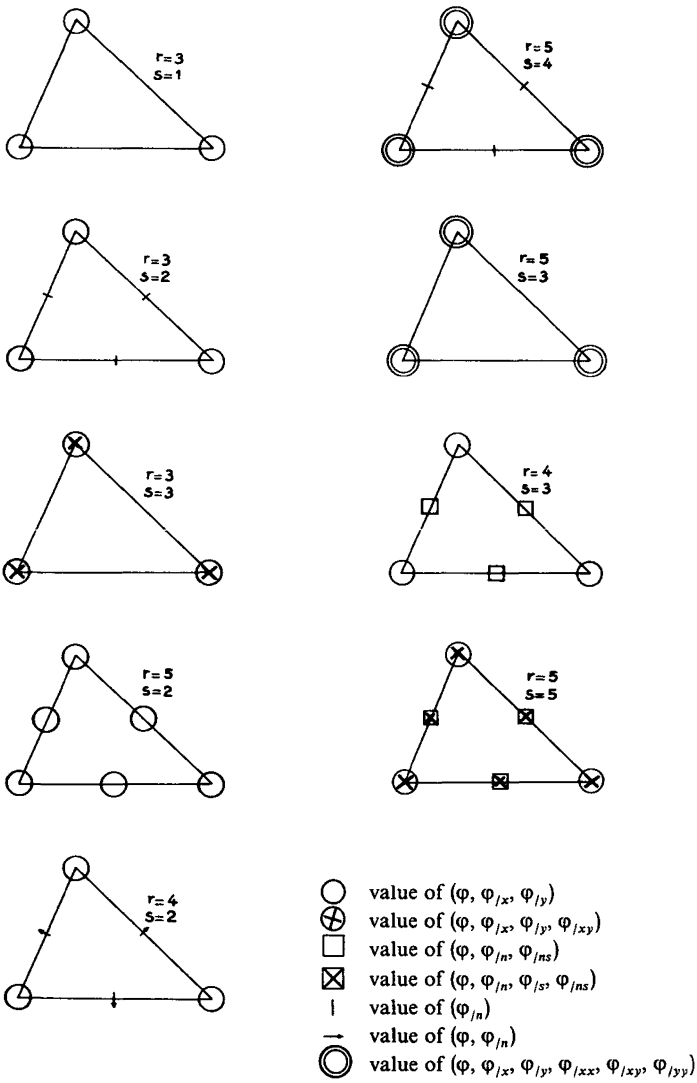

Figure 4

It follows that the topological structure of $\mathcal{K}$ is depending *only* on the choice of the degrees of freedom in W_h. We remark that, however, in order to have a C^1-continuity for the functions of $W(r, s, \mathcal{T}_h)$, we are compelled to choose, as degrees of freedom, *at least* the values of $(\varphi, \varphi_{/1}, \varphi_{/2})$ at the nodes,

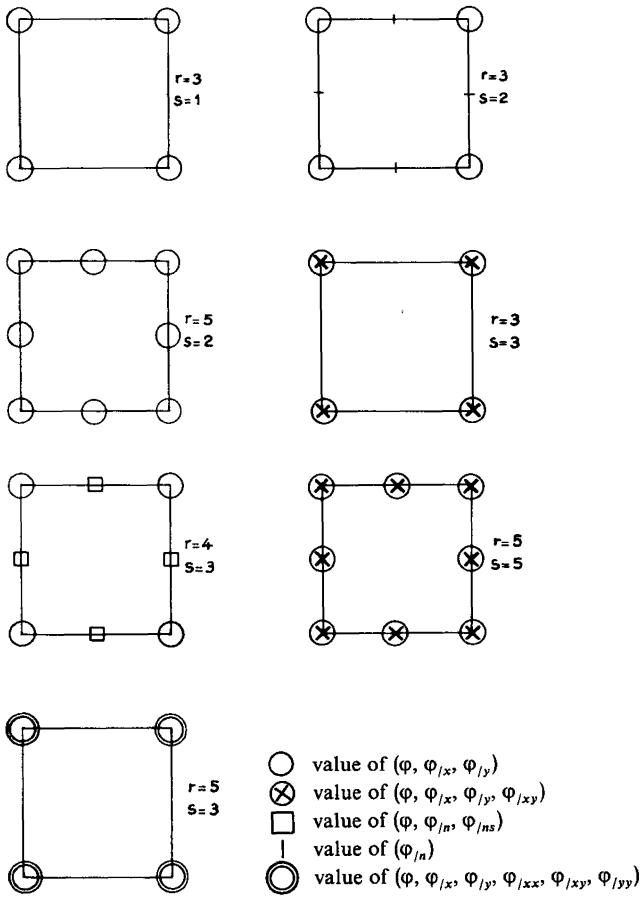

Figure 5

and this justifies conditions $r \geqslant 3$, $s \geqslant 1$. We also remark that in many cases the degrees of freedom of $W(r, s, \mathcal{T}_h)$ will coincide with those used in the classical, conforming and non conforming, finite element « displacement » methods for biharmonic equations. Nevertheless the coincidence of the degrees of freedom does not imply that the trial functions will coincide at the interior of the elements and, in many cases, even on the interelement boundaries, as in the non-conforming methods. Similarly, the structure of the matrices for

the displacement methods and for the hybrid methods will coincide, but not the values of non-zero coefficients; again the « known vectors » are computed in a similar fashion but they do not coincide. We also remark that, even if the topological structure of $\mathcal{K}$ remains unchanged, changing V_h, the values of the non-zero coefficients (and, of course, of the known vector) depend on the choice of V_h; we have therefore many « hybrid analogues » of the same displacement method.

We give in the following some examples of choices of degrees of freedom in $W(r, s, \mathcal{T}_h)$ for some value of r and s in the case $\hat{K}$ = triangle (fig. 4) and in the case $\hat{K}$ = unit square (fig. 5). The reader will recognize many of the classical « structures » used in the displacement methods (as e.g. Zienkiewicz, Clough-Tocher, Bramble-Zlamal, Adini, Fraeijs de Veubeke-Sander, Bogner-Fox-Schmit) and also other different « structures » that can be easily used by the « hybrid analogue » approach.

d) *Numerical results*

We shall report here some of the results obtained in the numerical experiments performed on the Honeywell 6030 of the « Centro di Calcoli Numerici dell'Università di Pavia », in the simplest case of a square plate; the « reference element $\hat{K}$ » was also a square and the choices $W(3, 1, \mathcal{T}_h)$ and $W(3, 3, \mathcal{T}_h)$ has been tested for W_h (that is, the hybrid analogous of the « Adini element » and of the « Bogner-Fox-Schmit element »). For the case $W_h = W(3, 1, \mathcal{T}_h)$, different choices for $\hat{V}$ have also been tested, that is :

$$\hat{V}1 \equiv \begin{cases} v_{11} = a_0 + a_1 x + a_2 y \\ v_{12} = v_{21} = b_0 + b_1 x + b_2 y \\ v_{22} = c_0 + c_1 x + c_2 y \end{cases}$$

$$\hat{V}2 \equiv \begin{cases} v_{11} = a_0 + a_1 x + a_2 xy \\ v_{12} = v_{21} = b_0 + b_1 x + b_2 y \\ v_{22} = c_0 + c_1 y + c_2 xy \end{cases}$$

$$\hat{V}3 \equiv \begin{cases} v_{11} = a_0 + a_1 x + a_2 y + a_3 xy \\ v_{12} = v_{21} = b_0 + b_1 x + b_2 y \\ v_{22} = c_0 + c_1 x + c_2 y + c_3 xy \end{cases}$$

$$\hat{V}4 \equiv \begin{cases} v_{11} = a_0 + a_1 x + a_2 y + a_3 xy \\ v_{12} = v_{21} = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 y^2 \\ v_{22} = c_0 + c_1 x + c_2 y + c_3 xy \end{cases}$$

(We note that, in the case $\hat{V}1$, hypothesis $C(\hat{V}, \hat{W})$ is not satisfied; the results, however, are « good », at least for the tested values of h). In the case $W_h = W(3, 3, \mathcal{T}_h)$ the only choice

$$\hat{V}5 \equiv \begin{cases} v_{11} = a_0 + a_1 x + a_2 y + a_3 xy + a_4 y^2, \\ v_{12} = v_{21} = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 y^2 \\ v_{22} = c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2, \end{cases}$$

has been tested. Figures 6, 7, 8 report the plots of the errors

$$\mathrm{ERR}^{(1)} \simeq \|w - \psi_h\|_{H^1} \|w\|_{H^1}^{-1},$$
$$\mathrm{ERR}^{(0)} \simeq \|w - \psi_h\|_{L^2} \|w\|_{L^2}^{-1},$$
$$\mathrm{ERR}^{(2)} \simeq \frac{|u - u_h|}{|u|} \text{ in the middle point of the plate,}$$

for the different choices of W_h and V_h, with different values of h (or, more precisely, of $N = \frac{1}{h}$ = number of elements on each side). Further results, with an accurate comparison between some of the classical displacement methods and their hybrid analogues will be published in a forthcoming paper by the same authors.

REFERENCES

[1] A. Adini and R. W. Clough, *Analysis of plate bending by the finite element method*, NSF Report G. 7337, 1961.

[2] F. K. Bogner, R. L. Fox and L. A. Schmit, The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas, *Conference on Matric Methods in Structural Mechanics*, Wright Patterson A.F.B., Ohio, 1965.

[3] J. H. Bramble and R. S. Hilbert, *Estimation of linear functionals on Sobolev Spaces with applications to Fourier transforms and spline interpolation*, SIAM J. Numer. Anal. 7, pp. 112-124, 1970.

[4] J. H. Bramble and R. S. Hilbert, *Bounds for a class of linear functionals with applications to Hermite interpolation*, Numer. Math., 16, pp. 362-369, 1971.

[5] J. H. Bramble and M. Zlamal, *Triangular Elements in the finite element method*, Math. Comp. 24, pp. 809-820, 1970.

(*Suite page 49*)

In figures 6-7-8 the following symbols are used :

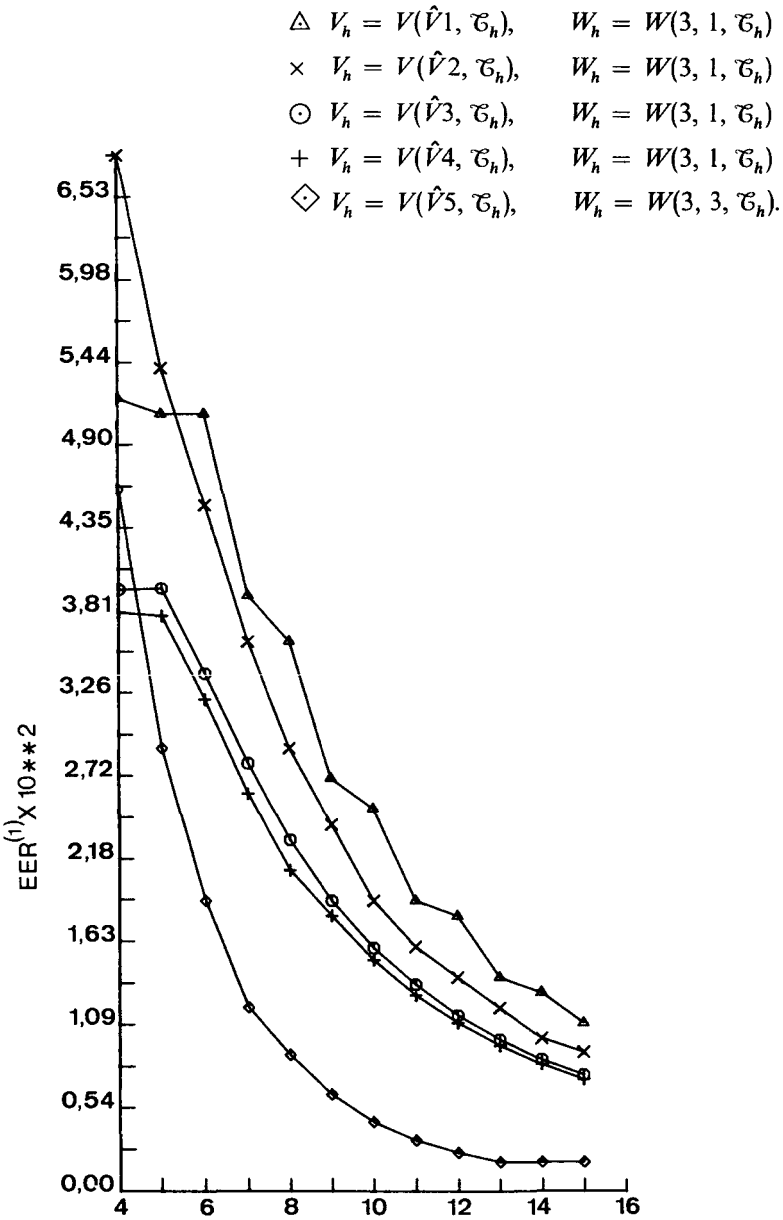

Figure 6

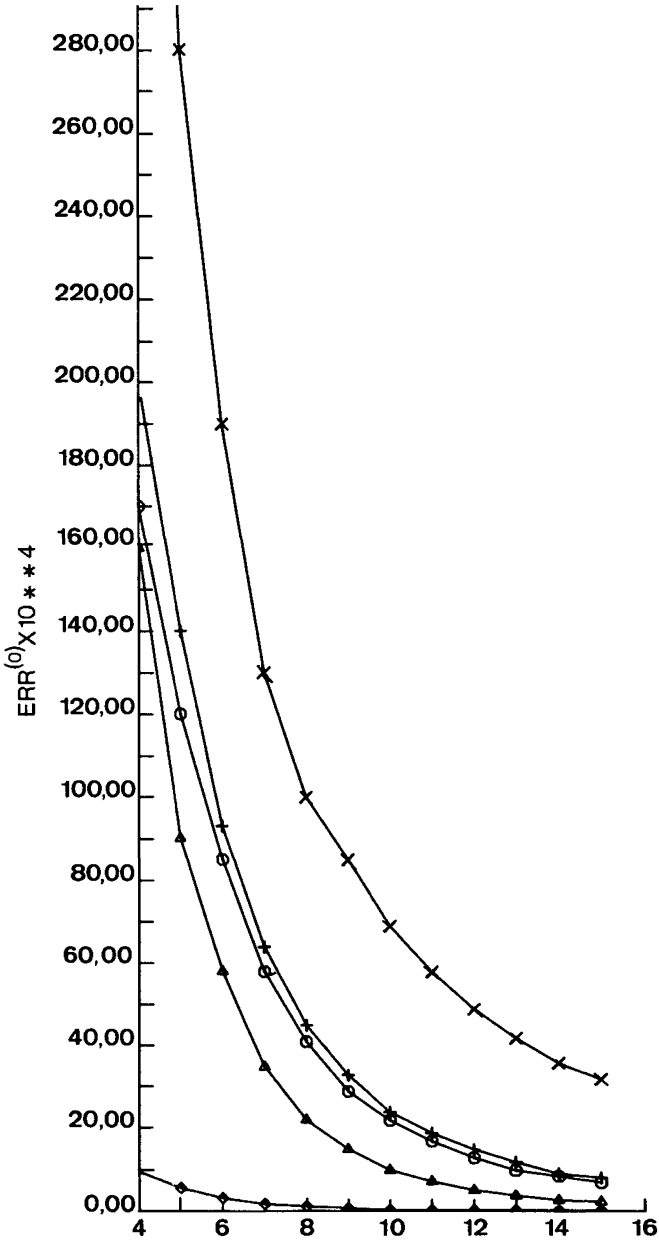
280,00
260,00
240,00
220,00
200,00
180,00
160,00
140,00
120,00
100,00
80,00
60,00
40,00
20,00
0,00
ERR(0)X10 * * 4
4
6
8
10
12
14
16

Figure 7

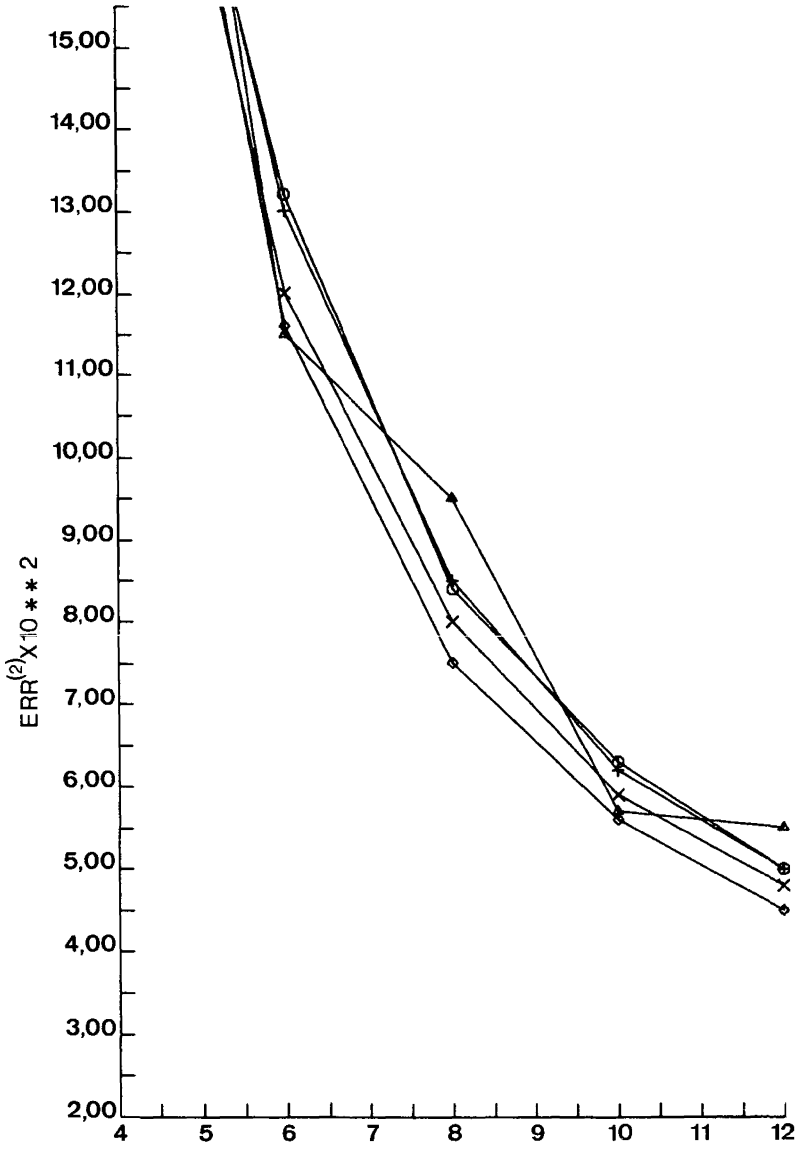
15,00
14,00
13,00
12,00
11,00
10,00
9,00
8,00
7,00
6,00
5,00
4,00
3,00
2,00
ERR(2)X10 * * 2
4
5
6
7
8
9
10
11
12

Figure 8

REFERENCES (*suite*)

[6] F. BREZZI, *Sur une méthode hybride pour l'approximation du problème de la torsion d'une barre élastique* Rend. Ist. Lombardo Sci. Lett. A, *108* (1974), 274-300.

[7] F. BREZZI, *Sur la méthode des éléments finis hybrides pour le problème biharmonique*. Num. Math. 24 (1975) 103-131.

[8] F. BREZZI, *On the existence uniqueness and approximation of saddle point problems arising from Lagrangian multipliers*. R.A.I.R.O. 8-R2 (1974) 129-151.

[9] P. G. CIARLET, *Quelques méthodes d'éléments finis pour le problème d'une plaque encastrée*, Colloque IRIA sur « Méthodes de calcul scientifique et technique », Roquencourt (Paris) 17-21 déc. 1973. (To appear in Springer Verlag).

[10] P. G. CIARLET, *Sur l'élément de Clough et Tocher*. (To appear).

[11] P. G. CIARLET and P. A. RAVIART, *La méthode des éléments finis pour les problèmes aux limites elliptiques*. (To appear).

[12] P. G. CIARLET and P. A. RAVIART, *A mixed finite element method for the biharmonic equation*, Symposium on Mathematical Aspect of Finite Elements in P.D.E., University of Wisconsin, Madison, April 01-03, 1974.

[13] P. G. CIARLET and P. A. RAVIART, *A nonconforming method for the plate problem*. (To appear).

[14] P. G. CIARLET and P. A. RAVIART, *General Lagrange and Hermite interpolation in R^n with applications to finite element method*, Arch. Rat. Mech. Anal. 46, pp. 177-189, 1972.

[15] J. F. CIAVALDINI and J. C. NEDELEC. (To appear).

[16] R. W. CLOUGH and J. L. TOCHER, Finite element stiffness matrices for analysis of plate in bending. *Conference on Matrix Methods in Structural Mechanics*, Wright Patterson A.F.B., Ohio, 1965.

[17] G. FICHERA, *Esistenza del minimo in un classico problema di calcolo delle variazioni*, Rend. Acc. Nazionale Lincei, s. VIII, v. XI, fasc. 1-2, pp. 34-39, 1951.

[18] G. FICHERA, *Linear elliptic differential systems and eigenvalue problems*, Lecture Notes, Springer Verlag, Berlin. 1965.

[19] B. FRAEIJS DE VEUBEKE, Displacements and equilibrium models in the finite element method, *Stress Analysis* (ed. by O. C. Zienkiewicz and G. S. Holister), chap. 9, Wiley, 1964.

[20] B. FRAEIJS DE VEUBEKE, Bending and stretching of plates, *Conference on Matrix Methods in Structural Mechanics*, Wright Patterson A.F.B., Ohio, 1965.

[21] B. FRAEIJS DE VEUBEKE, *Variational principles and the patch test*. (To appear on Internat. J. Numer. Methods Engrg., 8, 783-801, 1974.

[22] B. FRAEIJS DE VEUBEKE and O. C. ZIENKIEWICZ, *Strain-energy bounds in finite-element analysis by slab analogy*, Journal of strain analysis, Vol. 2 N. 4, 1967.

[23] R. GLOWINSKI, Approximations externes, par éléments finis de Lagrange d'ordre un et deux, du problème de Dirichlet pour l'opérateur biharmonique. Méthode itérative de résolution des problèmes approchés, *Conference on Numerical Analysis*, Royal Irish Academy, 1972.

[24] B. M. IRONS and A. RAZZAQUE, Experience with the patch test for convergence of finite elements, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (A. K. Aziz, Editor), pp. 557-587, Academic Press, New York, 1972.

[25] C. JOHNSON, *On the convergence of some mixed finite element methods in plate bending problems*, Num. Math, 21 (1), 43-62 (1973).

[26] C. JOHNSON, Convergence of another mixed finite element method for plate bending problems, Report N. 27, Department of Mathematics, *Chalmers Institute of Technology and the University of Goteborg*, 1972.

[27] P. LASCAUX and P. LESAINT, *Convergence de certains éléments finis non conformes pour le problème de la fiexion des plaques minces.* (To appear).

[28] J. L. LIONS and E. MAGENES, *Non homogeneous boundary value problems and applications*, I, II, Grund. B. 181-182, Springer-Berlin, 1970.

[29] B. MERCIER, *Numerical solution of the biharmonic problem by mixed finite elements of class C^0*. Boll. U.M.I., 4, 10, 1974.

[30] J. T. ODEN, Some contributions to the mathematical theory of mixed finite element approximations, *Tokyo Seminar on Finite Elements*, 1973.

[31] T. H. H. PIAN, *Hybrid Models*, Int. Symp. on Numerical and Computer Methods in Structural Mechanics, Urbana, Illinois, Sept. 1971.

[32] T. H. H. PIAN, Finite element formulation by variational principles with relaxed continuity requirements, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (A. K. Aziz, Editor), Academic Press, New York, 1972.

[33] T. H. H. PIAN and P. TONG, *Basis of finite element methods for solid continua*, Internat. J. Numer. Methods Engrg, Vol. I, 3-28, 1969.

[34] T. H. H. PIAN and P. TONG, *A variational principle and the convergence of a finite element method based on assumed stress distribution*, Int. J. Solids & Struct. 5, pp. 463-472, 1969.

[35] T. H. H. PIAN and P. TONG, Rationalization in Deriving element stiffness Matrix by assumed stress approach, *Proceedings Second Conf. on Matrix Methods in Structural Mechanics*, AFFDL TR-68-150. Wright Patterson, AFB, Ohio, pp. 441-469, 1968.

[36] P. A. RAVIART, *Cours de troisième cycle*, de l'Université de Paris VI, 1971-1972.

[37] P. A. RAVIART, Hybrid finite element methods for solving 2nd order elliptic equations. *Conference on Numerical Analysis*, Royal Irish Academy, Dublin, July 29-August 2, 1974.

[38] P. A. RAVIART and J. M. THOMAS, *Hybrid finite element methods and 2nd order elliptic problems.* (To appear.)

[39] G. SANDER, *Applications de la méthode des éléments finis à la flexion des plaques*, Université de Liège, Fac. de Sci. Appl., Pubblic. N. 15, 1969.

[40] G. STRANG, *Approximation in the finite element method*, Numer. Math., 19, pp. 81-98, 1972.

[41] G. STRANG, Variational Crimes in the finite element method, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (A. K. Aziz, Editor), pp. 689-710, Academic Press, New York, 1972.

[42] G. STRANG and G. FIX, *An Analysis of the Finite Element Method*, Prentice Hall Englewood Cliffs, 1973.

[43] L. TARTAR. (To appear.)

[44] S. TIMOSCHENKO, *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

[45] J. M. THOMAS. (To appear.)

[46] O. C. ZIENKIEWICZ, *The finite element method in engineering Science*, McGraw-Hill, London, 1971.

[47] M. ZLAMAL, *On the finite element method*, Numer. Math., 12, pp. 394-409, 1968.