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A RITZ METHOD BASED ON A COMPLEMENTARY VARIATIONAL PRINCIPLE (1)

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Abstract. — *We show how a finite element method based on a complementary variational principle for the approximation of a second order elliptic Dirichlet problem leads to the same equations and a more direct proof of estimates obtained for a finite element method using Lagrangian multipliers.*

1. INTRODUCTION

In this paper we wish to re-examine the application of Lagrange multipliers to the finite element method for the approximation of the Dirichlet problem for second order elliptic operators. A finite element method using this approach was first analyzed by Babuška in [4].

There are three purposes for this re-examination. The first is to see more clearly the sources of error arising from the approximation scheme by viewing this finite element method as an approximation based on a complementary variational principle (i. e. as an approximation to the dual problem) rather than as arising from a search for a stationary point. The second is to simplify the proofs of the error estimates. The final purpose is to show why one still obtains a good approximation to the solution when the method is applied on certain special domains (e. g. rectangles), even though the conditions imposed in [4] are violated.

We will be considering then the approximation of the problem

$$\left.\begin{aligned} Au &= f \text{ in } \Omega \\ u &= g \text{ on } \Gamma \end{aligned}\right\} \qquad (\bigstar)$$

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where

$$A u = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j} \right) + cu$$

and Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary Γ. We will assume that $a_{ij}(x)$ and $c(x)$ satisfy the following:

(H1) $$a_{ij}(x) \in C^1(\bar{\Omega}), \qquad c(x) \in L^\infty(\Omega),$$

(H2) $$a_{ij}(x) = a_{ji}(x), \qquad i, j = 1, \ldots, N,$$

(H3) $$c(x) \geqq c_0 > 0 \text{ for some constant } c_0,$$

(H4) $$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geqq \alpha \sum_{i=1}^{N} \xi_i^2,$$

for all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbf{R}^N$ and some constant $\alpha > 0$.

Define a bilinear form

$$a(u, v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_\Omega c(x) uv \, dx.$$

Under hypotheses (H 1)-(H 4) it is easy to verify that $\| v \|_E = [a(v, v)]^{1/2}$ is a norm equivalent to the $H^1(\Omega)$ norm, i. e. there exist positive constants d_1 and d_2 such that

$$d_1 \| v \|_1 \leqq \| v \|_E \leqq d_2 \| v \|_1. \tag{5}$$

Making use of a standard complementary variational principle, (e. g. *see* [1]) the solution u of (★) minimizes the functional

$$J(v) = \frac{1}{2} a(v, v) - \langle Bv, g \rangle$$

over all v satisfying $Av = f$ in Ω, where

$$B v = \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \nu_j, \qquad \nu = (\nu_1, \ldots, \nu_N)$$

is the outward normal and $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Gamma)$ inner product.

The approach we will take is to introduce a new variable α and define $u(\alpha)$ to be the solution of

$$a(u(\alpha), v) = (f, v) + \langle \alpha, v \rangle \quad \text{for all } v \in H^1(\Omega),$$

i. e., $u(\alpha)$ is the weak solution of the boundary value problem

$$Au(\alpha) = f \text{ in } \Omega,$$

$$Bu(\alpha) = \alpha \text{ on } \Gamma.$$

Hence we view the family of solutions of the constraint $Av = f$ as being parametrized by their conormal derivatives on Γ. We then observe that if θ minimizes the functional

$$L(\alpha) = \frac{1}{2} a(u(\alpha), u(\alpha)) - \langle \alpha, g \rangle$$

subject to $a(u(\alpha), v) = (f, v) + \langle \alpha, v \rangle$ for all $v \in H^1(\Omega)$, then $u(\theta)$ is the solution of (★). Writing out the variational equations, we obtain $a(u(\theta), u(\alpha) - u(0)) = \langle \alpha, g \rangle$ for all $\alpha \in H^{-1/2}(\Gamma)$, or using the definition of $u(\alpha)$,

$$\langle u(\theta) - g, \alpha \rangle = 0 \quad \text{for all } \alpha \in H^{-1/2}(\Gamma).$$

Now the solution u of ★ satisfies $\langle u - g, \alpha \rangle = 0$ for all $\alpha \in H^{-1/2}(\Gamma)$ and $a(u, v) = (f, v)$ for all $v \in H_0^1(\Omega)$. Hence

$$u - u(\theta) \in H_0^1(\Omega) \qquad \text{and} \qquad a(u - u(\theta), v) = 0$$

for all $v \in H_0^1(\Omega)$. Choosing $v = u - u(\theta)$, we obtain $\| u - u(\theta) \|_E = 0$ or $u = u(\theta)$.

We will consider a Ritz method based on this last minimization problem:

Problem (P_h) : Find $\theta_h \in T_{h_2}(\Gamma)$ such that

$$L_h(\theta_h) = \inf_{\alpha_h \in T_{h_2}(\Gamma)} L_h(\alpha_h)$$

where

$$L_h(\alpha_h) = \frac{1}{2} a(u_h(\alpha_h), u_h(\alpha_h)) - \langle \alpha_h, g \rangle$$

and $u_h(\alpha_h) \in T_{h_1}(\Omega)$ is the solution of

$$a(u_h(\alpha_h), v_h) = (f, v_h) + \langle \alpha_h, v_h \rangle \quad \text{for all } v_h \in T_{h_1}(\Omega).$$

The finite dimensional subspaces $T_{h_1}(\Omega)$ and $T_{h_2}(\Gamma)$ contain functions defined on Ω and Γ respectively, and will be defined later.

Writing out the variational equations for this approximate problem, we obtain

$$a(u_h(\theta_h), u_h(\alpha_h) - u_h(0)) = \langle \alpha_h, g \rangle \quad \text{for all } \alpha_h \in T_{h_2}(\Gamma)$$

and

$$a(u_h(\alpha_h), v_h) = (f, v_h) + \langle \alpha_h, v_h \rangle \quad \text{for all } v_h \in T_{h_1}(\Omega).$$

Since

$$a(u_h(\theta_h), u_h(\alpha_h) - u_h(0)) = \langle \alpha_h, u_h(\theta_h) \rangle,$$

we obtain

$$\langle u_h(\theta_h) - g, \alpha_h \rangle = 0 \quad \text{for all } \alpha_h \in T_{h_2}(\Gamma) \tag{6)}$$

and

$$a(u_h(\theta_h), v_h) = (f, v_h) + \langle \theta_h, v_h \rangle \quad \text{for all } v_h \in T_{h_1}(\Omega). \tag{7}$$

These are the same equations obtained in [4].

In the next section we describe the notation and the principle ideas to be used in the derivation of the error estimates.

2. NOTATION

For $s \geqq 0$ let $H^s(\Omega)$ and $H^s(\Gamma)$ denote the Sobolev spaces or order s of functions on Ω and Γ respectively, with associated norms $\|.\|_s$ and $|.|_s$, respectively. For definitions and characterizations of these spaces, the conventions of [6] are adopted. We will also be using the space $H^s(\Gamma)$ for $s < 0$, normed by

$$|g|_s = \sup_{v \in H^{-s}(\Gamma)} \frac{\langle g, v \rangle}{|v|_{-s}}.$$

We will need the following facts, proved in [6], with constants C independent of w, f, and g.

LEMMA 1: *Let $w \in H^k(\Omega)$, $k > 1/2$. Then its boundary values satisfy*

$$|w|_{k-(1/2)} \leqq C \|w\|_k.$$

LEMMA 2: *Let $w \in H^k(_r)$, $k > 3/2$. Then there exists a trace $\partial w/\partial n$ on Γ and,*

$$\left|\frac{\partial w}{\partial n}\right|_{k-(3/2)} \leqq C \|w\|_k.$$

LEMMA 3: *Let $f \in H^k(\Omega)$, $k \geqq 0$, $g \in H^l(\Gamma)$, $l \geqq 1/2$. Then there exists in $H^1(\Omega)$ exactly one weak solution of the equation $Au = f$ with boundary condition $u = g$. Furthermore, $u \in H^s(\Omega)$ where $s = \min(k+2, l+(1/2))$ and*

$$\|u\|_s \leqq C[\|f\|_k + |g|_l].$$

LEMMA 4: *Let* $f \in H^k(\Omega)$, $g \in H^l(\Gamma)$, $l \geqq -1/2$. *Then there exists in* $H^1(\Omega)$ *exactly one weak solution of the equation* $Au = f$ *with boundary condition* $Bu = g$. *Furhthermore,* $u \in H^s(\Omega)$ *where* $s = \min(k+2, l+(3/2))$ *and*

$$\|u\|_s \leqq C[\|f\|_k + |g|_l].$$

LEMMA 5: *Let* $g \in H^{-1/2}(\Gamma)$ *and let* $u \in H^1(\Omega)$ *be the weak solution of the equation* $Au = 0$ *with boundary condition* $Bu = g$. *Then* $|g|_{-1/2} \leqq C\|u\|_1$.

Proof: For $v \in H^{1/2}(\Gamma)$ let $w \in H^1(\Omega)$ be the weak solution of the equation $Aw = 0$ with boundary condition $w = v$. Then

$$|g|_{-1/2} = \sup_{v \in H^{1/2}(\Gamma)} \frac{\langle g, v \rangle}{|v|_{1/2}} = \sup_{v \in H^{1/2}(\Gamma)} \frac{a(u, w)}{|v|_{1/2}} \leqq \sup_{v \in H^{1/2}(\Gamma)} \frac{C\|u\|_1 \|w\|_1}{|v|_{1/2}}$$

$$\leqq C\|u\|_1 \quad \text{since} \quad \|w\|_1 \leqq C|v|_{1/2}$$

by Lemma 3.

We now introduce the finite dimensional subspaces we will be using in our approximation scheme. Following Babuška [4] and [5] we will define for all $0 < h < 1$ two one-parameter families of finite dimensional spaces which we will denote $S_h^{t,k}(\Omega)$ and $S_h^{t,k}(\Gamma)$. We call $S_h^{t,k}(\Omega)$ [resp. $S_h^{t,k}(\Gamma)$] a (t, k) system for $t > k \geqq 0$ [resp. $t > k \geqq -1/2$] if

(A1) $S_h^{t,k}(\Omega) \subset H^k(\Omega)$ [resp. $S_h^{t,k}(\Gamma) \subset H^k(\Gamma)$];

(A2) if

$$f \in H^l(\Omega)\,[\text{resp. } f \in H^l(\Gamma)] \qquad \text{and} \qquad 0 \leqq s \leqq k \leqq l\,[\text{resp. } -\tfrac{1}{2} \leqq s \leqq k \leqq l]$$

then there exists $g \in S_h^{t,k}(\Omega)$ [resp. $g \in S_h^{t,k}(\Gamma)$] such that $\|g-f\|_s \leqq Ch^\mu \|f\|_l$ [resp. $|g-f|_s \leqq Ch^\mu |f|_l$] where $\mu = \min(l-s, t-s)$ and C does not depend on s, h, or f. Note that the function g may be different for different s.

If the function g can be chosen independently of s, then the system will be called regular.

Finally we say that the regular system $S_h^{t,k}(\Gamma)$ is strongly regular if its members satisfy $|g|_s \leqq Ch^{-(s-q)}|g|_q$ for $-1/2 \leqq q \leqq s \leqq k$. Such systems are constructed in [2] and [3].

We now proceed with the derivation of the error estimates.

3. ERROR ESTIMATES

THEOREM 1: *Suppose* $f \in H^{r-2}(\Omega)$, $g \in H^{r-(1/2)}(\Gamma)$, *and* $u = u(\theta)$ *is the solution of the Dirichlet problem* (★). *Let* $u_h(\theta_h)$ *be the solution of Problem* P_h *with* $T_{h_1}(\Omega)$ *a* (t_1, k_1) *system and* $T_{h_2}(\Gamma)$ *a strongly* (t_2, k_2) *regular system with* $t_1 \geqq 2$, $k_1 \geqq 1$, *and* $k_2 \geqq 1/2$. *If* $h_2 \geqq Kh_1$ *for* K *sufficiently large* (K *a* *constant independent of* h_1), *then there exists a constant C independent of h and u such that*

$$\| u(\theta)-u_h(\theta_h)\|_1+|\theta-\theta_h|_{-1/2} \leqq C h^{\mu}[\| f \|_{r-2}+| g |_{r-(1/2)}],$$

where $h = \max(h_1, h_2)$ *and* $\mu = \min(r-1, t_1-1, t_2+(1/2))$.

Proof: Since $u(\theta)$ is the solution of Problem (★),

$$\langle u(\theta)-g, \alpha \rangle = 0 \quad \text{for all } \alpha \in H^{-1/2}(\Gamma).$$

By (6),

$$\langle u_h(\theta_h)-g, \alpha_h \rangle = 0 \quad \text{for all } \alpha_h \in T_{h_2}(\Gamma).$$

Hence

$$\langle u(\theta)-u_h(\theta_h), \alpha_h \rangle = 0 \quad \text{for all } \alpha_h \in T_{h_2}(\Gamma).$$

Now

$$\begin{aligned} &\| u(\theta)-u_h(\theta_h)\|_E^2 \\ &\quad = a(u(\theta)-u_h(\theta_h), u(\theta)-u(\alpha_h)) \\ &\qquad + a(u(\theta)-u_h(\theta_h), u(\alpha_h)-u(\theta_h))+a(u(\theta)-u_h(\theta_h), u(\theta_h)-u_h(\theta_h)). \end{aligned}$$

We first observe that

$$a(u(\theta)-u_h(\theta_h), u(\alpha_h)-u(\theta_h)) = \langle \alpha_h-\theta_h, u(\theta)-u_h(\theta_h) \rangle = 0,$$

by the result obtained above. Applying the Schwarz inequality to the remaining two terms and collecting terms, we obtain

$$\frac{1}{2}\| u(\theta)-u_h(\theta_h)\|_E^2 \leqq \| u(\theta)-u(\alpha_h)\|_E^2+\| u(\theta_h)-u_h(\theta_h)\|_E^2.$$

By Lemma (4) and Statement 5:

$$\| u(\theta)-u(\alpha_h)\|_E \leqq C|\theta-\alpha_h|_{-1/2}.$$

Hence we obtain

$$\| u(\theta)-u_h(\theta_h)\|_E \leqq C[|\theta-\alpha_h|_{-1/2}+\| u(\theta_h)-u_h(\theta_h)\|_E].$$

Now by Lemma (5),

$$|\theta-\theta_h|_{-1/2} \leqq C\| u(\theta)-u(\theta_h)\|_E \leqq [\| u(\theta)-u_h(\theta_h)\|_E+\| u_h(\theta_h)-u(\theta_h)\|_E].$$

Combining these results, we get

$$|\theta-\theta_h|_{-1/2}+\| u(\theta)-u_h(\theta_h)\|_E \leqq C[|\theta-\alpha_h|_{-1/2}+\| u(\theta_h)-u_h(\theta_h)\|_E]. \quad (8)$$

We remark that this espression reflects precisely the two sources of error arising in the approximation scheme. The first term reflects the fact that the energy functional is being minimized over only a finite dimensional subspace and the second term occurs due to the fact that the functional itself has been modified by replacing $u(\theta_h)$ by its "Galerkin" approximation.

Using the approximation assumption (A2), the first term on the right of (8) is bounded by $Ch_2^{\mu_2} |\theta|_{r-(3/2)}$, where $\mu_2 = \min(r-1, t_2+(1/2))$. The second term is the troublesome one and is typical of this type of analysis. In this case the troubles can be overcome by making use of the strong hypotheses we have made in the theorem about the approximation properties of the subspaces and their relationship. Now

$$\| u(\theta_h) - u_h(\theta_h) \|_E \leqq \| [u(\theta_h) - u(\theta)] - [u_h(\theta_h) - u_h(\theta)] \|_E + \| u(\theta) - u_h(\theta) \|_E .$$

Since $a(u(\alpha) - u_h(\alpha), z_h) = 0$ for all $z_h \in T_{h_1}(\Omega)$, we have for z_h and $v_h \in T_{h_1}(\Omega)$ that

$$\begin{aligned}\| u(\theta_h) - u_h(\theta_h) \|_E &\leqq \| u(\theta_h) - u(\theta) - z_h \|_E + \| u(\theta) - v_h \|_E \\ &\leqq C h_1 \| u(\theta_h) - u(\theta) \|_2 + C h_1^{\mu_1} \| u(\theta) \|_r ,\end{aligned}$$

where $\mu_1 = \min(r-1, t_1-1)$ (by using A2). Now

$$\begin{aligned}\| u(\theta_h) - u(\theta) \|_2 &\leqq C |\theta - \theta_h|_{1/2} \quad \text{(by Lemma 4)} \\ &\leqq C [|\theta - \alpha_h|_{1/2} + |\alpha_h - \theta_h|_{1/2}] \\ &\leqq C \left[|\theta - \alpha_h|_{1/2} + \frac{C}{h_2} |\alpha_h - \theta_h|_{-1/2} \right] \quad \text{(by strong regularity)} \\ &\leqq C \left[|\theta - \alpha_h|_{1/2} + \frac{C}{h_2} |\alpha_h - \theta|_{-1/2} + \frac{C}{h_2} |\theta - \theta_h|_{-1/2} \right].\end{aligned}$$

Applying the approximability assumption (A2), we obtain

$$\begin{aligned}\| u(\theta_h) - u_h(\theta_h) \|_E &\leqq C h_1 \left[h_2^{\mu_2 - 1} |\theta|_{r-(3/2)} + \frac{C}{h_2} |\theta - \theta_h|_{-1/2} \right] + C h_1^{\mu_1} \| u(\theta) \|_r \\ &\leqq C h_1 h_2^{\mu_2 - 1} |\theta|_{r-(3/2)} + \frac{C}{K} |\theta - \theta_h|_{-1/2} + C h_1^{\mu_1} \| u(\theta) \|_r .\end{aligned}$$

Since $|\theta|_{r-(3/2}$ and $\| u(\theta) \|_r$ are bounded by $C[\| f \|_{r-2} + |g|_{r-(1/2)}]$ by Lemmas 2 and 3, we have after collecting terms that for K sufficiently large

$$\begin{aligned}&|\theta - \theta_h|_{-1/2} + \| u(\theta) - u_h(\theta_h) \|_E \\ &\quad \leqq C [h_2^{\mu_2} + h_1 h_2^{\mu_2 - 1} + h_1^{\mu_1}] [\| f \|_{r-2} + |g|_{r-(1/2)}].\end{aligned}$$

THEOREM 2: *Suppose* $f \in H^{r-2}(\Omega)$ *and* $u = u(\theta)$ *is the solution of the Dirichlet problem* (★). *Let* $u_h(\theta_h)$ *be the solution of Problem* (P_h) *where* $T_{h_1}(\Omega)$ *is a* (t_1, k_1)*-system and* $T_{h_2}(\Gamma)$ *is a* (t_2, k_2) *system with* $k_1 \geqq 1$, $k_2 \geqq 1/2$. *(Note that* $T_{h_2}(\Gamma)$ *need not be strongly regular.) If there exists* $v_h \in T_{h_1}(\Omega)$ *with* $\langle g - v_h, \alpha_h \rangle = 0$ *for all* $\alpha_h \in T_{h_2}(\Gamma)$ *such that*

$$\| u - v_h \|_1 \leqq C h_1^{\mu_1} \| u(\theta) \|_r$$

where

$$\mu_1 = \min(r-1, t_1 - 1),$$

then

$$\| u(\theta) - u_h(\theta_h) \|_1 \leqq C h^{\mu} [\| f \|_{r-2} + | g |_{r-(1/2)}]$$

where

$$\mu = \min\left(r-1, t_1 - 1, t_2 + \frac{1}{2}\right) \qquad \text{and} \qquad h = \max(h_1, h_2).$$

Proof: Using the same argument as in Theorem 1, we have

$$\| u(\theta) - u_h(\theta_h) \|_E^2 = a(u(\theta) - u_h(\theta_h), u(\theta) - u(\alpha_h)) + a(u(\theta) - u_h(\theta_h), u(\theta_h) - u_h(\theta_h)).$$

Since $a(z_h, u(\theta_h) - u_h(\theta_h)) = 0$ for all $z_h \in T_{h_1}(\Omega)$, it follows that for all $v_h \in T_{h_1}(\Omega)$,

$$\begin{aligned} & a(u(\theta) - u_h(\theta_h), u(\theta_h) - u_h(\theta_h)) \\ &= a(u(\theta) - v_h, u(\theta_h) - u_h(\theta_h)) \\ &= a(u(\theta) - v_h, u(\theta_h) - u(\theta)) + a(u(\theta) - v_h, u(\theta) - u_h(\theta_h)) \\ &= \langle u(\theta) - v_h, \theta_h - \theta \rangle + a(u(\theta) - v_h, u(\theta) - u_h(\theta_h)) \\ &= \langle u(\theta) - v_h, \theta_h - \alpha_h \rangle + \langle u(\theta) - v_h, \alpha_h - \theta \rangle + a(u(\theta) - v_h, u(\theta) - u_h(\theta_h)). \end{aligned}$$

Since $u(\theta) = g$ on Γ and $\theta_h - \alpha_h \in T_{h_2}(\Gamma)$, we have for all $v_h \in T_{h_1}(\Omega)$ with $\langle g - v_h, \alpha_h \rangle = 0$ for all $\alpha_h \in T_{h_2}(\Gamma)$ that

$$\begin{aligned} & \| u(\theta) - u_h(\theta_h) \|_E^2 \\ & \leqq a(u(\theta) - u_h(\theta_h), u(\theta) - u(\alpha_h)) \\ & \quad + \langle u(\theta) - v_h, \alpha_h - \theta \rangle + a(u(\theta) - v_h, u(\theta) - u_h(\theta_h)) \\ & \leqq \| u(\theta) - u_h(\theta_h) \|_E \| u(\theta) - u(\alpha_h) \|_E + | u(\theta) - v_h |_{1/2} | \theta - \alpha_h |_{-1/2} \\ & \quad + \| u(\theta) - v_h \|_E \| u(\theta) - u_h(\theta_h) \|_E . \end{aligned}$$

Applying the arithmetic-geometric mean inequality and Lemmas 1 and 4 we obtain

$$\|u(\theta)-u_h(\theta_h)\|_E \leqq C[\|u(\theta)-v_h\|_1+|\theta-\alpha_h|_{-1/2}]$$
$$\leqq C h_2^{\mu_2}|\theta|_{r-(3/2)}+C h_1^{\mu_1}\|u(\theta)\|_r$$

[by (A2) and the hypothesis of the Theorem], where $\mu_2 = \min(r-1, t_2+(1/2))$, and $\mu_1 = \min(r-1, t_1-1)$. Hence applying Lemmas (2) and (3) we obtain

$$\|u(\theta)-u_h(\theta_h)\|_E \leqq C h^{\mu}[\|f\|_{r-2}+|g|_{r-(1/2)}]$$

where $\mu = \min(r-1, t_1-1, t_2+(1/2))$ and $h = \max(h_1, h_2)$.

REMARK 1: The practical consequence of Theorem 2 is most easily seen in the case $g = 0$. If one then solves Problem (P_h) using a subspace $T_{h_2}(\Gamma)$ which is not strongly regular or where h_2 is not related to h_1 by the K of Theorem 1, then $u_h(\theta_h)$ will still be a good approximation to $u(\theta)$ in the energy norm provided there happens to be a function in $T_{h_1}(\Omega) \cap H_0^1(\Omega)$ which is a good approximation to $u(\theta)$. Note that the functions in $T_{h_1}(\Omega)$ do not have to lie in $H_0^1(\Omega)$. Thus, for example, one might expect good numerical results on model problems solved on rectangles even when the conditions of Theorem 1 are violated.

REMARK 2: In the case described above (i. e. without the inverse assumptions and relation between h_1 and h_2), it is no longer necessarily true that θ_h will be unique. However, from (7) it easily follows that if there is any solution θ_h to Problem (P_h) then $u_h(\theta_h)$ exists. Furthermore if θ_h^1, θ_h^2 are two solutions, then

$$\|u_h(\theta_h^1)-u_h(\theta_h^2)\|_E^2 = \langle u_h(\theta_h^1)-u_h(\theta_h^2), \theta_h^1-\theta_h^2\rangle$$
$$= \langle u_h(\theta_h^1)-g, \theta_h^1-\theta_h^2\rangle+\langle g-u_h(\theta_h^2), \theta_h^1-\theta_h^2\rangle = 0$$

by (6). Hence $u_h(\theta_h)$ will be unique.

Using the standard technique (e. g. *see* [4]) we can obtain the following estimate for the error in $L^2(\Omega)$.

THEOREM 3: *Under the hypotheses of Theorem* 1:

$$\|u(\theta)-u_h(\theta_h)\|_{L^2(\Omega)} \leqq C h^{\mu+1}[\|f\|_{r-2}+|g|_{r-(1/2)}]$$

where

$$h = \max(h_1, h_2) \quad \text{and} \quad \mu = \min\left(r-1, t_1-1, t_2+\frac{1}{2}\right).$$

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