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LOCAL H^{-1} GALERKIN AND ADJOINT LOCAL H^{-1} GALERKIN PROCEDURES FOR ELLIPTIC EQUATIONS (*)

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Abstract. — Two essentially dual, finite element methods for approximating the solution of the boundary value problem $Lu = \nabla \cdot (a \nabla u) + b \cdot \nabla u + cu = f$ on Ω , a rectangle, with $u = 0$ on $\partial\Omega$ are shown to give optimal order convergence. The local H^{-1} method is based on the inner product $(u, L^ v)$ and the adjoint method on (Lu, v) . Discontinuous spaces can be employed for the approximate solution in the local H^{-1} procedure and for the test space in the adjoint method.*

1. INTRODUCTION

Consider the elliptic boundary value boundary problem

$$\left. \begin{aligned} (Lu)(p) &= \nabla \cdot (a(p)\nabla u) + b(p) \cdot \nabla u + c(p)u = f(p), & p \in \Omega, \\ u(p) &= 0, & p \in \partial\Omega, \end{aligned} \right\} \quad (1)$$

where Ω is the square $I \times I$ and $I = (0, 1)$. We assume that $a, (\nabla a)_i, b_i, c \in C^1(\bar{\Omega})$, that $f \in L_2(\Omega)$, and that $0 < a_0 \leq a(p) \leq a_1, p \in \bar{\Omega}$, where a_0 and a_1 are constants. We further assume that, given $g \in L_2(\Omega)$, there exists a unique function $\phi \in H^2(\Omega)$ satisfying $L\phi = g$ in Ω and $\phi = 0$ on $\partial\Omega$.

We shall use the following notation. Let $\delta : 0 = x_0 < x_1 < \dots < x_N = 1$ be a partition of $[0, 1]$. Set $I_j = (x_{j-1}, x_j), h_j = x_j - x_{j-1}$, and $h = \max_{1 \leq j \leq N} h_j$.

For $E \subset I$ let $P_r(E)$ denote the functions defined on I whose restrictions to E coincide with polynomials of degree at most r . Let

$$\mathcal{M}(-1, r, \delta) = \bigcap_{j=1}^N P_r(I_j)$$

and, for k a non-negative integer,

$$\begin{aligned} \mathcal{M}(k, r, \delta) &= \mathcal{M}(-1, r, \delta) \cap C^k(I), \\ \mathcal{M}^0(k, r, \delta) &= \mathcal{M}(k, r, \delta) \cap \{v \mid v(0) = v(1) = 0\}, \\ \tilde{\mathcal{M}}(k-1, r-1, \delta) &= \{v' : v \in \mathcal{M}^0(k, r, \delta)\}. \end{aligned}$$

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We assume that δ is quasi-uniform and that $r \geq 1$. For brevity, we set

$$\mathcal{N} = \mathcal{M}^0(k+2, r+2, \delta) \otimes \mathcal{M}^0(k+2, r+2, \delta),$$

$$\mathcal{Q} = \tilde{\mathcal{M}}(k+1, r+1, \delta) \otimes \tilde{\mathcal{M}}(k+1, r+1, \delta),$$

and

$$\mathcal{M} = \mathcal{M}(k, r, \delta) \otimes \mathcal{M}(k, r, \delta).$$

Note that \mathcal{Q} and \mathcal{M} are the images of \mathcal{N} under the maps given by $\partial^2/\partial x \partial y$ and $\partial^4/\partial x^2 \partial y^2$, respectively.

The local H^{-1} Galerkin approximation is defined as the solution $U \in \mathcal{M}$ of the equations

$$(U, L^* \varphi) = (f, \varphi), \quad \varphi \in \mathcal{N}, \tag{2}$$

where the inner product is the standard $L_2(\Omega)$ one. The adjoint local H^{-1} Galerkin approximation is given by $W \in \mathcal{N}$ satisfying

$$(LW, \varphi) = (f, \varphi), \quad \varphi \in \mathcal{M}. \tag{3}$$

We first show that there exists a unique U and a unique W satisfying (2) and (3), respectively, for $L = \Delta$. Optimal L_2 error estimates are also obtained for the operator Δ . We then generalize our results to obtain optimal L_2 results for operators of the form given in (1).

Let $H^k(\Omega)$ be the Sobolev space of functions having $L_2(\Omega)$ -derivatives through order k . Denote the usual norm on $H^s(\Omega)$ by $\|\cdot\|_s$; for $s = 0$ the subscript will be omitted. We also use the norm

$$\|w\|_{-1} = \sup_{z \in H^1(\Omega)} \frac{(w, z)}{\|z\|_1}.$$

If the reader wishes to use any of the results derived below for non-integral indices, then standard interpolation theory [3] should be applied.

2. ERROR ESTIMATES FOR $L = \Delta$

First note that, since $\dim \mathcal{M} = \dim \mathcal{N}$, uniqueness implies existence.

LEMMA 1 : Suppose that $V \in \mathcal{M}$ satisfies

$$(V, \Delta \varphi) = 0, \quad \varphi \in \mathcal{N}.$$

Then, $V \equiv 0$.

Proof : Note that there exists a unique $Q \in \mathcal{N}$ such that $Q_{xxyy} = V$. Integrating by parts, we have

$$(\nabla Q_{xy}, \nabla w) = 0, \quad w \in \mathcal{Q}.$$

Since $Q_{xy} \in \mathcal{Q}$, we note that $Q_{xxy} = 0$ and $Q_{yyx} = 0$. Thus, $V = 0$.

Since the matrix arising in (3) is the adjoint of that of (2), there exists a unique W satisfying (3) for $L = \Delta$.

We now derive L_2 and negative norm error estimates for $U-u$ when $L = \Delta$. Let $Z \in \mathcal{N}$ satisfy $Z_{xxyy} = U$. Also let $z_{xxyy} = u$ in Ω and $z = 0$ on $\partial\Omega$. We observe from (1) and (2) with $\xi = Z-z$ that

$$(\nabla \xi_{xy}, \nabla w) = 0, \quad w \in \mathcal{Q}. \tag{4}$$

THEOREM 1 : *Let z and Z be as defined above, and let $z_{xy} \in H^s(\Omega)$ for some s such that $1 \leq s \leq r+2$. Then,*

$$\| (z-Z)_{xy} \| + h \| (z-Z)_{xy} \|_1 \leq C \| z_{xy} \|_s h^s.$$

Proof : It follows from (4) that

$$\| \nabla \xi_{xy} \| = \inf_{\chi \in \mathcal{Q}} \| \nabla (z_{xy} - \chi) \|. \tag{5}$$

Let $T : H^1(I) \rightarrow \mathcal{M}(k+1, r+1, \delta)$ be determined by the relations

$$\int_0^1 (g - Tg)' v \, dx = \int_0^1 (g - Tg) \, dx = 0, \quad v \in \mathcal{M}(k, r, \delta).$$

It is easy to see that $(g - Tg)(0) = (g - Tg)(1) = 0$, by taking $v = x$ or $1-x$. Since $(Tg)'$ is the $L_2(I)$ -projection of g' into $\mathcal{M}(k, r, \delta)$,

$$\| (g - Tg)' \|_{L_2(I)} \leq C \| g^{(s)} \|_{L_2(I)} h^{s-1}, \quad 1 \leq s \leq r+2.$$

Let

$$\begin{aligned} -\varphi'' &= \zeta = g - Tg, & x \in I, \\ \varphi'(0) &= \varphi'(1) = 0, \\ \int_0^1 \varphi \, dx &= 0. \end{aligned}$$

Then for $v \in \mathcal{M}(k, r, \delta)$ appropriately chosen

$$\| \zeta \|^2 = (\zeta, \varphi' - v) \leq C \| \zeta' \|_{L_2(I)} \| \zeta \|_{L_2(I)} h,$$

and

$$\| g - Tg \|_{L_2(I)} \leq C \| g^{(s)} \|_{L_2(I)} h^{(s)}, \quad 1 \leq s \leq r+2.$$

Consider $(T \otimes T) z_{xy} \in \mathcal{M}(k+1, r+1, \delta) \otimes \mathcal{M}(k+1, r+1, \delta)$. It is easy to see that $(T \otimes T) z_{xy} \in \mathcal{Q}$ and that

$$\| z_{xy} - (T \otimes T) z_{xy} \|_q \leq C \| z_{xy} \|_s h^{s-q}, \quad 2 \leq s \leq r+2, \quad 0 \leq q \leq 1, \tag{6}$$

since $T \otimes T - I \otimes I = (T-I) \otimes I + I \otimes (T-I) + (T-I) \otimes (T-I)$. Thus, from (5) and (6),

$$\| \nabla \xi_{xy} \| \leq C \| z_{xy} \|_s h^{s-1}, \quad 2 \leq s \leq r+2.$$

The inequality

$$\| \nabla \xi_{xy} \| \leq C \| \nabla z_{xy} \| \leq C \| z_{xy} \|_1,$$

is obvious, and the desired result follows:

$$\|\nabla \xi_{xy}\| \leq C \|z_{xy}\|_s h^{s-1}, \quad 1 \leq s \leq r+2.$$

Since ξ_{xy} has average value zero,

$$\|\xi_{xy}\|_1 \leq C \|z_{xy}\|_s h^{s-1}, \quad 1 \leq s \leq r+2.$$

To obtain the $L_2(\Omega)$ estimate, first let

$$\begin{aligned} -\Delta\varphi &= \xi_{xy}, & (x, y) \in \Omega, \\ \frac{\partial\varphi}{\partial n} &= 0, & (x, y) \in \partial\Omega. \end{aligned}$$

Since $(\xi_{xy}, 1) = 0$, there exists φ such that $(\varphi, 1) = 0$ and $\|\varphi\|_2 \leq C \|\xi_{xy}\|$. Then,

$$\|\xi_{xy}\|^2 = (\nabla \xi_{xy}, \nabla(\varphi - \chi)), \quad \chi \in \mathcal{Q},$$

and

$$\|\xi_{xy}\|^2 \leq C \|\nabla \xi_{xy}\| \inf_{\chi \in \mathcal{Q}} \|\nabla(\varphi - \chi)\|.$$

The function ξ_{xy} can be expanded in a double cosine series:

$$\xi_{xy} = \sum_{p, q=1}^{\infty} c_{pq} \cos \pi p x \cos \pi q y.$$

Thus,

$$\varphi = \frac{1}{\pi^2} \sum_{p, q=1}^{\infty} \frac{c_{pq}}{p^2 + q^2} \cos \pi p x \cos \pi q y.$$

It then follows by approximating each product of cosines in \mathcal{Q} that

$$\inf_{\chi \in \mathcal{Q}} \|\nabla(\varphi - \chi)\| \leq Ch \|\xi_{xy}\|,$$

and the theorem has been proved.

Denote by P the restriction of the projection T to the subclass of $H^1(I)$ consisting of functions having zero average value. Let $\mathcal{P} = P \otimes P$.

We wish to obtain a better H^1 estimate of $v = \mathcal{P} z_{xy} - Z_{xy}$ than would follow from (6) and theorem 1. We deduce from (4) that

$$(\nabla v, \nabla w) = (\nabla(\mathcal{P} z_{xy} - z_{xy}), \nabla w) = \tau_x + \tau_y, \quad w \in \mathcal{L}. \tag{7}$$

Using the definition of P and integration by parts, we see that, for $w \in \mathcal{L}$,

$$\begin{aligned} \tau_x &= (((I \otimes P)(P \otimes I) z_{xy} - z_{xy})_x, w_x) \\ &= (I \otimes (P - I) z_{xxy}, w_x) \\ &= -(I \otimes (P - I) z_{xxy}, w) \\ &\quad + \int_0^1 I \otimes (P - I) z_{xxy}(\cdot, y) w(\cdot, y) \Big|_0^1 dy. \end{aligned} \tag{8}$$

Note that z has the representation

$$\begin{aligned}
 z(x, y) = & \int_0^y \int_0^x (x-\alpha)(y-\beta) u(\alpha, \beta) d\alpha d\beta \\
 & - x \int_0^y \int_0^1 (1-\sigma)(y-\beta) u(\alpha, \beta) d\alpha d\beta \\
 & - y \int_0^1 \int_0^x (x-\alpha)(1-\beta) u(\alpha, \beta) d\alpha d\beta \\
 & + xy \int_0^1 \int_0^1 (1-\alpha)(1-\beta) u(\alpha, \beta) d\alpha d\beta.
 \end{aligned} \tag{9}$$

One can easily verify from (9) that the boundary terms in (8) are zero since $z_{xxy}(0, y) = 0$ and $z_{xxy}(1, y) = 0$. We also observe that

$$\int_0^1 z_{xxxy} dy = z_{xxx}(x, 1) - z_{xxx}(x, 0) = 0,$$

since z vanishes on the boundary. Similarly, $\int_0^1 z_{yyyyx} dx = 0$. Thus, we see that

$$\|v\|_1 \leq C \|\Psi\|_{-1}, \tag{10}$$

where

$$\Psi = I \otimes (I - P)(z_{xxxy}) + (I - P) \otimes I(z_{yyyy}). \tag{11}$$

It follows that

$$\begin{aligned}
 \|\Psi\|_{-1} \leq & \left(\int_0^1 \left\| I \otimes (I - P) \frac{\partial^4 z}{\partial x^3 \partial y} (x, \cdot) \right\|_{H^{-1}(I)}^2 dx \right)^{1/2} \\
 & + \left(\int_0^1 \left\| (I - P) \otimes I \frac{\partial^4 z}{\partial x \partial y^3} (\cdot, y) \right\|_{H^{-1}(I)}^2 dy \right)^{1/2}.
 \end{aligned}$$

It is easy to show that

$$\|(I - P)f\|_{H^{-1}(I)} \leq C \|f^{(s)}\|_{L^2(I)} h^{s+1},$$

provided that

$$\int_0^1 f dx = 0,$$

by using the auxiliary problem

$$\begin{aligned}
 -\varphi'' &= g - \int_0^1 g dx, & x \in I, \\
 \varphi'(0) &= \varphi'(1) = 0,
 \end{aligned}$$

where $g \in H^1(I)$. Thus,

$$\|\Psi\|_{-1} \leq C \left\{ \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^{s+2} \quad (12)$$

for $0 \leq s \leq r+1$.

THEOREM 2 : *Let u be the solution to (1) with $L = \Delta$, and let $U \in \mathcal{M}$ satisfy (2). Let \hat{U} be the L_2 projection of u into \mathcal{M} . Then,*

$$\|U - \hat{U}\| \leq C \left\{ \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^{s+1} \quad (13)$$

for $0 \leq s \leq r+1$.

Proof : Since \hat{U} satisfies

$$(\hat{U} - u, v) = 0, \quad v \in \mathcal{M},$$

one can easily verify that

$$\hat{U} = (\mathcal{P} z_{xy})_{xy}.$$

Thus, (13) follows from (10), (12), and the quasi-uniformity hypothesis on the partition δ .

COROLLARY : *The error $U - u$ satisfies the following bounds:*

$$\begin{aligned} \|U - u\| &\leq C \|u\|_s h^s, \quad 1 \leq s \leq r+1, \\ \|U - u\|_{L_\infty(\Omega)} &\leq C \left\{ \|u\|_{W_\infty^s(\Omega)} + \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^s. \\ &0 \leq s \leq r+1. \end{aligned}$$

Proof: The $L_2(\Omega)$ -estimate is a trivial consequence of (13). To obtain the $L_\infty(\Omega)$ -estimate, note first that (13) and the quasi-uniformity of δ imply that, for $0 \leq s \leq r+1$,

$$\|U - \hat{U}\|_{L_\infty(\Omega)} \leq \|v\|_{W_\infty^2(\Omega)} \leq C \left\{ \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^s.$$

It follows from inequality (28) of [2] or from [1] that

$$\|u - \hat{U}\|_{L_\infty(\Omega)} \leq C \|u\|_{W_\infty^2(\Omega)} h^s, \quad 0 \leq s \leq r+1.$$

We now wish to consider the adjoint local H^{-1} Galerkin procedure for $L = \Delta$. As noted earlier, there exists a unique $W \in \mathcal{N}$ satisfying

$$(\Delta W, v) = (f, v), \quad v \in \mathcal{M}. \quad (14)$$

THEOREM 3: *Let u be the solution to (1) with $L = \Delta$ and assume that $u_{xy} \in H^s(\Omega)$, $1 \leq s \leq r+2$. Let $W \in \mathcal{N}$ be defined by (14). Then,*

$$\|(W - u)_{xy}\| + h \|(W - u)_{xy}\|_1 \leq C \|u_{xy}\|_s h^s.$$

Proof: Just as in (4),

$$(\nabla(W-u)_{xy}, \nabla w_{xy}) = 0, \quad w \in \mathcal{N}$$

Since w_{xy} represents an arbitrary element of \mathcal{Q} , the theorem follows from the analysis of (4) given in the proof of theorem 1.

Next, we shall derive an $H^1(\Omega)$ -estimate of the error $W-u$. Note that

$$\begin{aligned} \|\nabla(W-u)\|^2 &= -(\Delta(W-u), W-u) \\ &= -(\Delta(W-u), W-u-\chi), \quad \chi \in \mathcal{M}. \end{aligned} \tag{15}$$

We choose $\chi \in \mathcal{M}$ as the local H^{-1} Galerkin approximation to $W-u$; i. e.,

$$(W-u-\chi, \Delta\phi) = 0, \quad \phi \in \mathcal{N}. \tag{16}$$

By the corollary to theorem 2,

$$\|W-u-\chi\| \leq C \|W-u\|_1 h.$$

From (15) and (16), we see that

$$\|\nabla(W-u)\|^2 = -(W-u-\chi, \Delta(W-u-\mu)), \quad \mu \in \mathcal{N}.$$

Hence,

$$\begin{aligned} \|\nabla(W-u)\|^2 &\leq Ch \|W-u\|_1 \inf_{\mu \in \mathcal{N}} \|u-\mu\|_2 \\ &\leq Ch^{s+1} \|W-u\|_1 \|u\|_{s+2}, \quad 0 \leq s \leq r+1. \end{aligned}$$

Since the boundary values of u were imposed strongly on the elements of \mathcal{N} , the $L_2(\Omega)$ -norm of the $\nabla(W-u)$ is equivalent to the $H^1(\Omega)$ -norm of $W-u$; thus,

$$\|W-u\|_1 \leq C \|u\|_{s+2} h^{s+1}, \quad 0 \leq s \leq r+1.$$

As a result of the quasi-uniformity of δ , it follows easily that

$$\|W-u\|_2 \leq C \|u\|_{s+2} h^s, \quad 0 \leq s \leq r+1. \tag{17}$$

Now, we shall seek an estimate of the error in $L_2(\Omega)$. Consider

$$\begin{aligned} \Delta\phi &= W-u && \text{on } \Omega, \\ \phi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then,

$$\begin{aligned} \|W-u\|^2 &= (W-u, \Delta\phi) \\ &= (\phi, \Delta(W-u)) \\ &= (\phi - \phi^*, \Delta(W-u)), \quad \phi^* \in \mathcal{M}. \end{aligned}$$

Thus, choosing an appropriate ϕ^* , we obtain the inequality

$$\begin{aligned} \|W-u\|^2 &\leq C \|\phi\|_2 h^2 \|\Delta(W-u)\| \\ &\leq C \|W-u\| \|\Delta(W-u)\| h^2; \end{aligned}$$

therefore,

$$\|W-u\| \leq C \|u\|_{s+2} h^{s+2}, \quad 0 \leq s \leq r+1.$$

Summarizing the above results, we have proved the following theorem.

THEOREM 4: *Let u be the solution to (1) with $L = \Delta$ and assume that $u \in H^s(\Omega)$, $2 \leq s \leq r+3$. Then, if W is defined by (14),*

$$\|W-u\|_q \leq C \|u\|_s h^{s-q}, \quad 0 \leq q \leq 2.$$

If $k \geq 0$, then the range on q in theorem 4 can be extended to $0 \leq q \leq \min(k+3, s)$ by repeated use of quasi-uniformity to obtain the analogue of (17) in $H^{k+3}(\Omega)$.

3. THE GENERAL CASE

Let $U \in \mathcal{M}$ be determined as the solution of (2), and introduce an auxiliary function $U_1 \in \mathcal{M}$ as the solution of

$$(U_1 - u, \Delta v) = 0, \quad v \in \mathcal{N}.$$

Let $\xi = U - U_1$, and let ψ be given by the Dirichlet problem

$$\begin{aligned} L^* \psi &= \xi \quad \text{on } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then, if $\psi^* \in \mathcal{N}$,

$$\begin{aligned} \|\xi\|^2 &= (\xi, L^* \psi) \\ &= (\xi, L^*(\psi - \psi^*)) + (\xi, L^* \psi^*) \\ &= (\xi, L^*(\psi - \psi^*)) + (\eta, L^* \psi^*), \end{aligned}$$

where $\eta = u - U_1$. We choose $\psi^* \in \mathcal{M}$ to satisfy

$$(\Delta(\psi - \psi^*), v) = 0, \quad v \in \mathcal{M}.$$

Thus, with \tilde{b} and \tilde{c} indicating the lower order coefficients of L^* ,

$$\begin{aligned} \|\xi\|^2 &= (a\xi, \Delta(\psi - \psi^*)) + (\xi, \tilde{b} \cdot \nabla(\psi - \psi^*)) \\ &\quad + (\xi, \tilde{c}(\psi - \psi^*)) + (\eta, L^* \psi^*) \\ &= (a\xi - \chi, \Delta(\psi - \psi^*)) + (\xi, \tilde{b} \cdot \nabla(\psi - \psi^*)) \\ &\quad + (\xi, \tilde{c}(\psi - \psi^*)) + (\eta, L^* \psi^*), \quad \chi \in \mathcal{M}. \end{aligned}$$

It is well-known that, since $a \in C^1(\bar{\Omega})$,

$$\inf_{\chi \in \mathcal{M}} \|a\xi - \chi\| \leq C \|\xi\| h.$$

Replacing u by ψ and W by ψ^* in theorem 4, we observe that

$$\|\psi - \psi^*\|_q \leq C \|\psi\|_2 h^{2-q}, \quad 0 \leq q \leq 2.$$

Since $\|\psi\|_2 \leq C \|\xi\|$,

$$\|\xi\|^2 \leq C \{h \|\xi\|^2 + \|\eta\| \|\xi\|\}.$$

Hence, for h sufficiently small,

$$\|\xi\| \leq C \|\eta\|.$$

Consequently, we have the following theorem.

THEOREM 5: *There exists $h_0 = h_0(L) > 0$ such that a unique solution $U \in \mathcal{M}$ of (2) exists for $h \leq h_0$; moreover, if $1 \leq s \leq r+1$ and if $u \in H^s(\Omega)$ is the solution of (1), then*

$$\|U - u\| \leq C \|u\|_s h^s.$$

We shall now consider error estimates for the adjoint local H^{-1} Galerkin procedure. Note that the ellipticity of L implies a Gårding inequality of the form

$$C_0 \|\varphi\|_1^2 \leq -(L\varphi, \varphi) + C_1 \|\varphi\|^2$$

for $\varphi \in H^2(\Omega)$ such that $\varphi = 0$ on $\partial\Omega$, where C_0 is some positive constant. Since (1) and (3) imply that $(L(W-u), \psi) = 0$ for $\psi \in \mathcal{M}$,

$$C_0 \|W-u\|_1^2 - C_1 \|W-u\|^2 \leq -(L(W-u), W-u-\psi), \quad \psi \in \mathcal{M}.$$

For h sufficiently small, theorem 5 when applied to the operator L^* instead of L implies the existence of $\psi \in \mathcal{M}$ such that

$$(Lv, W-u-\psi) = 0, \quad v \in \mathcal{N},$$

and

$$\|W-u-\psi\| \leq C \|W-u\|_1 h.$$

Thus, for any $\theta \in \mathcal{N}$:

$$\begin{aligned} C_0 \|W-u\|_1^2 - C_1 \|W-u\|^2 &\leq -(L(\theta-u), W-u-\psi) \\ &\leq C \|u-\theta\|_2 \|W-u\|_1 h. \end{aligned}$$

By noting that $\|W-u\|^2 \leq \|W-u\|_1 \|W-u\|$, we see that

$$\|W-u\|_1 \leq C (\|u\|_{s+2} h^{s+1} + \|W-u\|), \quad 0 \leq s \leq r+1.$$

Again by the quasi-uniformity of δ ,

$$\|W-u\|_2 \leq C(\|u\|_{s+2} h^s + h^{-1} \|W-u\|), \quad 0 \leq s \leq r+1.$$

In order to obtain an $L_2(\Omega)$ -estimate, we now consider the auxiliary Dirichlet problem given by

$$\begin{aligned} L^* \varphi &= W-u && \text{on } \Omega, \\ \varphi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then,

$$\begin{aligned} \|W-u\|^2 &= (W-u, L^* \varphi) = (L(W-u), \varphi) \\ &= (L(W-u), \varphi - \varphi^*), \quad \varphi^* \in \mathcal{M}. \end{aligned}$$

Thus, choosing an appropriate φ^* , we obtain the inequality

$$\begin{aligned} \|W-u\|^2 &\leq C \|W-u\|_2 \|\varphi\|_2 h^2 \\ &\leq C \|W-u\|_2 \|W-u\| h^2, \end{aligned}$$

and

$$\begin{aligned} \|W-u\| &\leq C \|W-u\|_2 h^2 \\ &\leq C(\|u\|_{s+2} h^{s+2} + \|W-u\| h), \quad 0 \leq s \leq r+1. \end{aligned}$$

Hence, we have proved the following theorem.

THEOREM 6: *There exists $h_0 = h_0(L) > 0$ such that there exists a unique solution $W \in \mathcal{N}$ of (3), and if $2 \leq s \leq r+3$ and if the solution u of (1) belongs to $H^s(\Omega)$, then*

$$\|W-u\|_q \leq C \|u\|_s h^{s-q}, \quad 0 \leq q \leq 2.$$

The range on q can be extended just as for theorem 4.

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