RAIRO
ANALYSE NUMÉRIQUE

HAROLD D. MEYER

Numerical analytic continuation of holomorphic functions in C^n

RAIRO – Analyse numérique, tome 11, n° 1 (1977), p. 75-92.
<http://www.numdam.org/item?id=M2AN_1977__11_1_75_0>

R.A.I.R.O. Analyse numérique/Numerical Analysis
(vol. 11, n° 1, 1977, p. 75 à 92)

NUMERICAL ANALYTIC CONTINUATION OF HOLOMORPHIC FUNCTIONS IN C^n (*)

by HAROLD D. MEYER

Communiqué par Jim DOUGLAS, Jr.

Summary. — *A procedure is presented for continuing numerically holomorphic functions in* $\mathbf{C}^n$. *To accomplish this, a representation involving an infinite series of boundary integrals is obtained. A numerical analog is then introduced, with measures replaced by parameters satisfying relevant bounds. Error estimates are derived using the three circle theorem for polydiscs. Continuation is on the unit polydisc and the method developed leads to a linear programming problem. A second related procedure is also discussed.*

1. INTRODUCTION

Let $\mathbf{C}^n$ represent n-dimensional complex space with typical point $z = (z_1, \ldots, z_n)$ where $z_k = x_k + iy_k$, x_k and y_k real. In this paper, we extend numerically to all of the unit polydisc holomorphic functions $f(z)$ $(= f(re^{i\theta}))$ whose values are known only approximately and only on the distinguished boundary of a polydisc located interior to and concentric with the unit polydisc. Since analytic continuation is an unstable process, in addition to boundary data, a global constraint is also provided. That instability is possible is seen by noting that the holomorphic function z_1^k goes to zero at any point in the unit polydisc as $k \to \infty$, but becomes infinitely large at any point outside.

To obtain analytic continuations, we first show that holomorphic functions in the open unit polydisc have representations

$$f(z) = \sum_{K_1=0}^{\infty} \cdots \sum_{K_n=0}^{\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} P_r^{(K)}(\theta - t)\, d\mu_K(t)$$

and

$$f(z) = \sum_{K_1=0}^{\infty} \cdots \sum_{K_n=0}^{\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} H_r^{(K)}(\theta - t)\, d\mu_K(t),$$

where P is the Poisson kernel and H is a related complex function. $P_r^{(K)}$ and $H_r^{(K)}$, $K = (K_1, \ldots, K_n)$, represent partial derivatives of different orders

(*) Reçu août 1975.

([1]) Department of Mathematics, Texas Tech University, Lubbock, Texas 79409. This paper is based, in part, on work in the author's doctoral thesis under Professor Jim Douglas, Jr. at the University of Chicago.

and the μ's are Radon measures whose total variations satisfy a boundedness condition. These representations, with the integrals replaced by truncated sums and the μ's by $(a+ib)$'s that satisfy boundedness conditions, are then used. They lead to linear programming procedures which give our approximations. The linear programming problem is briefly discussed at the end of the paper.

Error estimates are derived using an approach related to approaches in [1], [3], [4], and [13]. A three circle theorem for polydiscs is applied in order to accomplish this end. The approach applied here is inspired by work of Douglas for harmonic functions in R^2 [4], extending such ideas to analytic functions and to $\mathbf{C}^n$.

In the literature, there has been considerable attention given to the study of unstable problems. Of particular interest with relation to work here are the papers [1, 2, 3, 4, 10, 11, 12, 13, 14]. In these, representations are presented which can be used to treat unstable problems, different from the above, by methods related to those here.

It should be pointed out that though the approach used in what follows employs particular representations, domains, and distributions of data, the ideas can be extended to more general situations and also representations other than those above can be used. For instance, the approximation might be based on the Cauchy integral representation and data points might be randomly scattered. For further details, refer to [10].

2. NOTATION

In this paper,

$$D = \{ z \mid |z_k| < 1, \ k = 1, \ldots, n \}$$

is the open unit polydisc in $\mathbf{C}^n$,

$$\bar{D} = \{ z \mid |z_k| \leqq 1, \ k = 1, \ldots, n \}$$

is its closure, and

$$\dot{D} = \{ z \mid |z_k| = 1, \ k = 1, \ldots, n \}$$

is its distinguished boundary. For other polydiscs we use the notation

$$D_r = \{ z \mid |z_k| < r_k, \ k = 1, \ldots, n \}, \qquad \bar{D}_r = \{ z \mid |z_k| \leqq r_k, \ k = 1, \ldots, n \},$$

and

$$\dot{D}_r = \{ z \mid |z_k| = r_k, \ k = 1, \ldots, n \}, \qquad \text{where} \quad r = (r_1, \ldots, r_n).$$

Standard multi-index conventions will be used so that if K and J are multi-indices, then

$$K = (K_1, \ldots, K_n), \qquad J = (J_1, \ldots, J_n), \qquad K^J = K_1^{J_1} \cdot K_2^{J_2} \cdots K_n^{J_n},$$
$$K! = K_1!K_2! \cdots K_n!, \qquad |K| = K_1 + K_2 + \ldots + K_n,$$

$K \leqq J$ if $K_k < J_k$ for all k, and $K < J$ if $K_k < J_k$ for all k. Both capital and lower case letters will appear in multi-index form as subscripts, exponents, and radii. It will be clear from context whether or not the indices or symbols we are using should be taken in the multi-index sense or not, and also whether a symbol such as K stands for $(K_1, \ldots, K_n)$ or $K_1 \cdot K_2 \cdots K_n$. By $K+a$, where a is a scalar, we will mean $(K_1+a, \ldots, K_n+a)$; by $K+a_j$ we mean $(K_1, \ldots, K_j+a, \ldots, K_n)$. Sometimes we will denote $(a, \ldots, a)$ by a. We use interchangeably

$$\sum_{|K|=0}^{J} \quad \text{and} \quad \sum_{K_1=0}^{J_1} \cdots \sum_{K_n=0}^{J_n}, \quad \text{and regard} \quad \sum_{|K|=0}^{\infty}, \ \sum_{K>L}^{J},$$

and other such sums in a similar fashion.

The symbols z, $re^{i\theta}$, and $(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$ will be written interchangeably as will dt and $dt_1 \ldots dt_n$. A similar convention holds for dz and $d\theta$. Conjugate complex values will be indicated by bars.

3. THE CONTINUATION PROBLEM

We wish to approximate in D, i. e., in any D_R, $R_k < 1$, $k = 1, \ldots, n$, the function $f(z)$ which satisfies the following conditions:

(i) $f(z)$ is holomorphic in $\bar{D}$.

(ii) The $B(\alpha, f)$ of Theorem 5.1 or Theorem 5.2 (*see* Section 5) is known.

(iii) $|f(s_Q) - f_\eta(s_Q)| < \eta$, where

$$s_Q = (s_{Q_1}, \ldots, s_{Q_n}) = (\rho_1 e^{2\pi i Q_1/P_1}, \ldots, \rho_n e^{2\pi i Q_n/P_n}),$$
$$Q_k = 0, 1, \ldots, P_k - 1, \quad k = 1, \ldots, n,$$

$\eta > 0$ is a constant, ρ is given, and f_η is a complex-valued function defined for each s_Q. $P = (P_1, \ldots, P_n)$, with positive integer components, is given and fixed.

To accomplish the above, we will need some representation theorems. These will be derived in the next two sections and have been suggested by work of Johnson [9] for $\mathbf{C}$.

4. PRELIMINARY LEMMAS

We shall be encountering functions in the spaces $\mathscr{A} = \{f(z) \mid f(z)$ holomorphic in D and continuous in $\bar{D}\}$ and $\tilde{\mathscr{A}} = \{f(z) \mid f(z)$ real, pluriharmonic

in D, and continuous in $\bar{D}\}$. A norm for each is given by

$$\|w\| = \sup_{\substack{0 \leq t_j \leq 2\pi \\ j=1,\ldots,n}} |w(e^{it})|,$$

where w is complex for $\mathscr{A}$ and real for $\tilde{\mathscr{A}}$.

LEMMA 4.1: *Let $\{a_J\}$, $|J| > 0$, be a sequence of complex-valued numbers such that*

$$\limsup_{|J| \to \infty} (|a_J|^{1/|J|}) \leq 1. \tag{4.1}$$

Then there exist sequences $\{a_{J,K}\}$, for which

$$a_J = \sum_{|K|=0}^{J} a_{J,K}, \tag{4.2}$$

where

$$|a_{J,K}| \leq C(\varepsilon)\varepsilon^{|K|} \prod_{l \in R(J)} \frac{J_l^{K_l}}{(K_l+2)!}. \tag{4.3}$$

In (4.3), *J and K are multi-indices, and $C(\varepsilon)$ is a finite-valued positive constant depending on $\varepsilon > 0$. Further,*

$$R(J) = \{l \,|\, J_l \neq 0\}, \quad \text{and} \quad 0 \leq K_l \leq J_l, \quad l = 1, \ldots, n.$$

Proof: The proof is similar to that for single indices so we shall only briefly indicate the argument. We refer the reader to [9] for help in filling in omitted details.

Thanks to (4.1) and the binomial theorem, we have that

$$\begin{aligned} |a_J| &\leq \sup_{|K| \geq |J|} |a_k|^{1/|K|} \leq \prod_{l=1}^{n} (1+\varepsilon_{|J|})^{J_l} \\ &\leq \prod_{l \in R(J)} \left\{ 1 + \binom{J_l}{1}\varepsilon_{|J|} + \ldots + \binom{J_l}{K_l}\varepsilon_{|J|}^{K_l} + \ldots + \varepsilon_{|J|}^{J_l} \right\}, \end{aligned}$$

where $\varepsilon_m > 0$ and $\varepsilon_m \searrow 0$ as $m \to \infty$. It is therefore possible to put a_J in the form

$$a_J = \prod_{l \in R(J)} \{a_{J_l,0} + \ldots + a_{J_l,K_l} + \ldots + a_{J_l,J_l}\}$$

(where $|a_{J_l,K_l}| \leq J_l^{K_l}\,\varepsilon_{|J|}^{K_l}/K\,!$, $l = 1, \ldots, n$). Taking

$$a_{J,K} = \prod_{l \in R(J)} a_{J_l,K_l},$$

we then have (4.2). Further, using the bound on $|a_{J_l,K_l}|$ and proceeding in parallel to the derivation in [9], (4.3) follows easily with

$$C(\varepsilon) = \max_{l=1,\ldots,n} [C^*(\varepsilon)\varepsilon^2]^l, \tag{4.4}$$

where

$$C^*(\varepsilon) = \max\left\{\left(\max_{|J|\leq \tilde{J}(\varepsilon)} \max_{\substack{0\leq K_l\leq J_l\\ l\in R(J)}}\left[\frac{(K_l+2)(K_l+1)}{\varepsilon^2}\left(\frac{\varepsilon_{|J|}}{\varepsilon}\right)^{K_l}\right]\right), \frac{3}{\varepsilon^2}\right\}.$$

The Poisson formula in $\mathbf{C}^n$ takes the form given below:

LEMMA 4.2. *Let $f(z)\in\mathscr{A}$, $z = r\,e^{i\theta}$ $(= (r_1\,e^{i\theta_1}, \ldots, r_n\,e^{i\theta_n})\in D$. Then*

$$f(z) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} P_r(\theta-t)f(e^{it})\,dt, \tag{4.5}$$

where $P_r(\theta-t)$, the Poisson kernel, is defined by

$$P_r(\theta-t) = \prod_{j=1}^{n}\left\{\frac{1-r_j^2}{1-2\,r_j\cos(\theta_j-t_j)+r_j^2}\right\}. \tag{4.6}$$

The proof for this employs the $\mathbf{C}^n$ version of the Cauchy integral formula and is a straightforward extension of that for the one-dimensional case.

Next let us define

$$H_r(\theta) = 2\,C_r(\theta)-1, \tag{4.7}$$

where $C_r(\theta)$ is the Cauchy kernel,

$$C_r(\theta) = \prod_{j=1}^{n}\left(\frac{1}{1-r_j\,e^{i\theta_j}}\right). \tag{4.8}$$

Take $f(z) = u(z)+iv(z)$. A proof analogous to that for the one-dimensional case [8, p. 31] provides the following lemma:

LEMMA 4.3: *Let $f(z)\in\mathscr{A}$ and $z\in D$. If $f((0, \ldots, 0))$ is real, then*

$$f(z) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} u(e^{it})\,H_r(\theta-t)\,dt. \tag{4.9}$$

5. REPRESENTATION THEOREMS

We are now ready for the representations. Let us denote

$$\left(\frac{\partial}{\partial\theta}\right)^{K} F(\theta) = \left(\frac{\partial}{\partial\theta_1}\right)^{K_1}\cdots\left(\frac{\partial}{\partial\theta_n}\right)^{K_n} F(\theta) \quad \text{by } F^{(K)}(\theta).$$

Also, take $F^{(0)}(\theta) = F(\theta)$.

THEOREM 5.1: *If $f(z)$ is analytic in D, then there exist complex Radon measures $\{\mu_K(t)\}$ on $\{t\mid 0\leq t_j\leq 2\pi,\ j = 1, \ldots, n\}$ for which*

$$f(z) = \sum_{|K|=0}^{\infty}\int_0^{2\pi}\cdots\int_0^{2\pi} P_r^{(K)}(\theta-t)\,d\mu_K(t), \qquad z\in D. \tag{5.1}$$

Each of the measures μ_K *has bounded total variation* var (μ_K) *which satisfies*

$$\operatorname{var}(\mu_K) \leqq \frac{B(\alpha, f)\,\alpha^K}{K\,!} \tag{5.2}$$

for any $\alpha_j > 0, j = 1, \ldots, n$; $B(\alpha, f)$ *is a positive constant which depends on* α *and* f.

Proof: The proof is parallel to that for the one-dimensional complex case [9] so the discussion will be brief.

First note that

$$f(z) = a_0 + \sum_{|J|>0}^{\infty} a_J z^J = a_0 + \sum_{\substack{|K|>2n \\ K_l \geqq 2 \\ l=1,\ldots,n}} \sum_{J=K-2} a_{J,K-2} z^J, \tag{5.3}$$

where $z^J = z_1^{J_1} \cdot z_2^{J_2} \ldots z_n^{J_n}$. We have used Lemma 4.1, which is permissible since (4.1) holds [5, p. 51]. Let us define $f_0(z) = a_0$. Further, for the K's appearing in (5.3), we can take

$$\begin{aligned} f_K(z) &= (-i)^{|K|} \sum_{J=K-2}^{\infty} \frac{a_{J,K-2}}{\prod_{l \in R(J)} J_l^{K_l}} \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} P_r(\theta - t) f_K(e^{it})\, dt, \end{aligned} \tag{5.4}$$

where Lemma 4.2 has been used. The first expression above makes sense (and $f_K \in \mathscr{A}$), since by (4.3) it is easily seen that the sum there is uniformly convergent. In fact

$$\|f_K\| \leqq \left[\varepsilon^{-2n} C(\varepsilon) \sum_{J=K-2}^{\infty} \left\{ \prod_{l \in R(J)} \frac{1}{J_l^2} \right\} \right] \frac{\varepsilon^{|K|}}{K\,!}. \tag{5.5}$$

Also, we observe that the Poisson representation in (5.4) holds for f_0.

Differentiating (5.4), we have that

$$\begin{aligned} \left(\frac{\partial}{\partial\theta}\right)^K f_K(z) &= \sum_{J=K-2}^{\infty} a_{J,K-2} z^J \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} P_r^{(K)}(\theta - t) f_K(e^{it})\, dt. \end{aligned} \tag{5.6}$$

Combining this with (5.3), then

$$f(z) = \sum_{|K|=0}^{\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} P_r^{(K)}(\theta - t) f_K(e^{it})\, dt, \tag{5.7}$$

where we have taken $f_K(z) \equiv 0$ for all K such that $|K| > 0$, $K_l < 2$, $l = 1, \ldots, n$.

Taking

$$d\mu_K = \frac{1}{(2\pi)^n} f_K(e^{it})\,dt$$

in (5.7) and employing (5.5), (5.1) and (5.2) follow easily.

A second representation is given by the following theorem:

THEOREM 5.2: *If $f(z)$ is analytic in D, then there exist Radon measures* $\{\mu_K(t)\}$ on $\{t \mid 0 \leqq t_j \leqq 2\pi, j = 1, \ldots, n\}$ *for which*

$$f(z) = \sum_{|K|=0}^{\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} H_r^{(K)}(\theta - t)\,d\mu_K(t), \qquad z \in D. \tag{5.8}$$

In the above, each of the measures μ_K, $K \neq (0, \ldots, 0)$ must be real and have bounded total variation which satisfies

$$\operatorname{var}(\mu_K) \leqq \frac{B(\alpha, f)\,\alpha^K}{K!} \tag{5.9}$$

for any α with $\alpha_j > 0$, $j = 1, \ldots, n$, and for some positive constant $B(\alpha, f)$ which may vary with α and f. The measure $\mu_{(0,\ldots,0)}$ is real if $f((0, \ldots, 0))$ is real and complex otherwise. In either case, its total variation must have bound $B(\alpha, f)$.

Proof: One follows essentially the same steps as for the proof of the preceding theorem, only with the obvious modifications. Relation (5.4) is replaced by

$$f_K(z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} H_r(\theta - t)\,u_K(e^{it})\,dt, \tag{5.10}$$

using Lemma 4.3, where $f_K = u_K + iv_K$; the function $f_0 \equiv a_0$ has this same form (through the application of (4.9) to both the real and imaginary parts of a_0) only $f_0\,(e^{it})$ takes the place of $u_K\,(e^{it})$ on the right. The u_K's, $K \neq (0, ..., 0)$ are in $\tilde{\mathscr{A}}$ f_0 is in $\mathscr{A}$ or $\tilde{\mathscr{A}}$ depending on whether it is real or imaginary, and the bounding leading to estimates in the norms of these spaces is similar to before. Bounds on the measures follow from these.

6. THE APPROXIMATION

Pick multi-indices $N = (N_1, \ldots, N_n)$ and $L = (L_1, \ldots, L_n)$ having positive integer entrees, and let

$$\tau_{J_i} = \frac{2\pi i}{N_i}, \qquad J_i = 1, \ldots, N_i, \quad i = 1, \ldots, n. \tag{6.1}$$

Set $\tau_J = (\tau_{J_1}, \ldots, \tau_{J_n})$. Theorem 5.1 suggests an approximation of form

$$\begin{aligned} f(z) &= \sum_{|K|=0}^{L} \sum_{J=1}^{N} P_r^{(K)}(\theta-\tau_J)\mu_K\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+)1/2)})\right) \\ &= \sum_{|K|=0}^{L} \sum_{J=1}^{N} P_r^{(K)}(\theta-\tau_J) \\ &\quad\times\left\{\mu_K^+\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right)\right. \\ &\quad-\mu_K^-\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right) \\ &\quad+i\left(\nu_K^+\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right)\right. \\ &\quad\left.\left.-\nu_K^-\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right)\right)\right\}, \end{aligned} \tag{6.2}$$

which comes from substituting sums for integrals and truncating the infinite sum in (5.1). In (6.2),

$$\sum_{J=1}^{N} = \sum_{J_1=1}^{N_1} \cdots \sum_{J_n=1}^{N_n}$$

and the μ_K^+, μ_K^-, ν_K^+, ν_K^- come from the standard decomposition of μ_K, where μ_K^+ and μ_K^- are the positive and negative components of the real part of μ_K and ν_K^+ and ν_K^- are the same for the imaginary part.

We do not know the

$$\left\{\mu_K^+\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right)\right\}, \quad \left\{\mu_K^-\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right)\right\},$$
$$\left\{\nu_K^+\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right)\right\}, \quad \left\{\nu_K^-\left(\mathop{\mathsf{X}}_{i=1}^{n}(\tau_{J_i-(1/2)}, \tau_{J_i+(1/2)})\right)\right\}.$$

However, by (5.2),

$$\operatorname{var}(\mu_K^+),\ \operatorname{var}(\mu_K^-),\ \operatorname{var}(\nu_K^+),\ \operatorname{var}(\nu_K^-) \leqq \frac{B(\alpha, f)\alpha^K}{K!},$$

so it seems reasonable to try instead of (6.2) an approximation of form

$$\begin{aligned} &F_{L,N}(z, \{a_{J,K}\}, \ldots, \{d_{J,K}\}) \\ &\quad= \sum_{|K|=0}^{L} \sum_{J=1}^{N} P_r^{(K)}(\theta-\tau_J)[a_{J,K}-b_{J,K}+i(c_{J,K}-d_{J,K})], \end{aligned} \tag{6.3}$$

where

$$a_{J,K}, b_{J,K}, c_{J,K}, d_{J,K} \geqq 0, \tag{6.4}$$

and

$$\sum_{J=1}^{N} a_{J,K}, \sum_{J=1}^{N} b_{J,K}, \sum_{J=1}^{N} c_{J,K}, \sum_{J=1}^{N} d_{J,K} \leqq \frac{B((1-R)/2, f)((1-R)/2)^K}{K!}. \tag{6.5}$$

In (6.5), R is the poly-radius that appeared in Section 3,

$$\frac{1-R}{2} = \left(\frac{1-R_1}{2}, \ldots, \frac{1-R_n}{2}\right),$$

and

$$\left(\frac{1-R}{2}\right)^K = \left(\frac{1-R_1}{2}\right)^{K_1} \cdot \left(\frac{1-R_2}{2}\right)^{K_2} \cdots \left(\frac{1-R_n}{2}\right)^{K_n}.$$

Use of $B((1-R)/2, f)$ in (6.5) serves as the global constraint mentioned earlier as needed because of the unstable nature of the continuation problem. The choice of $\alpha = (1-R)/2$ in B is desirable for later error bounding.

It still remains to stipulate what the values of $\{a_{J,K}\}, \ldots, \{d_{J,K}\}$ should be before (6.3) will provide the approximation we wish to use. These are selected to be any set of sequences, $\{\tilde{a}_{J,K}\}$, $\{\tilde{b}_{J,K}\}$, $\{\tilde{c}_{J,K}\}$, $\{\tilde{d}_{J,K}\}$, not necessarily unique, which satisfy the relation

$$F_{L,N}(z, \{\tilde{a}_{J,K}\}, \ldots, \{\tilde{d}_{J,K}\}) = \inf_{\hat{F}_{L,N}} \{\sup_{Q} |F_{L,N}(s_Q, \{a_{J,K}\}, \ldots, \{d_{J,K}\}) - f_\eta(s_Q)|\}, \tag{6.6}$$

where

$$\hat{F}_{L,N} = \{F_{L,N}(s_Q, \{a_{J,K}\}, \ldots, \{d_{J,K}\}) \mid (6.4) \text{ and } (6.5) \text{ are satisfied}\}. \tag{6.7}$$

Finally, then, our approximation $F_{L,N}(z)$ is given by

$$F_{L,N}(z) = F_{L,N}(z, \{\tilde{a}_{J,K}\}, \ldots, \{\tilde{d}_{J,K}\}). \tag{6.8}$$

Theorem 5.2 suggests an approximation $G_{L,N}(z)$ which is similar to the preceding. We consider sequences of real numbers $\{a_{J,K}\}$, $\{b_{J,K}\}$, $\{c_{J,0}\}$, $\{d_{J,0}\}$ satisfying

$$a_{J,K}, b_{J,K}, c_{J,0}, d_{J,0} \geqq 0, \tag{6.9}$$

$$\sum_{J=1}^{N} a_{J,K}, \sum_{J=1}^{N} b_{J,K} \leqq \frac{B((1-R)/2, f)((1-R)/2)^K}{K!}, \tag{6.10}$$

$$\sum_{J=1}^{N} c_{J,0}, \sum_{J=1}^{N} d_{J,0} \leqq B((1-R)/2, f). \tag{6.11}$$

Further, we let

$$\hat{G}_{L,N} = \{G_{L,N}(s_Q, \{a_{J,K}\}, \{b_{J,K}\}, \{c_{J,0}\}, \{d_{J,0}\}) \mid (6.9)\text{-}(6.11) \text{ are satisfied}\}, \tag{6.12}$$

where, in (6.12),

$$G_{L,N}(z, \{a_{J,K}\}, \{b_{J,K}\}, \{c_{J,0}\}, \{d_{J,0}\})$$
$$= \sum_{|K|=0}^{L} \sum_{J=1}^{N} H_r^{(K)}(\theta-\tau_J)[a_{J,K}-b_{J,K}]$$
$$+ \sum_{J=1}^{N} H_r^{(K)}(\theta-\tau_J)[i(c_{J,0}-d_{J,0})], \tag{6.13}$$

with $z = s_Q$. Also in (6.12), we take $c_{J,0} \equiv c_{J,(0,\ldots,0)}$ and $d_{J,0} \equiv d_{J,(0,\ldots,0)}$ to be zero if $f((0, \ldots, 0))$ is real. Then the approximation $G_{L,N}(z)$ is the same as that for $F_{L,N}(z)$ only with $G_{L,N}(s_Q, \{a_{J,K}\}, \ldots, \{d_{J,0}\})$ and $\hat{G}_{L,N}$ replacing $F_{L,N}(s_Q, \{a_{J,K}\}, \ldots, \{d_{J,K}\})$ and $\hat{F}_{L,N}$.

7. FURTHER PRELIMINARIES

The next set of results will be needed in Section 8 for the derivation of error bounds. Both here and later, we shall use the notation

$$|f|_\rho = \sup_{z \in D_\rho} |f(z)|. \tag{7.1}$$

THEOREM 7.1: *Let $f(z)$ be holomorphic in D_R and suppose ρ, ρ', and r are polydisc radii satisfying*

$$0 < \rho_k \leqq r_k \leqq \rho'_k < R_k, \qquad k = 1, \ldots, n.$$

Then

$$|f|_r \leqq |f|_\rho^{\alpha(\rho, r, \rho')} |f|_{\rho'}^{1-\alpha(\rho, r, \rho')}, \tag{7.2}$$

where

$$\alpha(\rho, r, \rho') = \prod_{k=1}^{n} \left\{\frac{\ln(r_k/\rho_k)}{\ln(\rho_k/\rho'_k)}\right\}. \tag{7.3}$$

This result is well known. Hille's argument [7] for the one-dimensional complex case carries over to this situation.

The lemmas below are considered for the case of $F_{L,N}(z)$. The approach for results here and in some of the later sections is related to work of Douglas [4] for **C**.

LEMMA 7.1: *Let $f(z)$ satisfy conditions* (i) *and* (ii) *of Section* 3. *Then*

$$\left| f(z) - \sum_{|K|=0}^{L} \sum_{J=1}^{N} P_r^{(K)}(\theta-\tau_J)\,\mu_K\left(\mathop{\times}_{l=1}^{n} (\tau_{J_l-(1/2)}, \tau_{J_l+(1/2)}) \right) \right|_r$$
$$\leqq C(r, f)[2^{-|L|} + \hat{N}^{-1}] \tag{7.4}$$

for $z = r\,e^{i\theta} \in D$, where $N = \min \hat{N}_j$ and $C(r; f)$ is defined in the proof below.

Proof: Using the series representation for

$$P_{r_j}(\theta - t_j) = \frac{1-r_j^2}{1-2r_j\cos(\theta_j - t_j)+r_j^2}, \tag{7.5}$$

we have that

$$\begin{aligned}\left|\frac{\partial^{K_j} P_{r_j}}{\partial \theta_j^{K_j}}\right| &= \left|\frac{\partial^{K_j}}{\partial t_j}\left[1+2\sum_{m_j=1}^{\infty} r_j^{m_j}\cos m_j(t_j-\theta_j)\right]\right| \\ &\leqq \left|2\sum_{m_j=1}^{\infty} m_j^{k_j} r_j^{m_j}\right| < \frac{2K_j!}{(1-r_j)^{K_j+1}}.\end{aligned} \tag{7.6}$$

Since

$$P_r^{(K)}(\theta - t) = \prod_{j=1}^{n} P_r^{(K_j)}(\theta_j - t_j),$$

then

$$\left|\int_0^{2\pi}\cdots\int_0^{2\pi} P_r^{(K)}(\theta - t)\,d\mu_k(t)\right|_r \leqq \frac{2^n B(\alpha, f)\,\alpha^K}{(1-r)^{K+1}}, \tag{7.7}$$

and taking $\alpha = (1-r)/2$,

$$\begin{aligned}&\left|f(z) - \sum_{|K|=0}^{L}\int_0^{2\pi}\cdots\int_0^{2\pi} P_r^{(K)}(\theta - t)\,d\mu_K(t)\right|_r \\ &\leqq \left|\sum_{K>L}^{\infty}\int_0^{2\pi}\cdots\int_0^{2\pi} P_r^{(K)}(\theta - t)\,d\mu_K\right|_r \\ &\leqq \frac{4^n B((1-r)/2, f)}{1-r}2^{-|L|}.\end{aligned} \tag{7.8}$$

Further,

$$\begin{aligned}&\left|\int_0^{2\pi}\cdots\int_0^{2\pi} P_r^{(K)}(\theta - t)\,d\mu_K(t)\right. \\ &\left.\quad - \sum_{|J|=0}^{N-1} P_r^{(K)}(\theta - \tau_J)\,\mu_K\left(\mathop{\times}_{l=1}^{n}(\tau_{J_l-1/2}, \tau_{J_l+1/2})\right)\right|_r \\ &\leqq \operatorname{var}(\mu_K)\sum_{l=1}^{n} |P_r^{(K+1_l)}|_r\left(\frac{2\pi}{N_l}\right) \\ &\leqq \frac{\pi B((1-r)/2, f)\,2^{-|K|+n+1}}{1-r}\sum_{l=1}^{n}\frac{(K_l+1)}{(1-r_l)N_l},\end{aligned} \tag{7.9}$$

so that combining the last two relations and performing some algebra provides (7.4). The constant $C(r, f)$ turns out to be

$$C(r, f) = \max\left\{\frac{4^n B((1-r)/2, f)}{1-r}, \left[\max_{l=1,\ldots,n}\left(\frac{2^{n+1}\pi B((1-r)/2, f)}{1-r}\sum_{|K|=0}^{L}\frac{(K_l+1)2^{-|K|}}{1-r_l}\right)\right]n\right\}. \quad (7.10)$$

A parallel result to (7.4) holds when P_r is replaced by H_r. Then H_r is written as

$$\prod_{j=1}^{n} H_{r_j}(\theta_j - t_j)$$

and the H_{r_j}'s have series representation

$$H_{r_j}(\theta_j - t_j) = 2\sum_{m_j=0}^{\infty} r_j^{m_j}[\cos(\theta_j - t_j) + i\sin(\theta_j - t_j)] - 1. \quad (7.11)$$

The same type of approach that led to (7.7) gives $\sqrt{2}$ times the bound found there and we have

$$\left|f(z) - \sum_{|K|=0}^{L}\int_0^{2\pi}\cdots\int_0^{2\pi} H_r^{(K)}(\theta - t)\, d\mu_K(t)\right| \leq \frac{2^{(5/2)n} B((1-r)/2, f)}{1-r}. \quad (7.12)$$

Continuing essentially as before, we then obtain (7.4) only with $H_r^{(K)}$ in place of $P_r^{(K)}$ and

$$C(r, f) = \max\left\{\frac{2^{(5/2)n} B((1-r)/2, f)}{1-r}, \left[\max_{l=1,\ldots,n}\left(\frac{2^{(3/2)n+1}\pi B((1-r)/2, f)}{1-r} \times \sum_{|K|=0}^{L}\frac{(K_l+1)2^{-|K|}}{1-r_l}\right)\right]n\right\}. \quad (7.13)$$

LEMMA 7.2: *Suppose that* $w(z)\ (= w(r\, e^{i\theta}))$ *is an holomorphic function in* $\bar{D}$ *which has derivatives* $\partial^2 w/\partial\theta_j^2$ *satisfying*

$$\left|\frac{\partial^2 w}{\partial\theta_j^2}\right| \leq K, \qquad j = 1, \ldots, n, \quad (7.14)$$

on $\dot{D}_\rho$. *Further, assume that*

$$|w(s_Q)| \leq \delta \quad (7.15)$$

for all of the s_Q *of condition* (iii), *Section* 3. *Then*

$$|w|_\rho \leqq \delta + \frac{5}{2} n \pi^2 K \hat{P}^{-2}, \tag{7.16}$$

where $\hat{P} = \min_j P_j$ *and the* P_j *are those appearing in condition* (iii).

Proof: Let $z = \rho e^{i\theta}$ be any point on $\dot{D}_\rho$ and consider the function

$$W(\theta) \equiv w(z) = w(\rho e^{i\theta}),$$

depending on θ alone. Each of the points s_Q has corresponding to it an angular value $\theta^Q = (\theta_1^Q, \ldots, \theta_n^Q) = (2\pi Q_1/P_1, \ldots, 2\pi Q_n/P_n)$, and if z_0 is any fixed point on $\dot{D}_\rho$, there corresponds to it the angular value $\theta^0 = (\theta_1^0, \ldots, \theta_n^0)$.

If we consider ourselves in n-dimensional θ-space, then θ^0 falls in some $2\pi/P_1$ by $2\pi/P_2$ ... by $2\pi/P_n$ n-dimensional cube with a θ^Q at each vertex. Let the vertices be $\{ \theta^{\tilde{Q}}, \theta^{\tilde{Q}+1_1}, \theta^{\tilde{Q}+1_2}, \ldots, \theta^{\tilde{Q}+1_1+1_2}, \ldots, \theta^{\tilde{Q}+1_1+\ldots+1_{n-1}}, \theta^{\tilde{Q}+1}$ (where we have assumed that all the $(\tilde{Q}_j+1)'s$ are $\leqq P_j - 1$; the treatment requires just trivial modification if this is not the case). For any given θ^0, there will be a vertex θ^Q such that $|\theta_j^Q - \theta_j^0| \leqq \pi/P_j$, $j = 1, \ldots, n$. We shall assume that for our θ^0, this vertex is $\theta^{\tilde{Q}}$. If the vertex were another, the proof would proceed in essentially the same manner.

We wish to bound W at the point $(\theta_1^0, \theta_2^{\tilde{Q}}, \ldots, \theta_n^{\tilde{Q}})$ which lies on the line segment connecting $\theta^{\tilde{Q}}$ and $\theta^{\tilde{Q}+1_1}$. It is not difficult to show that

$$\begin{aligned} W(\theta_1^0, \theta_2^{\tilde{Q}}, \ldots, \theta_n^{\tilde{Q}}) = {} & W(\theta^{\tilde{Q}}) + [W(\theta^{\tilde{Q}+1}) - W(\theta^{\tilde{Q}})]\, t_0 \\ & + \frac{1}{2}\psi_1 K (\theta_1^{\tilde{Q}+1_1} - \theta_1^{\tilde{Q}})^2 t_0 + \frac{1}{2}\psi_2 K (\theta_1^0 - \theta_1^{\tilde{Q}})^2, \end{aligned}$$

where ψ_1 and ψ_2 are complex numbers with $|\psi_1|, |\psi_2| \leqq 1$, and t_0 is defined from the relation

$$\theta_1^0 = \theta_1^{\tilde{Q}} + t_0 (\theta_1^{\tilde{Q}+1_1} - \theta_1^{\tilde{Q}}).$$

Thus

$$|W(\theta_1^0, \theta_2^{\tilde{Q}}, \ldots, \theta_n^{\tilde{Q}})| \leq (1-t_0) |W(\theta^{\tilde{Q}})| + t_0 |W(\theta^{\tilde{Q}+1_1})| + \frac{5}{2} K \left(\frac{\pi}{P_1}\right)^2,$$

since

$$\theta_1^{\tilde{Q}+1_1} - \theta_1^{\tilde{Q}} = \frac{2\pi}{P_1}, \qquad \text{and} \qquad \theta_1^0 - \theta_1^{\tilde{Q}} < \frac{\pi}{P_1}.$$

Hence

$$|W(\theta_1^0, \theta_2^{\tilde{Q}}, \ldots, \theta_n^{\tilde{Q}})| \leq \delta + \frac{5}{2}\pi^2 K \hat{P}^{-2}.$$

Similarly, we can show

$$| W(\theta_1^0, \theta_2^{\tilde{Q}+1_2}, \theta_3^{\tilde{Q}}, \ldots, \theta_n^{\tilde{Q}}) | \leqq \delta + \frac{5}{2}\pi^2 K\hat{P}^{-2}.$$

Repeating what we have done for the other pertinent cases, we have the generalizations

$$| W(\theta_1^0, \theta_2^{\tilde{Q}}, \theta^{n-1}) | \leqq \delta + \frac{5}{2}\pi^2 K\hat{P}^{-2},$$

$$| W(\theta_1^0, \theta^{\tilde{Q}+1_2}, \theta^{n-1}) | \leqq \delta + \frac{5}{2}\pi^2 K\hat{P}^{-2},$$

where θ^{n-1} represents any possible $(\theta_3, \ldots, \theta_n)$ having each of its components θ_j either $\theta_j^{\tilde{Q}}$ or $\theta_j^{\tilde{Q}+1_j}$.

For each of the possible θ^{n-1}, look at the line segment connecting $(\theta_1^0, \theta_2^{\tilde{Q}}, \theta^{n-1})$ to $(\theta_1^0, \theta_2^{\tilde{Q}+1_2}, \theta^{n-1})$. If $\theta^{n-2} = (\theta_4, \ldots, \theta_n)$, with the θ_j the same as for θ^{n-1}, then $(\theta_1^0, \theta_2^0, \theta^{n-2})$ falls on this segment, and the same analysis as above gives

$$| W(\theta_1^0, \theta_2^0, \theta_3^{\tilde{Q}}, \theta^{n-3}) |, | W(\theta_1^0, \theta_2^0, \theta_3^{\tilde{Q}+1_3}, \theta^{n-3}) |$$
$$\leqq \left(\delta + \frac{5}{2}\pi^2 K\hat{P}^{-2} \right) + \frac{5}{2}\pi^2 K\hat{P}^{-2} = \delta + \frac{5}{2}.2\pi^2 K\hat{P}^{-2},$$

where θ^{n-3} is defined just like θ^{n-1} and θ^{n-2}.

Continuing this process and, after obtaining the final bound, converting back from W to w, we arrive at the conclusion.

8. THE ERROR ESTIMATE

Consider the $F_{L,N}(z)$ case. If $z = r\, e^{i\theta}$ is any point in D, then a bound for the error is given by a bound for $|f(z) - F_{L,N}(z)|_r$. To obtain this, we first find a bound on $\dot{D}_\rho$, then on $\dot{D}_R$ for ρ and R such that

$$0 < \rho_k \leqq r_k \leqq R_k, \qquad k = 1, \ldots, n,$$

and R as before. Then we apply Theorem 7.1.

Since

$$\sum_{|K|=0}^{L} \sum_{J=1}^{N} P_r^{(K)}(\theta - \tau_J)\, \mu_K\left(\mathop{\mathsf{X}}_{l=1}^{n} (\tau_{J_l - (1/2)}, \tau_{J_l + (1/2)}) \right)$$
$$= F_{L,N}\left(z, \left\{ \mu_K^+\left(\mathop{\mathsf{X}}_{l=1}^{n} (\tau_{J_l - (1/2)}, \tau_{J_l + (1/2)}) \right) \right\}, \ldots, \right.$$
$$\left. \times \left\{ \nu_K^-\left(\mathop{\mathsf{X}}_{l=1}^{n} (\tau_{J_l - (1/2)}, \tau_{J_l + (1/2)}) \right) \right\} \right) \in \hat{F}_{L,N}, \qquad (8.1)$$

we have

$$\begin{aligned}
&|f_\eta(s_Q)-F_{L,N}(s_Q)| \\
&\leqq \left| f_\eta(s_Q)-\sum_{|K|=0}^{L}\sum_{J=1}^{N} P_r^{(K)}(\theta^Q-\tau_J)\mu_K\left(\mathop{\mathsf{X}}_{l=1}^{n}(\tau_{J_l-(1/2)},\tau_{J_l+(1/2)})\right)\right| \\
&\leqq |f_\eta(s_Q)-f(s_Q)| \\
&\quad+\left| f(s_Q)-\sum_{|K|=0}^{L}\sum_{J=1}^{N} P_r^{(K)}(\theta^Q-\tau_J)\mu_K\left(\mathop{\mathsf{X}}_{l=1}^{n}(\tau_{J_l-(1/2)},\tau_{J_l+(1/2)})\right)\right|. \qquad (8.2)
\end{aligned}$$

Using condition (iii) of Section 3 and (7.4), then

$$|f_\eta(s_Q)-F_{L,N}(s_Q)| \leqq \eta + C(\rho,f)[2^{-|L|}+\hat{N}^{-1}]. \qquad (8.3)$$

Noting that

$$\frac{\partial^2 P_r^{(K)}}{\partial\theta_j^2}=\frac{\partial^2 P_r^{(K)}}{\partial t_j^2}=P_r^{(K+2_j)},$$

there follows

$$\frac{\partial^2 f(z)}{\partial\theta_j^2}=\sum_{|K|=0}^{\infty}\int_0^{2\pi}\cdots\int_0^{2\pi}P_r^{(K+2_j)}(\theta-t)\,d\mu_K(t). \qquad (8.4)$$

If one proceeds in a similar fashion to that leading to (7.7), with $r=\rho$, and uses (5.2), with $\alpha = 1-R/2$, then

$$\left|\frac{\partial^2 f}{\partial\theta_j^2}\right|_\rho \leqq \frac{1}{5}C_1(\rho,R,f), \qquad (8.5)$$

where

$$\begin{aligned}
C_1(\rho,R,f) = 5\cdot 2^n B\left(\frac{1-R}{2},f\right)&\left[\max_{j=1,\ldots,n}\left(\frac{1}{1-\rho}\right)^{1+2_j}\right. \\
&\left.\times\sum_{|K|=0}^{\infty}\frac{(K_j+2)(K_j+1)}{2^{|K|}}\right]\left(\frac{1-R}{1-\rho}\right)^{K}. \qquad (8.6)
\end{aligned}$$

Similarly, $\partial^2 F_{L,N}(z)/\partial\theta_j^2$ has four times this bound on $\dot{D}_\rho$. Thus

$$\left|\frac{\partial^2 f(z)}{\partial\theta_j^2}-\frac{\partial^2 F_{L,N}(z)}{\partial\theta_j^2}\right| \leqq C_1(\rho,R,f), \qquad j=1,\ldots,n. \qquad (8.7)$$

Since

$$|f(s_Q)-F_{L,N}(s_Q)| \leqq |f(s_Q)-f_\eta(s_Q)| + |f_\eta(s_Q)-F_{L,N}(s_Q)|, \qquad (8.8)$$

if one employs Lemma 7.2 with $w(z) = f(z)-F_{L,N}(z)$, as well as condition (iii), (8.3), and (8.7), it follows that

$$|f-F_{L,N}|_\rho \leqq M_1(\rho,R,f), \qquad (8.9)$$

where

$$M_1(\rho, R, f) = 2\eta + \max\left\{C(\rho, f), \frac{5\pi^2 C_1(\rho, R, f)n}{2}\right\}[2^{-|L|} + \hat{N}^{-1} + \hat{P}^{-2}]. \quad (8.10)$$

For a bound on $\dot{D}_R$, note that

$$|f|_R \leqq \sum_{|K|=0}^{\infty} |P_r^{(K)}|_R \operatorname{var}(\mu_K)$$
$$\leqq \frac{2^n B((1-R)/2, f)}{1-R} \sum_{|K|=0}^{\infty} 2^{-|K|} = 4^n \frac{B((1-R)/2, f)}{1-R}. \quad (8.11)$$

Similarly,

$$|F_{L,N}|_R \leqq \frac{4^{n+1} B((1-R)/2, f)}{1-R}. \quad (8.12)$$

Hence,

$$|f - F_{L,N}|_R \leqq M_2(R, f), \quad (8.13)$$

where

$$M_2(R, f) = \frac{5 \cdot 4^n B((1-R)/2, f)}{1-R}. \quad (8.14)$$

Finally, applying Theorem 7.1,

$$|f(z) - F_{L,N}(z)|_r \leqq M_1^{\prod_{j=1}^{n} \{\ln(r_j/R_j)/\ln(\rho_j/R_j)\}} M_2^{1 - \prod_{j=1}^{n} \{\ln(r_j/R_j)/\ln(\rho_j/R_j)\}}. \quad (8.15)$$

Alternatively, it is easy to see this can be written as

$$|f(z) - F_{L,N}(z)|_r \leqq \mathscr{C}(\rho, r, R, f)[\eta + 2^{-|L|} + \hat{N}^{-1} + \hat{P}^{-2}]^{1-\varepsilon(\rho, r, R)}, \quad (8.16)$$

where

$$\mathscr{C}(\rho, r, R, f) = [\max(2, C_1)]^{\prod_{j=1}^{n} \{\ln(r_j/R_j)/\ln(\rho_j/R_j)\}} M_2^{1 - \prod_{j=1}^{n} \{\ln(r_j/R_j)/\ln(\rho_j/R_j)\}} \quad (8.17)$$

and

$$\varepsilon(\rho, r, R) = 1 - \prod_{j=1}^{n} \left\{\frac{\ln(r_j/R_j)}{\ln(\rho_j/R_j)}\right\} = 1 - \prod_{j=1}^{n} \left\{1 - \frac{\ln(\rho_j/r_j)}{\ln(\rho_j/R_j)}\right\}. \quad (8.18)$$

As $r \equiv \min_j R_j$ approaches one, ε decreases but $\mathscr{C}$ becomes large. We summarize with the following theorem:

THEOREM 8.1: *Let $f(z)$ satisfy conditions* (i)-(iii) *of Section* 3 *and suppose $F_{L,N}(z)$, as given by* (6.6) *is the approximation for $f(z)$. Then for*

$$\rho_j < r_j < R_j, \qquad j = 1, \ldots, n,$$

$$|f(z) - F_{L,N}(z)|_r \leqq M_1^{\prod_{j=1}^{n} \{\ln(r_j/R_j)/\ln(\rho_j/R_j)\}} M_2^{1-\prod_{j=1}^{n} \{\ln(r_j/R_j)/\ln(\rho_j/R_j)\}}, \tag{8.19}$$

where M_1 and M_2 are defined, respectively, by (8.10) *and* (8.14). *Alternatively, this can be written as*

$$|f(z) - F_{L,N}(z)| \leqq \mathscr{C}(\rho, r, R, f)[\eta + 2^{-|L|} + \hat{N}^{-1} + \hat{P}^{-2}]^{1-\varepsilon(\rho, r, R)}, \tag{8.20}$$

where $\mathscr{C}$ and ε are given, respectively, by (8.17) *and* (8.18). *Also $\hat{N} = \min_j N_j$ and $\hat{P} = \min_j P_j$. To pick an optimum $\hat{r} = \min_j R_j$ for use in* (8.20), *one must balance off the tendency of $\mathscr{C}$ to increase and ε to decrease as $\hat{r}$ approaches one.*

It further should be pointed out that not only do we have a bound for r, $\rho_j \leqq r_j \leqq R_j$, but by the maximum principle [5, p. 51], the estimate on $\dot{D}_\rho$ gives a bound for r, $r_j \leqq \rho_j$.

Also note that analysis similar to that above gives a parallel result for $G_{L,N}$. The difference for this second case comes from using in place of (7.7) its H_r analog. This means that $C(r, \rho)$ is replaced by the C given by (7.13), the new $C_1(\rho, R, f)$ is $2^{n/2}$ times the old one in (8.6), and M_1 is specified by (8.10) only with the C's this time the new ones. Also, M_2 is $2^{n/2}$ times that given by (8.14). We have the following theorem:

THEOREM 8.2: *Let $f(z)$ be the same as in Theorem* 8.1. *Then a bound for $|f(z) - G_{L,N}(z)|_r$ is given by either* (8.15) *or* (8.16) *only with M_1 and M_2 changed as discussed just above.*

9. LINEAR PROGRAMMING

The actual determination of $F_{L,N}(z)$ or $G_{L,N}(z)$ reduces to a linear programming problem. Consider, again, the situation for $F_{L,N}(z)$. We require

$$a_{J,K}, b_{J,K}, c_{J,K}, d_{J,K} \geqq 0, \tag{9.1}$$

$$\sum_{J=1}^{N} a_{J,K}, \sum_{J=1}^{N} b_{J,K}, \sum_{J=1}^{N} c_{J,K}, \sum_{J=1}^{N} d_{J,K} \leqq \frac{B((1-R)/2, f)((1-R)/2)^K}{K!}, \tag{9.2}$$

along with the restrictions

$$|\mathrm{Re}[F_{L,N}(s_Q, \{a_{J,K}\}, \ldots, \{d_{J,K}\}) - f_\eta(s_Q)]| \leqq \alpha, \tag{9.3}$$

$$|\operatorname{Im}[F_{L,N}(s_Q, \{a_{J,K}\}, \ldots, \{d_{J,K}\}) - f_\eta(s_Q)]| \leqq \alpha, \tag{9.4}$$

$$\alpha \geqq 0, \tag{9.5}$$

where J, K, and s_Q range through their usual values. Relations (9.3) and (9.4) are not quite the same as (7.6). However, as can be seen easily, the difference has only trivial effect on the error bounds. The linear programming problem consists of minimizing α and requires only straightforward application of the simplex method [6]. For the case of $G_{L,N}(z)$, the procedure is essentially the same,

It is possible (*see* [3, 4] for the general idea) to find *a posteriori* bounds on the error once α and the corresponding $\{a_{J,K}\}, \ldots, \{d_{J,K}\}$ (or $\{a_{J,K}\}, \ldots, \{d_{J,0}\}$) are known. Knowing these allows the determination of sharper values of M_1 and M_2.

REFERENCES

1. J. R. Cannon, *Error Estimates for Some Unstable Continuation Problems*, S.I.A.M. J., Vol. 12, 1964, pp. 270-284.
2. J. R. Cannon and J. Douglas, Jr., *The Approximation of Harmonic and Parabolic Functions on Half-Spaces from Interior Data*, Numerical Analysis of Partial Differential Equations (C.I.M.E., 2e Ciclo, Ispra, 1967), pp. 193-230.
3. J. Douglas, Jr., *A Numerical Method for Analytic Continuation*, in *Boundary Value Problems in Differential Equations*, R. E. Langer (Editor), University of Wisconsin Press, Madison, 1960, pp. 179-189.
4. J. Douglas, Jr., *Unstable Physical Problems and their Numerical Approximation*, Lecture Notes, Rice University, Houston, Texas.
5. B. A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables*, OGIZ, Moscow, 1948; English transl., Amer. Math. Soc., Transl., Vol. 8, 1963.
6. S. I. Gass, *Linear Programming Methods and Applications*, McGraw-Hill, New York, 1958.
7. E. Hille, *Analytic Function Theory*, Vol. II, Blaisdell, Waltham, Mass., 1962.
8. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
9. G. Johnson, Jr., *Harmonic Functions on the Unit Disc I*, Illinois J. Math., Vol. 12, 1968, pp. 366-385.
10. H. D. Meyer, Thesis, The University of Chicago, Chicago, Illinois.
11. H. D. Meyer, *A Representation for a Distributional Solution of the Heat Equation*, SIAM J. Math. Anal., Vol. 5, 1974.
12. H. D. Meyer, *Half-plane Representations and Harmonic Continuation*, SIAM J. Math. Anal., Vol. 7. 1976, pp. 713-722; also (with SIAM permission) in *Improperly Posed Boundary Value Problems*, Pitman Publishing, London, 1975, pp. 24-38.
13. R. Saylor, Thesis, Rice University, Houston, Texas, 1966.
14. R. Saylor, *Numerical Elliptic Continuation*, SIAM J. Numer. Anal., Vol. 4, 1967, pp. 575-581.