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MICHEL CROUZEIX

FRANÇOISE RUAMPS

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ON RATIONAL APPROXIMATIONS TO THE EXPONENTIAL ⁽¹⁾

by Michel CROUZEIX and Françoise RUAMPS ⁽²⁾

Communiqué par P.-A. RAVIART

Abstract. — *We give in this paper a simple characterization of the A-acceptability property for a family of rational approximations to e^{-z} . This result, which is implicitly contained in Norsett [3] is obtained here in a direct way. As a corollary, we obtain the results of Ehle [1] on Pade approximations to e^{-z} .*

Let $r(z) = \frac{1 + a_1 z + \dots + a_n z^n}{1 + b_1 z + \dots + b_n z^n}$, $a_i, b_i \in \mathbf{R}$, be a rational approximation to e^{-z} . The function $r(z)$ is said to be *A-acceptable* iff

$$|r(z)| \leq 1 \quad \text{for } z \in \mathbf{C} \operatorname{Re} z \geq 0. \quad (1)$$

This paper is devoted to the proof of the following.

THEOREM : *Assume that the rational function $r(z)$ satisfies*

$$r(z) = e^{-z} + o(z^{2n-1}) \quad (z \rightarrow 0) \quad (2)$$

Then $r(z)$ is A-acceptable iff

$$|a_n| \leq b_n \quad (3)$$

or

$$r(z) \text{ is reducible.} \quad (4)$$

Proof : We set $p(z) = 1 + a_1 z + \dots + a_n z^n$
 $q(z) = 1 + b_1 z + \dots + b_n z^n$

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⁽²⁾ U.E.R. Mathématiques et Informatique Université de Rennes.

1. THE CONDITION IS NECESSARY

Assume that $r(z)$ is A -acceptable. Then the poles of $r(z)$ belong to the open half-plane $\text{Re } z < 0$ and we have $\lim_{|z| \rightarrow \infty} |r(z)| \leq 1$. Moreover, if $r = p/q$ is irreducible, we have $b_n \geq 0$ (otherwise the polynomial q would have a positive root) and $|a_n| \leq b_n$.

2. THE CONDITION IS SUFFICIENT

Let us prove it by induction. We first notice that the result is clearly true for $n = 1$. Let us assume that the property is true for $n - 1$. We consider two cases :

1st case r is reducible or $a_n = b_n = 0$. Then it follows from (2) that r is the $(n - 1/n - 1)$ -th Padé approximation to e^{-z} and, from Hummel and Seebeck [2] and Padé [4] $r(z)$ may be written on the form $\frac{A_{n-1}(-z)}{A_{n-1}(z)}$ with

$$A_{n-1}(z) = \sum_{k=0}^{n-1} \frac{(2n - 2 - k)! (n - 1)!}{(2n - 2)! k! (n - 1 - k)!} z^k.$$

Therefore r satisfies the theorem hypothesis for $n - 1$ and it follows from the induction hypothesis that r is A -acceptable.

2nd case $b_n \neq 0$ and $|a_n| \leq b_n$. We set $(a, b) = (a_1, \dots, a_n, b_1, \dots, b_n)$ and $E_n = \{ (a, b); r(z) = e^{-z} + 0(z^{2n-1}), |a_n| \leq b_n \text{ and } b_n \neq 0 \}$.

The subset E_n of \mathbf{R}^{2n} is convex and therefore connected.

For all $(a, b) \in E_n$ we have

$$|r(iy)|^2 = \frac{1 + \alpha_1 y^2 + \alpha_2 y^4 + \dots + \alpha_n y^{2n}}{1 + \beta_1 y^2 + \beta_2 y^4 + \dots + \beta_n y^{2n}} = 1 + 0(y^{2n-1})$$

and then we have $\alpha_1 = \beta_1, \dots, \alpha_{n-1} = \beta_{n-1}$. From the inequality $|a_n| \leq b_n$ it follows that $\alpha_n \leq \beta_n$ and therefore we have for all $(a, b) \in E_n$:

$$|r(iy)| \leq 1 \quad \text{for all } y \in \mathbf{R}. \tag{5}$$

And so r has no pole on the axis $\text{Re } z = 0$. Therefore q cannot have a root on this axis : indeed $q(iy_0) = 0$ implies $q(-iy_0) = 0$ and $y_0 \neq 0$ for $q(0) = 1$; since iy_0 and $-iy_0$ cannot be poles of r , r may be written $r(z) = p_1(z)/q_1(z)$ with $d^0 p_1 \leq n - 2$ and $d^0 q_1 \leq n - 2$, which is incompatible with (2).

Now we set

$$F_n = \{ (a, b); (a, b) \in E_n \text{ and the roots of } q \text{ belong to the closed half plane } \text{Re } z \leq 0 \}$$

Since q has no root on the imaginary axis, we have

$$F_n = \{ (a, b); (a, b) \in E_n \text{ and the roots of } q \text{ belong to the open half plane } \operatorname{Re} z < 0 \}$$

We notice that F_n is not empty; indeed the $(n - 1/n - 1) - th$ Padé approximation to e^{-z} is irreducible and it follows from the induction hypothesis that its denominator A_{n-1} has no pole in the half plane $\operatorname{Re} z \geq 0$. We set

$$p(z) = A_{n-1}(-z)(1+z) \quad \text{and} \quad q(z) = A_{n-1}(z)(1+z);$$

then we have $(a, b) \in F_n$.

From (6) and (7) it follows that F_n is an open and closed subset of E_n and then we have $F_n = E_n$. Therefore if $(a, b) \in E_n$, r is analytic in a neighbourhood of the half plane $\operatorname{Re} z \geq 0$; hence from the maximum principle and from (5) it follows that r is A -acceptable.

COROLLARY 1 : *The Padé approximations to e^{-z} of type (n/n) , $(n - 1/n)$, $(n - 2/n)$ are A -acceptable.*

COROLLARY 2 : *The poles of Padé approximations to e^{-z} of type (n/n) , $(n - 1/n)$, $(n - 2/n)$ belong to the open left half plane $\operatorname{Re} z < 0$.*

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