

RAIRO

ANALYSE NUMÉRIQUE

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RAIRO – Analyse numérique, tome 12, n° 1 (1978), p. 59-74.

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AN ELASTO-PLASTIC CONTACT PROBLEM (*) (1)

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Communiqué par P. G. Ciarlet

Abstract. — We study the problem of finding the stresses and the displacements in an elasto-plastic body \mathcal{E} in frictionless contact with a rigid body which is pressed against \mathcal{E} . We prove existence of a solution and then we consider finite element methods for finding approximate solutions of the problem.

INTRODUCTION

Duvaut [1] has studied the problem of finding the stresses in an elasto-plastic body \mathcal{E} in frictionless contact with a rigid body \mathcal{B} which is pressed against \mathcal{E} . In this note we extend the study of Duvaut by looking also for the displacements of \mathcal{E} and \mathcal{B} . We shall consider a stationary case corresponding to Henky's law. For simplicity we shall assume that \mathcal{E} is isotropic.

In Section 1 we prove existence of a solution to the contact problem assuming that \mathcal{E} is elastic-perfectly plastic. In this case the displacements of \mathcal{E} may be discontinuous (and even non-unique) and we have to use a formulation requiring little regularity of the displacements. One way of obtaining more regular displacements is to assume a suitable hardening of the elasto-plastic material. Such a case is studied in Section 2. Then in Sections 3 and 4 we consider finite element methods for finding approximate solutions of the contact problem.

1. ELASTIC-PERFECTLY PLASTIC MATERIAL

Suppose that initially the elasto-plastic body \mathcal{E} occupies the bounded region $\Omega \subset \mathbf{R}^3$ with boundary Γ and that Γ contains an open set Γ_1 in the plane $\{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = 0\}$. Moreover, suppose that initially the rigid body \mathcal{B} occupies the region

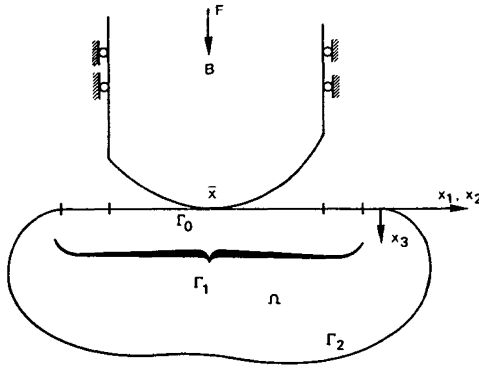
$$B = \{x \in \mathbf{R}^3 : -x_3 \geq \varphi(x_1, x_2), (x_1, x_2) \in \bar{\Gamma}_0\},$$

where Γ_0 is an open set compactly contained in Γ_1 with smooth boundary and $\varphi : \Gamma_1 \rightarrow \mathbf{R}$ is smooth, nonnegative and $\varphi(\bar{x}) = 0$ for some $\bar{x} \in \Gamma_0$ (see *Fig.*).

(*) Reçu avril 1977, révisé août 1977.

(1) Texte d'une conférence présentée aux Journées « Éléments Finis », Rennes, 4-6 mai 1977

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Let the boundary of \mathcal{E} be fixed on the portion $\Gamma_2 = \Gamma \setminus \Gamma_1$ and free on Γ_1 . Let \mathcal{B} be acted upon by the vertical force $F (F > 0)$ and suppose that \mathcal{B} is free to move vertically, whereas rotation and horizontal displacement are prevented. We want to find the vertical displacement U of \mathcal{B} , the stress $\sigma = \{ \sigma_{ij} \}$, $i, j = 1, 2, 3$, in \mathcal{E} , and the displacement $u = \{ u_i \}$, $i = 1, 2, 3$, of \mathcal{E} , where u_i is the displacement in the x_i -direction. The reference configuration is the one in *figure*. We shall assume that the displacements are small; in particular this means that the relation $u_3 \cong U - \varphi$ can be used to describe the compatibility of the displacements of \mathcal{B} and \mathcal{E} .

We shall use the following notation: For m a positive integer and $1 \leq p \leq \infty$, let $\| \cdot \|_{m,p}$ denote the norm in the usual Sobolev space $[W_p^m(\Omega)]^n$ with n a positive integer. If $m = 0$ we omit this index and write $\| \cdot \|_p$ instead of $\| \cdot \|_{0,p}$. Let (\cdot, \cdot) and $\| \cdot \|$ denote the scalar product and norm in $[L^2(\Omega)]^n$. Further we define

$$H = \{ \tau = \{ \tau_{ij} \} \in [L^2(\Omega)]^9 : \tau_{ij} = \tau_{ji}, i, j = 1, 2, 3, \},$$

$$\mathcal{W} = [W]^3, \quad W = \{ w \in W_2^1(\Omega) : w = 0 \text{ on } \Gamma_2 \},$$

$$K = \{ (u, U) \in \mathcal{W} \times \mathbf{R} : u_3 + \varphi - U \geq 0 \text{ on } \bar{\Gamma}_0 \},$$

and the deformation $\varepsilon(u) = \{ \varepsilon_{ij}(u) \}$ associated with $u \in \mathcal{W}$ by

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3,$$

where $w_{,j} = \partial w / \partial x_j$. We recall Green's formula:

$$(\tau, \varepsilon(v)) = \int_{\Gamma} \tau_{ij} n_j u_i ds - (\text{div } \tau, v), \tag{1.1}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to Γ and

$$\begin{aligned} \operatorname{div} \tau &= ((\operatorname{div} \tau)_i), \quad i = 1, 2, 3, \\ (\operatorname{div} \tau)_i &= \tau_{ij,j}. \end{aligned}$$

Here and below we use the summation convention: repeated indices indicate summation from 1 to 3.

Let $D \subset \mathbf{R}^9$ be a given closed convex set with $0 \in D$ and define the set of plastically admissible stresses

$$P = \{ \tau \in H : \tau^d(x) \in D \text{ a.e. in } \Omega \},$$

where

$$\begin{aligned} \tau^d &= \tau - \frac{1}{3} \operatorname{tr}(\tau) \delta, \\ \operatorname{tr}(\tau) &= \tau_{kk}, \\ \delta &= \{ \delta_{ij} \}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

The tensor τ^d is the so called *stress deviatoric* associated to τ . If $\sigma(x) \in$ (interior of D), then \mathcal{E} is in an elastic state at the point $x \in \Omega$ and then we have the linear strain-stress relation

$$\begin{aligned} \varepsilon(u) &= A \sigma, \\ (A \sigma)_{ij} &= \lambda \operatorname{tr}(\sigma) \delta_{ij} + \nu \sigma_{ij}^d, \end{aligned}$$

where λ and ν are positive constants. For notational simplicity we shall assume below that $\nu = 1$. We define the bilinear form

$$a(\sigma, \tau) = \int_{\Omega} (\lambda \operatorname{tr}(\sigma) \operatorname{tr}(\tau) + \sigma_{ij}^d \tau_{ij}^d) dx,$$

and we note that

$$a(\tau, \tau) \geq \alpha \|\tau\|^2, \quad \tau \in H, \tag{1.2}$$

where $\alpha = \min(1, 9\lambda)$.

A natural formulation of the contact problem is now the following: Find $(\sigma, (u, V)) \in P \times K$ such that

$$a(\sigma, \tau - \sigma) - (\varepsilon(u), \tau - \sigma) \geq 0, \quad \forall \tau \in P, \tag{1.3 a}$$

$$(\sigma, \varepsilon(v - u)) \geq F(V - U), \quad \forall (v, V) \in K, \tag{1.3 b}$$

or, equivalently, find a saddle point $(\sigma, (u, V)) \in P \times K$ for the functional $L : P \times K \rightarrow \mathbf{R}$ defined by

$$L(\tau, (v, V)) = \frac{1}{2} \|\tau\|_a^2 - (\tau, \varepsilon(v)) + FV,$$

where $\| \cdot \|_a^2 = a(\cdot, \cdot)$. However, the regularity of the displacement u needed in this formulation is in general not possible to achieve in the perfectly-plastic case. Therefore, we shall instead consider a formulation requiring less regularity of u (cf. [3]). To be more precise, we shall seek u in the space $Y_{3/2}$, where for $1 \leq p \leq \infty$,

$$Y_p = [L_p(\Omega)]^3.$$

To motivate this formulation we first note that be Green's formula (1.1), it follows that (1.3 b) is formally equivalent to the following relations:

$$\operatorname{div} \sigma = 0 \quad \text{in } \Omega, \quad (1.4)$$

$$-\int_{\Gamma_0} \sigma_{33} ds = F, \quad (1.5)$$

$$\sigma_{13} = \sigma_{23} = 0, \quad \sigma_{33} \leq 0 \quad \text{on } \Gamma_1, \quad (1.6)$$

$$\sigma_{33} = 0 \quad \text{on } \Gamma_1 \setminus \Gamma_0, \quad (1.7)$$

$$\sigma_{33}(x) = 0 \quad \text{if } (u_3 + \varphi - U)(x) > 0, \quad x \in \Gamma_0, \quad (1.8)$$

which is the intuitive way of formulating the statical relationship in the contact problem.

We shall seek the stress σ in the space $\mathcal{P}_3 = P \cap \mathcal{H}_3$, where for $2 \leq q < \infty$,

$$\mathcal{H}_q = \{ \tau \in H : \operatorname{div} \tau \in Y_q \text{ and } \tau \text{ satisfies (1.6) and (1.7)} \}.$$

Here (1.6) and (1.7) are to be understood in the following sense:

$$\int_{\Gamma} \tau_{13} w ds = \int_{\Gamma} \tau_{23} w ds = 0, \quad w \in \mathcal{W}, \quad (1.9)$$

$$\int_{\Gamma} \tau_{33} w ds \leq 0, \quad w \in \mathcal{W}, \quad w \geq 0, \quad (1.10)$$

$$\int_{\Gamma} \tau_{33} w ds = 0 \quad \text{for } w \in \mathcal{W} \text{ such that } w = 0 \text{ on } \Gamma_0. \quad (1.11)$$

Note that if $\tau \in H$ and $\operatorname{div} \tau \in Y_2$, then (see [2]) $\tau_{ij} n_j \in H^{-\frac{1}{2}}(\Gamma)$ so that (1.9)-(1.11) are meaningful. Now, taking $\tau \in \mathcal{P}_3$ in (1.3 a) using Green's formula and (1.6), we find with $\psi \in \mathcal{W}$ satisfying $\psi = 1$ on Γ_0 , that

$$\begin{aligned} 0 &\leq a(\sigma, \tau - \sigma) - (\varepsilon(u), \tau - \sigma) \\ &= a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} u_3 (\tau_{33} - \sigma_{33}) ds \\ &= a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} (u_3 + \psi(\varphi - U)) (\tau_{33} - \sigma_{33}) ds \\ &\quad + \int_{\Gamma} \psi(U - \varphi) (\tau_{33} - \sigma_{33}) ds, \end{aligned}$$

so that

$$a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} \psi(U - \varphi)(\tau_{33} - \sigma_{33}) ds \geq 0, \quad (1.12)$$

since by (1.7)-(1.8),

$$\int_{\Gamma} (u_3 + \psi(\varphi - U)) \sigma_{33} ds = 0,$$

and by (1.10) and (1.11) assuming that $(u, U) \in K$,

$$\int_{\Gamma} (u_3 + \psi(\varphi - U)) \tau_{33} ds \leq 0.$$

We are thus led to the following formulation of the contact problem: Find $(\sigma, u, U) \in \mathcal{P} \times Y \times \mathbf{R}$ such that

$$a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} \psi(U - \varphi)(\tau_{33} - \sigma_{33}) ds \geq 0, \quad \tau \in \mathcal{P}, \quad (1.13 a)$$

$$(v, \operatorname{div} \sigma) = 0, \quad v \in Y, \quad (1.13 b)$$

$$-\int_{\Gamma} \psi \sigma_{33} ds = F, \quad (1.13 c)$$

where $\mathcal{P} = \mathcal{P}_3$ and $Y = Y_{3/2}$.

Remark: Note that the condition $u_3 + \varphi - U \geq 0$ on Γ_0 does not appear explicitly in this formulation, which is natural since the trace of $u \in Y$ on Γ may not be defined.

To prove existence of a solution of (1.13) we shall need the following "safe load hypothesis":

$$\begin{aligned} &\text{There exists } \delta > 0 \quad \text{and} \quad \chi \in \mathcal{P} \cap E \\ &\text{such that } \operatorname{dist}(\chi(x), \partial D) > \delta \text{ for } x \in \Omega, \end{aligned} \quad (1.14)$$

where ∂D denotes the boundary of D and

$$E = \left\{ \tau \in H : \operatorname{div} \tau = 0 \text{ in } \Omega, -\int_{\Gamma} \psi \tau_{33} = F \right\}.$$

Note that with $\delta = 0$, this is a necessary condition for existence of a solution.

THEOREM 1: *If (1.14) holds then there exists $(\sigma, u, U) \in \mathcal{P} \times Y \times \mathbf{R}$, satisfying (1.13). Moreover σ is uniquely determined.*

Proof: The proof will be divided into three parts: First we prove existence of a solution of a regularized problem depending on a parameter $\mu > 0$. Then we establish some *a priori* estimates for the solution of this problem and finally we obtain a solution of the original problem by passing to the limit as μ tends to zero. The uniqueness of σ is easy to prove.

(a) The regularized problem

For $\mu > 0$ we consider the following problem: Find a saddle point $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$ for the regularized Lagrangian $L_\mu : H \times K \rightarrow \mathbf{R}$ defined by

$$L_\mu(\tau, (v, V)) = \frac{1}{2} \|\tau\|_a^2 + J_\mu(\tau) - (\tau, \varepsilon(v)) + FV,$$

where

$$J_\mu(\tau) = \frac{1}{2\mu} \|\tau - \pi\tau\|^2,$$

and π is the orthogonal projection in \mathbf{R}^9 onto D . In other words, we seek $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$ satisfying

$$a(\sigma_\mu, \tau) + (J'_\mu(\sigma_\mu), \tau) - (\tau, \varepsilon(u_\mu)) = 0, \quad \tau \in H, \quad (1.15a)$$

$$(\sigma_\mu, \varepsilon(v - u_\mu)) \geq F(V - U_\mu), \quad (v, V) \in K, \quad (1.15b)$$

where $J'(\tau) = (1/\mu)(\tau - \pi\tau)$. Existence of a saddle point $(\sigma_\mu, (u_\mu, U_\mu))$ will follow easily if we can show that the problem

$$\sup_{(v, V) \in K} g_\mu(v, V), \quad (1.16)$$

where

$$g_\mu(v, V) = \inf_{\tau \in H} L_\mu(\tau, (v, V)), \quad (1.17)$$

has a solution. The infimum in (1.17) is attained for $\tau = \bar{\tau}$ satisfying

$$\varepsilon(v) - A\bar{\tau} = \frac{1}{\mu}(\bar{\tau} - \pi\bar{\tau}),$$

i. e.,

$$\varepsilon(v)^d - \bar{\tau}^d = \frac{1}{\mu}(\bar{\tau}^d - \pi\bar{\tau}^d), \quad (1.18)$$

$$\text{tr}(\varepsilon(v)) - \lambda \text{tr}(\bar{\tau}) = 0,$$

since by the definition of D , $\text{tr}(\tau - \pi\tau) = 0$ for $\tau \in H$. But (1.18) implies that $\pi\bar{\tau}^d = \pi\varepsilon(v)^d$ and thus

$$\bar{\tau} = \frac{1}{\lambda} \text{tr}(\varepsilon(v)) + \frac{\mu}{1+\mu} \varepsilon(v)^d + \frac{1}{1+\mu} \pi\varepsilon(v)^d,$$

which gives after a simple computation

$$g_\mu(v, V) = \frac{1}{2(1+\mu)} \|\varepsilon(v)^d - \pi\varepsilon(v)^d\|^2 - \frac{1}{2} \|\varepsilon(v)^d\|^2 - \frac{1}{2\lambda} \|\text{tr}(\varepsilon(v))\|^2 + FV.$$

Since $g_\mu : \mathcal{W} \times \mathbf{R} \rightarrow \mathbf{R}$ is concave (being the infimum of a set of linear functions) and continuous and K is closed and convex in $\mathcal{W} \times \mathbf{R}$, to prove existence of a solution of the problem (1.16) it remains only to prove that g_μ is coercive, i. e.,

$$g_\mu(v, V) \rightarrow -\infty \quad \text{as} \quad \|(v, V)\|_{\mathcal{W} \times K} \rightarrow \infty, \quad (v, V) \in \mathcal{W} \times K.$$

But this follows easily from Korn's inequality (see [2]),

$$\|v\|_{\mathcal{W}} \leq C \|\varepsilon(v)\|, \quad v \in \mathcal{W},$$

the trace inequality,

$$\|v_3\|_{L^1(\Gamma_0)} \leq C \|v\|_{\mathcal{W}}, \quad v \in \mathcal{W},$$

and the fact that

$$V \leq v_3 + \varphi \quad \text{on} \quad \Gamma_0,$$

if $(v, V) \in K$. Thus the problem (1.16) has a solution $(u_\mu, U_\mu) \in K$.

The extremality relation can be written

$$(\sigma_\mu, \varepsilon(v - u_\mu)) \geq F(V - U_\mu), \quad (v, V) \in K,$$

with

$$\sigma_\mu = \frac{1}{\lambda} \text{tr}(\varepsilon(u_\mu)) + \frac{\mu}{1+\mu} \varepsilon(u_\mu)^d + \frac{1}{1+\mu} \pi\varepsilon(u_\mu)^d.$$

Thus (1.15 b) holds and it is easy to check that also (1.15 a) is satisfied and therefore $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$ is a saddle point for L_μ .

(b) A priori estimates

By varying $(v, V) \in X$ in (1.15 b) one concludes that σ_μ satisfies the relations (1.9)-(1.11) and

$$\text{div} \sigma_\mu = 0 \quad \text{in} \quad \Omega, \tag{1.19}$$

$$-\int_{\Gamma_1} \psi \sigma_{\mu, 33} ds = 0, \tag{1.20}$$

$$\int_{\Gamma_1} (u_{\mu, 3} + \psi(\varphi - U_\mu)) \sigma_{\mu, 33} ds = 0. \tag{1.21}$$

Thus, replacing τ in (1.15 a) by $\tau - \sigma_\mu$ where $\tau \in \mathcal{P}$ and applying Green's formula, paralleling the proof of (1.12) we find that

$$a(\sigma_\mu, \tau - \sigma_\mu) + (J'_\mu(\sigma_\mu), \tau - \sigma_\mu) + (u_\mu, \operatorname{div} \tau - \operatorname{div} \sigma_\mu) + \int_\Gamma \psi(U_\mu - \varphi)(\tau_{33} - \sigma_{33, \mu}) ds \geq 0. \quad (1.22)$$

If we now take $\tau = \chi$, where χ is given by assumption (1.14) and use the fact σ_μ as well as χ satisfies (1.19) and (1.20), we get

$$\begin{aligned} & \|\sigma_\mu\|_a^2 + (J'_\mu(\sigma_\mu), \sigma_\mu - \chi) \\ & \leq \int_\Gamma \psi \varphi (\chi_{33} - \sigma_{33, \mu}) ds + a(\sigma_\mu, \chi). \end{aligned} \quad (1.23)$$

But, as is easily seen, (1.14) implies that

$$\|J'_\mu(\sigma_\mu)\|_1 \leq \frac{1}{\delta} (J'_\mu(\sigma_\mu), \sigma_\mu - \chi).$$

Using also the estimate (see [2]):

$$\|\tau\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C(\|\tau\| + \|\operatorname{div} \tau\|), \quad (1.24)$$

and (1.19), we now conclude from (1.23) that

$$\|\sigma_\mu\| \leq C(\|\psi \varphi\|_{H^{1/2}(\Gamma)} + \|\chi\|) \leq C, \quad (1.25)$$

$$\|J'_\mu(\sigma_\mu)\|_1 \leq C, \quad (1.26)$$

with C independent of μ . Using the equation (1.15 a), it then follows that

$$\|\varepsilon(u_\mu)\|_1 \leq C, \quad (1.27)$$

so that

$$\|u_\mu\|_{3/2} \leq C, \quad (1.28)$$

by using the estimate

$$\|v\|_{3/2} \leq C\|\varepsilon(v)\|_1, \quad v \in \mathcal{W}.$$

A proof of this result in the case $\Gamma_2 = \psi$ can be found in [3]. The proof can easily be modified to cover also the present case.

Further, to bound U_μ we note that by the easy to prove trace inequality

$$\int_{\Gamma_0} |w| ds \leq C\|w, 3\|_1, \quad w \in W,$$

and (1.27), we have

$$\int_{\Gamma_0} |u_{\mu 3}| ds \leq C.$$

Since $U_\mu \leq u_{\mu 3} + \varphi$ on Γ_0 we thus find that $U_\mu \leq C$. Moreover, taking $\tau = \sigma_\mu$ in (1.15 a) and $(v, V) = 0$ in (1.15 b) and adding we see that

$$\|\sigma_\mu\|_a^2 + (J'_\mu(\sigma_\mu), \sigma_\mu) \leq F U_\mu.$$

But since J'_μ is monotone and $J'_\mu(0) = 0$, $(J'_\mu(\sigma_\mu), \sigma_\mu) \geq 0$ and thus $U_\mu \geq 0$ so that

$$|U_\mu| \leq C. \tag{1.29}$$

(c) Passage to the limit

From (1.19), (1.24)-(1.27), (1.28) and (1.29) it follows that there exists $(\sigma, (u, U)) \in \mathcal{P} \times Y \times \mathbf{R}$ with $\text{div } \sigma = 0$ and a sequence μ tending to zero such that

$$\left. \begin{aligned} \sigma_\mu &\rightarrow \sigma \text{ weakly in } H, \\ \sigma_{33, \mu} &\rightarrow \sigma_{33} \text{ weakly in } H^{-1/2}(\Gamma), \\ u_\mu &\rightarrow u \text{ weakly in } Y, \\ U_\mu &\rightarrow U. \end{aligned} \right\} \tag{1.30}$$

Passing to the limit in the relation

$$\begin{aligned} a(\sigma_\mu, \tau - \sigma_\mu) + (u_\mu, \text{div } \tau - \text{div } \sigma_\mu) \\ + \int_\Gamma \psi(U_\mu - \varphi)(\tau_{33} - \sigma_{33, \mu}) ds \geq 0, \quad \tau \in \mathcal{P}, \end{aligned}$$

which follows from (1.22) using the monotonicity of J'_μ , we now obtain (1.13 a) without any difficulty recalling that $\text{div } \sigma_\mu = \text{div } \sigma = 0$. Finally, (1.13 c) follows from (1.30) by passing to the limit in (1.20). This completes the proof.

2. HARDENING MATERIAL

To describe the hardening of the elasto-plastic material (cf. [4]) we shall use a hardening parameter $\xi = \{\xi_i\}$, $i = 1, \dots, m$, where m is a positive integer. We shall use the notation

$$\begin{aligned} \hat{H} &= \{\hat{\sigma} = (\sigma, \xi) : \sigma \in H, \xi \in [L^2(\Omega)]^m\}, \\ [\hat{\sigma}, \hat{\tau}] &= a(\sigma, \tau) + \gamma(\xi, \eta), \quad \hat{\sigma} = (\sigma, \xi), \tau = (\tau, \eta) \in \hat{H}, \\ \|\tau\|_a &= [\hat{\tau}, \hat{\tau}]^{1/2}, \quad \hat{\tau} \in \hat{H}, \end{aligned}$$

where γ is a positive constant. Let now \hat{D} be a closed convex set in R^{9+m} , the set of admissible combinations of stress deviatorics and hardening, such that $0 \in \hat{D}$ and define

$$\hat{P} = \{\hat{\tau} \in \hat{H} : (\tau^d, \eta)(x) \in \hat{D} \text{ a. e. in } \Omega\}.$$

The elasto-plastic contact problem can now be formulated in the following way: Find $(\hat{\sigma}, (u, U)) \in \hat{P} \times X$ such that

$$[\hat{\sigma}, \hat{\tau} - \hat{\sigma}] - (\varepsilon(u), \tau - \sigma) \geq 0, \quad \forall \hat{\tau} \in \hat{P}, \quad (2.1 a)$$

$$(\sigma, \varepsilon(v-u)) \geq F(V-U), \quad \forall (v, V) \in K, \quad (2.1 b)$$

or, equivalently, find a saddle point $(\hat{\sigma}, (u, U))$ for the functional $\hat{L} : \hat{P} \times K \rightarrow \mathbf{R}$ defined by

$$\hat{L}(\hat{\tau}, (v, V)) = \frac{1}{2} \|\hat{\tau}\|_a^2 - (\tau, \varepsilon(v)) + FV.$$

To prove existence of a solution of (2.1) we shall use the same method as that used in Section 1 for the regularized problem (1.15). Thus, we consider the problem

$$\sup_{(v, V) \in K} g(v, V), \quad (2.2)$$

where

$$\begin{aligned} g(v, V) &\equiv \inf_{\hat{\tau} \in \hat{P}} \hat{L}(\hat{\tau}, (v, V)) \\ &= \frac{1}{2} \|\hat{\varepsilon}(v) - \hat{\pi}\hat{\varepsilon}(v)\|^2 - \frac{1}{2} \|\varepsilon(v)\|^2 + FV, \end{aligned}$$

with

$$\hat{\varepsilon}(v) = (\varepsilon(v), 0) \in \hat{H},$$

and $\hat{\pi}$ being the projection in \hat{H} onto \hat{P} . Since $g(v, V)$ is clearly concave and continuous on $\mathcal{W} \times \mathbf{R}$, existence of a solution of (2.2) will follow if g is coercive on K , i. e., if

$$g(v, V) \rightarrow -\infty \quad \text{if} \quad \|(v, V)\|_{\mathcal{W}} \rightarrow \infty, \quad (v, V) \in K. \quad (2.3)$$

A solution $(u, U) \in K$ of (2.2) is characterized by the variational inequality

$$(\hat{\pi}\hat{\varepsilon}(u), \hat{\varepsilon}(v-u)) \geq F(V-U), \quad (v, V) \in K. \quad (2.4)$$

Thus, having a solution (u, U) of (2.2) we obtain $(\hat{\sigma}, (u, U)) \in \hat{P} \times K$ satisfying (2.1) by setting $\hat{\sigma} = \hat{\pi}\hat{\varepsilon}(u)$. We therefore have the following result:

THEOREM 2: *If (2.3) holds, then there exists $(\hat{\sigma}, U) \in \hat{P} \times X$ satisfying (2.1). Moreover, $\hat{\sigma}$ is uniquely determined.*

Remark. It is easy to verify that (2.3) holds in the following two cases important in applications (for definiteness we use here the von Mises yield criterion, cf. [4]).

(i) *Isotropic hardening:* In this case $m = 1$ and

$$\hat{B} = \{(\tau, \eta) \in \mathbf{R}^9 \times \mathbf{R} : |\tau^d| \leq 1 + \gamma\eta\}.$$

(ii) *Kinematic hardening*: In this case $m = 9$ and

$$B = \{(\tau, \eta) \in \mathbf{R}^9 \times \mathbf{R}^9 : |\tau^d - \eta| \leq 1\}.$$

It is easy to see that also (u, U) is uniquely determined in these two cases.

3. FINITE ELEMENT METHODS: HARDENING MATERIAL

We shall only briefly discuss the case of a hardening material assuming that the coercivity condition (2.3) holds. In this case we simply take a finite dimensional subspace \mathcal{W}_h of \mathcal{W} , define $K_h = \mathcal{W}_h \cap K$ and seek a solution (u_h, U_h) of the problem

$$\sup_{(v, V) \in K_h} g(v, V),$$

or, equivalently, we seek $(u_h, U_h) \in K_h$ satisfying

$$(\hat{\pi}\hat{\varepsilon}(u_h), \hat{\varepsilon}(v - u_h)) \geq F(V - U_h), \quad (v, V) \in K_h. \tag{3.1}$$

Since g is coercive it follows that such $a(u_h, U_h)$ exists.

It is also easy to obtain an estimate for $\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|$ in the following way: Using the fact that since \hat{P} is convex,

$$\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|^2 \leq (\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h), \hat{\varepsilon}(u) - \hat{\varepsilon}(u_h)),$$

and adding (2.4) and (3.1) with $(v, V) = (u_h, U_h)$ in (2.4), we obtain for all $(v, V) \in K_h$,

$$\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|^2 \leq (\hat{\pi}\hat{\varepsilon}(u_h), v - u) + F(U - V).$$

Having an *a priori* estimate of the form

$$\|\hat{\pi}\hat{\varepsilon}(u_h)\| \leq C, \tag{3.2}$$

this will then give an estimate for $\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|$. In the particular cases discussed in Section 2, (3.2) is easily seen to hold and in these cases we also have

$$\|\varepsilon(u) - \varepsilon(u_h)\| \leq C \|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|.$$

4. FINITE ELEMENT METHODS: PERFECTLY PLASTIC MATERIAL

We shall now consider a finite element method based on the formulation (1.13). We shall restrict ourselves to a two-dimensional problem (plane stress or plane strain) and we shall then use the notation of Section 1 with the obvious change from three to two dimensions. The x_2 -axis will now correspond to the x_3 -axis in the three-dimensional case. For simplicity we shall assume that Γ_1 and Γ_0 are line segments (cf. Fig.). In the two-dimensional case Theorem 1 holds with $\mathcal{P} = \mathcal{P}_2$ and $Y = Y_2$.

The finite element method will be the following: Given finite dimensional spaces $\mathcal{P}_h \subset \mathcal{P}_2$ and $Y_h \subset Y_2$, find $(\sigma_h, u_h, U_h) \in \mathcal{P}_h \times Y_h \times \mathbf{R}$ such that

$$a(\sigma_h, \tau - \sigma_h) + (u_h, \operatorname{div} \tau - \operatorname{div} \sigma_h) + \int_{\Gamma} \psi(U_h - \varphi)(\tau_{22} - \sigma_{22,h}) ds \geq 0, \quad \tau \in \mathcal{P}_h, \quad (4.1 a)$$

$$(v, \operatorname{div} \sigma_h) = 0, \quad v \in Y_h, \quad (4.1 b)$$

$$-\int_{\Gamma} \psi \sigma_{22,h} ds = 0. \quad (4.1 c)$$

We shall now consider a particular choice of the space \mathcal{P}_h and Y_h . For simplicity we shall assume that Ω is polygonal. Let $\{\mathcal{C}_h\}$, $0 < h < 1$, be a regular family of triangulations of Ω

$$\Omega = \bigcup_{K \in \mathcal{C}_h} K,$$

indexed by the parameter h denoting as usual the maximum of the diameters of the triangles $K \in \mathcal{C}_h$. We assume that nodes are placed at the endpoints of Γ_0 and Γ_1 . We shall construct a finite dimensional space $\mathcal{H}_h \subset \mathcal{H}_2$ and then define $\mathcal{P}_h = \mathcal{H}_h \cap P$. The finite element method will be an equilibrium method, i. e., the spaces \mathcal{H}_h and Y_h will satisfy:

$$\text{If } \tau \in \mathcal{H}_h \text{ and } (\operatorname{div} \tau, v) = 0 \text{ for } v \in Y_h, \text{ then } \operatorname{div} \tau = 0 \text{ in } \Omega. \quad (4.2)$$

Methods of this type including the present one have been studied in [5] in the case of linear elasticity. To define \mathcal{H}_h each triangle K is divided into three subtriangles T_k , $k = 1, 2, 3$, by connecting the center of gravity with the three nodes of K . For each $K \in \mathcal{C}_h$ we introduce the finite dimensional space H_k defined by

$$H_k = \{ \tau = \{ \tau_{ij} \} : \tau_{ij} = \tau_{ji} \text{ is linear on } T_k, \\ k = 1, 2, 3, i, j = 1, 2, \text{ and } \operatorname{div} \tau \in [L^2(K)]^2 \}.$$

One can prove (see [5]) that an element $\tau \in H_K$ is uniquely determined by the following 15 degrees of freedom:

$$\text{the value of } \tau \cdot n \text{ at two points of each side of } K, \quad (4.3)$$

$$\int_K \tau_{ij} dx, \quad i, j = 1, 2, \quad (4.4)$$

where $\tau \cdot n = (\tau_{11} n_1 + \tau_{12} n_2, \tau_{21} n_1 + \tau_{22} n_2)$ and $n = (n_1, n_2)$ is the outward unit normal to the boundary of K . The space \mathcal{H}_h can now be defined:

$$\mathcal{H}_h = \{ \tau : \tau|_K \in H_K, K \in \mathcal{C}_h, \text{ and } \operatorname{div} \tau \in Y_2 \}.$$

If $\tau|_K \in H_K$, $K \in \mathcal{C}_h$, then $\operatorname{div} \tau \in Y_2$ if and only if $\tau \cdot n$ is continuous at the interelement boundaries, i. e., if for any side S common to the triangles K and K' ,

$$\tau|_K \cdot n = \tau|_{K'} \cdot n \quad \text{on } S,$$

where n is a normal to S . Therefore the degrees of freedom for an element $\tau \in H_h$ can be chosen as follows: the value of $\tau \cdot n$ at two points at each side of \mathcal{C}_h and the values given by (4.4) for $K \in \mathcal{C}_h$.

Finally, defining

$$Y_h = \{v \in Y_2 : v \text{ is linear on } K, K \in \mathcal{C}_h\},$$

the particular case of the finite element method (4.1) we want to consider has been fully described.

In addition to (4.2) the spaces \mathcal{H}_h and Y_h have the following property (see [5]) important for the analysis: There is an interpolation operator $\pi_h : \mathcal{H}_2 \rightarrow \mathcal{H}_h$ such that

$$(\operatorname{div} \pi_h \tau, v) = (\operatorname{div} \tau, v), \quad v \in Y_h, \tag{4.5}$$

and

$$\|\tau - \pi_h \tau\| \leq Ch^2 \|\tau\|_{2,2}.$$

Given $\tau \in \mathcal{H}_2$, sufficiently regular e. g. $\tau \in [W_1(\Omega)]^4$, the interpolant $\pi_h \tau$ is defined to be the unique element in H_h satisfying for any side S of \mathcal{C}_h with normal n ,

$$\int_K ((\tau - \pi_h \tau) \cdot n) \cdot v \, ds = 0 \quad \text{for } v \text{ linear,}$$

and for any $K \in \mathcal{C}_h$,

$$\int_K (\tau_{ij} - (\pi_h \tau)_{ij}) \, ds = 0, \quad i, j = 1, 2.$$

We observe that if $\operatorname{div} \tau = 0$ in Ω then by (4.2) also $\operatorname{div} \pi_h \tau = 0$ in Ω .

Existence of a solution of the finite element problem can be proved under the following “discrete safe load hypothesis”:

$$\left. \begin{aligned} \text{There exists } \delta > 0 \text{ and } \chi_h \in \mathcal{P}_h \cap E \text{ such that} \\ \operatorname{dist}(\chi_h(x), \partial D) \geq \delta, \quad x \in \Omega, \\ \|\chi_h\| \leq C, \end{aligned} \right\} \tag{4.6}$$

where C and δ are independent of h . We note that if the χ in the safe load hypothesis (1.14) for the continuous problem is sufficiently smooth (e. g. if χ is continuous), then (4.6) will be true for h sufficiently small if we choose $\chi_h = \pi_h \chi$.

Let us now consider the convergence of the finite element method (4.1). We have the following result on weak convergence:

THEOREM 3: *If (4.6) holds, then for any p , $1 \leq p < 2$, there exists $(\sigma, u, U) \in \mathcal{P}_q \times Y_p \times R$ satisfying (1.13) with $\mathcal{P} = \mathcal{P}_q$ and $Y = Y_p$, where $(1/q) + (1/p) = 1$, and a sequence $\{h_i\}$ tending to zero, such that*

$$\sigma_h \rightarrow \sigma \text{ weakly in } H \text{ as } h \rightarrow 0,$$

$$u_{h_i} \rightarrow u \text{ weak star in } Y_p,$$

$$U_{h_i} \rightarrow U.$$

Proof: The theorem follows easily from the following *a priori* estimates:

$$\|\sigma_h\| \leq C, \quad (4.7)$$

$$\|u_h\| \leq C, \quad (4.8)$$

$$|U_h| \leq C. \quad (4.9)$$

The estimate (4.7) follows directly by taking $\tau = \chi_h$ in (4.1 a), where χ_h is given by (4.6).

Next, (4.9) follows by choosing $\tau = \chi_h + \pi_h \bar{\chi}$ in (4.1 a), where $\bar{\chi} \in C^\infty(\Omega) \cap H$ satisfies

$$\operatorname{div} \bar{\chi} = 0 \text{ in } \Omega,$$

$$\int_{\Gamma_1} \bar{\chi}_{22} ds \neq 0,$$

$$\|\bar{\chi}\|_\infty \leq \frac{\delta}{2}.$$

Such a $\bar{\chi}$ can easily be constructed by solving a suitable linear elastic problem.

Finally, (4.8) will follow by choosing $\tau = \chi_h + \pi_h \tilde{\chi}$ in (4.1 a), where $\tilde{\chi} \in \mathcal{H}_2$ satisfies

$$\operatorname{div} \tilde{\chi} = g,$$

where g ranges over the ball $\{g \in Y_q : \|g\|_q \leq \mu\}$, with $\mu > 0$ sufficiently small. For instance one can choose $\tilde{\chi} = \varepsilon(\tilde{u})$, where \tilde{u} satisfies

$$\begin{aligned} \operatorname{div}(\varepsilon(\tilde{u})) &= g \text{ in } \bar{\Omega}, \\ \varepsilon(\tilde{u}) \cdot n &= 0 \text{ on } \partial\bar{\Omega}, \end{aligned}$$

and $\bar{\Omega}$ is a domain with smooth boundary such that $\Omega \subset \bar{\Omega}$, $\Gamma_1 \subset \partial\bar{\Omega}$ and g is suitably extended outside Ω . To see that $\tau = \chi_h + \pi_h \tilde{\chi} \in \mathcal{P}_h$ for μ sufficiently

small, we note that by elliptic regularity (see [6]) one has

$$\|\tilde{\chi}\|_{1,q} \leq C \|g\|_q,$$

which by Sobolev's imbedding theorem implies that $\tilde{\chi}$ is continuous and

$$\|\tilde{\chi}\|_{\infty} \leq C \|g\|_q.$$

Now, (4.1 a) and (4.5) together with the previously obtained estimates (4.7) and (4.9) show that

$$(u_h, g) = (u_h, \operatorname{div} \chi) = (u_h, \operatorname{div} \pi_h \tilde{\chi}) \leq C,$$

if $\|g\|_q \leq \mu$, which proves (4.8). This completes the proof.

Finally, we shall obtain an estimate for $\sigma - \sigma_h$ in terms of the quantity

$$\alpha = \inf \{ \beta : \exists \tau_h \in \mathcal{H}_h \cap E \text{ such that } (1 - \beta) \tau_h \in P \text{ and } \|\sigma - \tau_h\| \leq \beta \}.$$

If σ is sufficiently regular then by choosing $\tau_h = \pi_h \sigma$ we see that $\alpha \rightarrow 0$ as $h \rightarrow 0$.

THEOREM 4: *If (4.6) holds then there exists a constant C independent of h such that for $\alpha < 1$,*

$$\|\sigma - \sigma_h\| \leq C \sqrt{\alpha}.$$

Proof: Choosing $\tau = (1 - 2\alpha) \tau_h$ in (4.1 a) where $\tau_h \in \mathcal{H}_h \cap E$ satisfies $\|\sigma - \tau_h\| \leq 2\alpha$, and $\tau = \sigma_h$ in (1.13 a) and adding, we easily find that

$$\frac{1}{C} \|\sigma - \sigma_h\|^2 \leq \|\sigma - \sigma_h\|_a^2 \leq a(\sigma_h, (1 - 2\alpha) \tau_h - \sigma).$$

$$+ 2\alpha U_h F + \int_{\Gamma_1} \varphi(\sigma_{22} - \tau_{22,h}) ds + 2\alpha \int_{\Gamma_1} \varphi \tau_{22,h} ds.$$

Thus, using the estimates (4.7), (4.9) and (1.24) we obtain the desired estimate.

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