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A MIXED METHOD FOR 4TH ORDER PROBLEMS USING LINEAR FINITE ELEMENTS (*)

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Communiqué par J. A. Nitsche

Abstract. — *The solution of the fourth order problem* $\Delta^2 u = f$ *in* Ω, $u = \partial u/\partial \nu = 0$ *on* $\partial\Omega$, Ω *bounded in* $\mathbf{R}^2$, *and its Laplacian are approximated by linear finite elements.* L_2-and L_∞-*error estimates are given.*

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Let $\Omega \subseteq \mathbf{R}^2$ be a bounded domain with sufficiently smooth boundary. We consider the fourth order boundary value problem

$$\left.\begin{aligned} \Delta^2 u = f \quad &\text{in } \Omega, \\ u = \partial u/\partial \nu = 0 \quad &\text{on } \partial\Omega \end{aligned}\right\} \tag{1}$$

with $f \in L_2$.

The basic idea of the mixed method considered here — due to Ciarlet-Raviart [3] — is to write the equation (1) as a system

$$\left.\begin{aligned} &\left.\begin{aligned} -\Delta u_2 &= f \\ -\Delta u_1 &= u_2 \end{aligned}\right\} \text{ in } \Omega, \\ &u_1 = \partial u_1/\partial \nu = 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{2}$$

and to approximate u_1 and u_2 simultaneously by suitably chosen subspaces. (Another mixed method can be used if one is interested to approximate u and all second derivatives of u. In this context we refer to Brezzi-Raviart [1] and the references given there.)

Using finite element spaces of piecewise polynomials of degree $r \geqq 2$ as approximating subspaces the first L_2-error estimates were given by Ciarlet-Raviart [3]. In [9] improved L_2-estimates have been obtained, and in the case of quadratic finite elements Rannacher [8] proved an L_∞-estimate.

In this note we show that also in the case of linear finite elements the mixed method approximations are convergent, and we derive an error estimate in the L_2- as well as in the L_∞- norm.

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For $h > 0$ let Γ_h be a $\varkappa$-regular partition of Ω in generalized triangles, that means:

(i) $\Delta \in \Gamma_h$ is a triangle if Δ and $\partial\Omega$ have at most one common point, otherwise one of the sides of Δ may be curved;

(ii) there is a fixed $\varkappa > 0$ such that for each $\Delta \in \Gamma_h$ two circles $\underline{K}$ and $\overline{K}$ exists with radii $\varkappa^{-1} h$ and $\varkappa h$ and $\underline{K} \subseteq \Delta \subseteq \overline{K}$.

Let $S_h = S_h(\Gamma_h)$ be the space of continuous functions which are linear in each triangle of Γ_h with the usual modification for the curved elements (*see* Ciarlet-Raviart [2], Zlamal [10]). The space $\mathring{S}_h$ is the intersection of S_h and $\mathring{W}^1_2$, The approximation properties of the spaces S_h and $\mathring{S}_h$ are well-known; confer Ciarlet-Raviart [2] for example.

Further we denote with $(.,.)$ and $D(.,.)$ the inner product in L_2 and the Dirichlet integral. Finally the Ritz operators $R_h : W^1_2 \to S_h$ and $\mathring{R}_h : \mathring{W}^1_2 \to \mathring{S}_h$ are defined by

$$D(v - R_h v, \eta) = 0 \quad \text{for all } \eta \in S_h,$$

$$\int_\Omega (v - R_h v)\, ds = 0$$

and

$$D(u - \mathring{R}_h u, \xi) = 0 \qquad \text{for all } \xi \in \mathring{S}_h$$

respectively.

The following lemma is fundamental for the derivation of our estimates. The proof rests on L_∞-error estimates for the Ritz approximation of second order problems (*see* Nitsche [7], confer also Frehse-Rannacher [4], Nitsche [6]).

LEMMA: *For all $u \in \mathring{W}^1_2 \cap W^2_\infty$ and for all $\eta \in S_h$ we have*

$$| D(u - \mathring{R}_h u, \eta) | \leqq C h^{1/2} | \ln h | \, \| \Delta u \|_{L_\infty} \| \eta \|_{L_2}, \tag{3}$$

with C independent of u, η and h.

Proof: Let Ω_h be the union of all triangles Δ with $\Delta \cap \partial\Omega \neq 0$. With $\xi \in \mathring{S}_h$ we denote the function which interpolates η at the interior grid-points of the triangulation Γ_h. For $\varphi := \eta - \xi$ it follows

$$\varphi(s) = 0, \qquad s \in \Omega - \Omega_h.$$

Setting $\varepsilon := u - \mathring{R}_h u$ and using the inverse estimate

$$\| \varphi \|_{W^1_\infty(\Delta)} \leqq C h^{-1} \| \varphi \|_{L_\infty(\Delta)}$$

we therefore get

$$\begin{aligned} |D(\varepsilon, \eta)| &= |D(\varepsilon, \varphi)| \\ &= \left|\sum \int_\Delta \operatorname{grad} \varepsilon . \operatorname{grad} \varphi \, ds\right| \\ &\leqq C h^2 \|\varepsilon\|_{W^1} \sum \|\varphi\|_{W^1_\infty(\Delta)} \\ &\leqq C h \|\varepsilon\|_{W^1_\infty} \sum \|\varphi\|_{L_\infty(\Delta)}, \end{aligned} \tag{4}$$

where the sum has to be taken over all triangles $\Delta \subseteq \Omega_h$. Now it is easy to see that also in the curved triangles

$$\|\varphi\|_{L_\infty(\Delta)} \leqq C \max_{k=1,2,3} |\varphi(s_k)|,$$

where s_k denotes the vertices of Δ. Therefore we find

$$\|\varphi\|_{L_\infty(\Delta)} \leqq C \|\eta\|_{L_\infty(\Delta)},$$

and with the inverse estimate

$$\|\eta\|_{L_\infty(\Delta)} \leqq C h^{-1} \|\eta\|_{L_2(\Delta)}$$

we get from (4)

$$\begin{aligned} |D(\varepsilon, \eta)| &\leqq C \|\varepsilon\|_{W^1_\infty} \sum \|\eta\|_{L_2(\Delta)} \\ &\leqq C h^{-1/2} \|\varepsilon\|_{W^1_\infty} \|\eta\|_{L_2(\Omega_h)}, \end{aligned}$$

since the number of triangles in Ω_h is of order h^{-1}.

With the L_∞-estimate for the Ritz-approximation (Nitsche [7])

$$\|\varepsilon\|_{W^1_\infty} = \|u - \mathring{R}_h u\|_{W^1_\infty} \leqq C h |\ln h| \|\Delta u\|_{L_\infty}$$

the lemma is proven.

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The mixed finite element approximation $(u_1^h, u_2^h) \in \mathring{S}_h \times S_h$ for the solution (u_1, u_2) of the problem (2) is given by

$$\left.\begin{aligned} D(u_2^h, \xi) &= (f, \xi) \quad \text{for all } \xi \in \mathring{S}_h \\ D(u_1^h, \eta) &= (u_2^h, \eta) \quad \text{for all } \eta \in S_h \end{aligned}\right\} \tag{5}$$

(*see* Ciarlet-Raviart [3], Scholz [9].)

Since $\mathring{S}_h \subseteq S_h$ holds, the system (5) is uniquely solvable, and with $e_i := u_i - u_i^h$, $i = 1, 2$, we can rewrite (5) in the following form

$$\left.\begin{aligned} D(e_2, \xi) &= 0 \quad \text{for all } \xi \in \mathring{S}_h, \\ D(e_1, \eta) &= (e_2, \eta) \quad \text{for all } \eta \in S_h. \end{aligned}\right\} \tag{5'}$$

In the L_2-norm we get the following estimates.

THEOREM 1: *The differences* $e_i = u_i - u_i^h$, $i = 1, 2$, *between the exact solution of the problem* (2) *and the mixed finite element approximation can be estimated by*

$$\| e_1 \|_{L_2} + h^{1/2} | \ln h | \| e_2 \|_{L_2} \leqq C h | \ln h |^2 \| u_1 \|_{W_2^4}, \tag{6}$$

where C *is independent of* (u_1, u_2) *and* h.

COROLLARY: *As a consequence of* (6) *combined with the second part of* (5') *we get for* e_1 *in the* W_2^1*-norm*

$$\| e_1 \|_{W_2^1} \leqq C h^{3/4} | \ln h |^{3/2} \| u_1 \|_{W_2^4}. \tag{7}$$

Proof: Let $\varphi_1 \in \mathring{S}_h$ and $\varphi_2 \in S_h$ be the Ritz approximations of u_1 respective u_2 as defined above. Using the equations (5') we find

$$\begin{aligned} \| u_2^h - \varphi_2 \|_{L_2}^2 &= (u_2^h - \varphi_2, u_2^h - \varphi_2) - D(u_1^h - \varphi_1, u_2^h - \varphi_2) \\ &\quad + D(u_1^h - \varphi_1, u_2^h - \varphi_2) \\ &= (u_2 - \varphi_2, u_2^h - \varphi_2) - D(u_1 - \varphi_1, u_2^h - \varphi_2) \\ &\quad + D(u_1^h - \varphi_1, u_2 - \varphi_2). \end{aligned} \tag{8}$$

With the standard error estimates for the Ritz approximations in the L_2-norm the first term on the right-hand side can be estimated by

$$| (u_2 - \varphi_2, u_2^h - \varphi_2) | \leqq C h^2 \| u_2 \|_{W_2^2} \| u_2^h - \varphi_2 \|_{L_2}.$$

For the second term we get with the help of the lemma

$$| D(u_1 - \varphi_1, u_2^h - \varphi_2) | \leqq C h^{1/2} | \ln h | \| \Delta u_1 \|_{L_\infty} \| u_2^h - \varphi_2 \|_{L_2},$$

and the third term is equal to zero by definition of φ_2. Combining these inequalities with (8) the estimate for e_2 follows.

The other part of (6) is proven by a duality argument. Let $w \in \mathring{W}_2^2 \cap W_2^4$ be the solution of

$$\Delta^2 w = e_1 \quad \text{in } \Omega,$$
$$w = \partial w / \partial \nu = 0 \quad \text{on } \partial\Omega.$$

Observing (5') we get

$$\| e_1 \|_{L_2}^2 = - D(e_1, \Delta w - R_h \Delta w) + (e_2, \Delta w - R_h \Delta w) + D(e_2, w - \mathring{R}_h w). \tag{9}$$

Since u_1^h and φ_1 are elements of $\mathring{S}_h$, we obtain for the first term

$$\begin{aligned} | D(e_1, \Delta w - R_h \Delta w) | &= | D(u_1 - \varphi_1, \Delta w - R_h \Delta w) | \\ &\leqq C h^2 \| u_1 \|_{W_2^2} \| w \|_{W_2^4}, \end{aligned}$$

and the second term can be estimated by

$$| (e_2, \Delta w - R_h \Delta w) | \leqq C h^2 \| e_2 \|_{L_2} \| w \|_{W_2^4}.$$

Finally we get—again with the help of (3)—

$$|D(e_2, w-\mathring{R}_h w)| \leqq C h^2 \| u_2 \|_{W_2^3}(\| w \|_{W_2^3} + \| \Delta w \|_{L_\infty}) + C h^{1/2} |\ln h| \| e_2 \|_{L_2} \| \Delta w \|_{L_\infty}.$$

Using Sobolev's embedding theorem we obtain

$$\| \Delta w \|_{L_\infty} \leqq C \| w \|_{W_2^4}.$$

Together with the estimate for e_2 we find

$$|D(e_2, w-\mathring{R}_h w)| \leqq C h |\ln h|^2 \| u_2 \|_{W_2^3} \| w \|_{W_2^4}.$$

Combining these inequalities with (9) and observing

$$\| w \|_{W_2^4} \leqq C \| e_1 \|_{L_2},$$

the theorem is proven.

The second result is an L_∞-estimate for e_1.

THEOREM 2: *The error $u_1 - u_1^h$ in the L_∞-norm can be estimated by*

$$\| e_1 \|_{L_\infty} \leqq C h |\ln h|^3 \| u_1 \|_{W_2^4}, \tag{10}$$

where C is independent of (u_1, u_2) and h.

Proof: We choose $\Delta \in \Gamma_h$ such that

$$\| e_1 \|_{L_\infty} = \| e_1 \|_{L_\infty(\Delta)}.$$

With standard arguments we get

$$\| e_1 \|_{L_\infty(\Delta)} \leqq C(h^{-1} \| e_1 \|_{L_2(\Delta)} + h^2 \| u_1 \|_{W_\infty^2}) \tag{11}$$

and it suffices to estimate $\| e_1 \|_{L_2(\Delta)}$. This again is done by a duality technique. Let $w \in \mathring{W}_2^2 \cap W_2^4$ be the solution of

$$\Delta^2 w = e_1 \chi_\Delta \quad \text{in } \Omega,$$
$$w = \partial w / \partial \nu = 0 \quad \text{on } \partial\Omega,$$

where χ_Δ is the characteristic function of the triangle Δ. With the same arguments as above we obtain

$$\| e_1 \|_{L_2(\Delta)}^2 \leqq C(h^2 \| w \|_{W_2^4} + h |\ln h|^2 \| \Delta w \|_{L_\infty}) \| u_1 \|_{W_2^4}, \tag{12}$$

and from *a priori*-estimates the inequality

$$\| w \|_{W_2^4} \leqq C h \| e_1 \|_{L_\infty} \tag{13}$$

immediately follows. Further with the help of Sobolev's integral identity (*see* for instance Mikhlin [5], pp. 60-66) we find for all real $\varepsilon > 0$:

$$\| \Delta w \|_{L_\infty} \leqq C(\| \Delta w \|_{L_2} + \varepsilon \| \Delta w \|_{W_\infty^1} + |\log \varepsilon|^{1/2} \| \Delta w \|_{W_2^1}) \tag{14}$$

with C independent of ε. From Sobolev's imbedding theorem it follows for all $p > 2$:

$$\|\Delta w\|_{W^1_\infty} \leqq C_p \|w\|_{W^3_p} \leqq C_p h^{2/p} \|e_1\|_{L_\infty}, \tag{15}$$

C_p independent of h, and from Frehse-Rannacher [4], Theorem 4.B we derive

$$\|\Delta w\|_{W^1_2} \leqq C h^2 |\ln h|^{1/2} \|e_1\|_{L_\infty}. \tag{16}$$

Fixing $p > 2$ and choosing $\varepsilon := h^{2-2/p}$, we get from (13), (14), (15), and (16):

$$\|e_1\|^2_{L_2(\Delta)} \leqq C h^3 |\ln h|^3 \|e_1\|_{L_\infty} \|u_1\|_{W^4_2}.$$

Together with (11) the last inequality gives the desired result.

Remark: An analogue of the estimate (3) can be shown for finite element spaces of higher degree. Therefore the error estimates in this case can be improved too, especially in the case of quadratic finite elements, provided that the solution is sufficiently smooth.

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