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# ON SPECTRAL APPROXIMATION
# PART 2. ERROR ESTIMATES FOR THE GALERKIN METHOD (*)

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Abstract. — One considers an isolated eigenvalue $\lambda$ of finite multiplicity of an operator $A$ which is approximated by a Galerkin method. Using Osborn's technics, one derives several error estimates for $\lambda$.

## 1. SITUATION AND RESULTS

In part 1 of this paper [3], we have been concerned with the problem of convergence in spectral approximation; since the theory we have developped has received concrete applications for non compact operators only in connection with the Galerkin method, we shall now restrict ourself to this case.

Let $X$ be a complex Banach space of norm $\| \ \|$ and $\{ X_h \}$ be a sequence of finite dimensional subspaces of $X$. One gives two continuous sesquilinear forms $a$ and $b$ on $X$ and one supposes a coercive. Then, by Lax-Milgram, one can define the continuous operators $A : X \to X$ and $A_h : X_h \to X_h$ by

$$a(Au, v) = b(u, v), \quad \forall u, v \in X, \quad a(A_h u, v) = b(u, v), \quad \forall u, v \in X_h.$$

All along this paper we shall suppose that the two following conditions are satisfied (*see* [3]):

$$\text{P1}: \quad \lim_{h\to 0} \| (A - A_h) |_{X_h} \| = 0; \quad \text{P2}: \quad \forall x \in X, \quad \lim_{h\to 0} \inf_{x_h \in X_h} \| x - x_h \| = 0.$$

Let $\lambda \in \mathbf{C}$ be an isolated eigenvalue of $A$ of finite algebraic multiplicity $m$; since a is coercive $\lambda \neq 0$ and there exists a closed disc $\Delta$ of center $\lambda$ and boundary $\Gamma$ such that $0 \notin \Delta$ and $\Delta \cap \sigma(A) = \{ \lambda \}$ where $\sigma(A)$ denotes the spectrum of $A$. Let $\mu_{1h}, \ldots, \mu_{m(h),h}$ be the eigenvalues of $A_h$, repeated following their algebraic multiplicities and contained in $\Delta$. In [3], section 2,

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we have proved:

*a*) $m(h) = m$ for $h$ small enough;

*b*) $\lim_{h \to 0} \mu_{ih} = \lambda$, $i = 1, 2, \ldots, m$.

The purpose of this part 2 of our paper is to give estimates of $\lambda$ by the $\mu_{ih}$'s. In fact, we shall adapt to the situation described above Osborn's method [5]; note that, independently of the fact that $A_h$ is a Galerkin approximation, we have simplified the presentation of Osborn's main argument and strengthened his results. See also Grigorieff [4].

At this point, we recall some standard notations. For an operator $D$, $R_z(D) = (z - D)^{-1}$ is the resolvent operator. Let $Y$ and $Z$ be closed subspaces of $X$; then for $x \in X$,

$$\delta(x, Z) = \inf_{z \in Z} \| x - z \|, \quad \delta(Y, Z) = \sup_{\substack{y \in Y \\ \| y \| = 1}} \delta(y, Z)$$

and

$$\hat{\delta}(Y, Z) = \max(\delta(Y, Z), \delta(Z, Y)).$$

Let us also open a short parenthesis on duality. Let $X^*$ be the adjoint space of $X$, i. e. the set of antilinear continuous forms on $X$. By Lax-Milgram, the operator $C : X^* \to X$ defined by the relation $a(v, C\varphi) = \bar{\varphi}(v)$, $\forall v \in X$, $\varphi \in X^*$, is an isomorphism between $X^*$ and $X$ which allows to identify these two spaces. With this identification if $D : X \to X$ is a bounded linear operator, its adjoint $D^* : X \to X$ will be characterized by the relation $a(Du, v) = a(u, D^* v)$, $\forall u, v \in X$; one verifies also immediately the relation $\| D^* \| \leqq \| C \| . \| C^{-1} \| . \| D \|$.

We need, for the following, to introduce some further operators. $\Pi_h : X \to X$ is the projector with range $X_h$ defined by the relation $a(\Pi_h u - u, v) = 0$, $\forall v \in X_h$. One has $A_h = \Pi_h A |_{X_h}$ and we set $B_h = \Pi_h A \Pi_h : X \to X$; exept for zero, $B_h$ has the same spectrum as $A_h$ and the same corresponding invariant subspaces. $E = (2 \Pi i)^{-1} \int_\Gamma R_z(A)\, dz$ is the spectral projector of $A$ relative to $\lambda$ and, for $h$ small enough, $F_h = (2 \Pi i)^{-1} \int_\Gamma R_z(B_h)\, dz$ is the spectral projector of $B_h$ relative to $\mu_{1h}, \ldots, \mu_{mh}$. Now consider the adjoints of these operators as defined above. $A^*$ has the isolated eigenvalue $\bar{\lambda}$ of algebraic multiplicity $m$; $\Pi_h^*$ will be the projector with range $X_h$ satisfying the relation $a(v, \Pi_h^* u - u) = 0$, $\forall v \in X_h$; $E^*$ and $F_h^*$ will be the spectral projectors of $A^*$ and $B_h^* = \Pi_h^* A^* \Pi_h^*$ associated respectively to $\bar{\lambda}$ and to the set $\bar{\mu}_{1h}, \ldots, \bar{\mu}_{mh}$;

they will satisfy the relations

$$E^* = (2\Pi i)^{-1} \int_{\bar{\Gamma}} R_z(A^*)\,dz \qquad \text{and} \qquad F_h^* = (2\Pi i)^{-1} \int_{\bar{\Gamma}} R_z(B_h^*)\,dz$$

where $\bar{\Gamma}$ is the conjugate circle of $\Gamma$ (positively oriented).

In applications $E(X)$ and $E^*(X)$, the $m$-dimensional invariant subspaces of $A$ and $A^*$ corresponding to $\lambda$ and $\bar{\lambda}$, will be often composed of smooth functions so that it is reasonable to introduce the quantities

$$\gamma_h = \delta(E(X), X_h), \qquad \gamma_h^* = \delta(E^*(X), X_h).$$

We can now state the results.

THEOREM 1 : *There exists a constant c, independent of h such that*

$$\hat{\delta}(F_h(X), E(X)) \leqq c\,\gamma_h; \quad \hat{\delta}(F_h^*(X), E^*(X)) \leqq c\,\gamma_h^*.$$

In section 2, we shall show that $F_h|_{E(X)}$ defines for $h$ small enough, a bijection between $E(X)$ and $F_h(X)$; let $\Lambda_h$ be this bijection; $\hat{A} = A|_{E(X)}$ and $\hat{B}_h = \Lambda_h^{-1} B_h \Lambda_h$ will be considered as operators in $E(X)$; $\hat{A}$ has the eigenvalue $\lambda$ of algebraic multiplicity $m$ and $\hat{B}_h$ has the eigenvalues $\mu_{1h}, \ldots, \mu_{mh}$.

THEOREM 2 : *There exists a constant c, independent of h such that*

$$\| \hat{A} - \hat{B}_h \|_{E(X)} \leqq c\,\gamma_h \gamma_h^*.$$

By the choice of a basis in $E(X)$, theorem 2 reduces our original task to a pure matricial problem. Let $f$ be a holomorphic function defined in a neighborhood of $\lambda$; writting $f(\hat{A})$ and $f(\hat{B}_h)$ by the mean of Dunford integrals, one verifies immediately that

$$\| f(\hat{A}) - f(\hat{B}_h) \|_{E(X)} \leqq c\, \| \hat{A} - \hat{B}_h \|_{E(X)}$$

where $c$ depends on $f$ but not on $h$; using the classical properties of traces and determinants, one obtains theorem 3 $a$, $b$; theorem 3 $c$, $d$ is a direct application of results quoted in [7], pp. 80-81; here $\alpha$ is the ascent of the eigenvalue $\lambda$ of $\hat{A}$, $\beta$ is the number of Jordan blocs of the canonical form of $\hat{A}$.

THEOREM 3 : *There exists a constant c, independent of h such that for h small enough:*

*a)* $$\left| f(\lambda) - \frac{1}{m} \sum_{i=1}^{m} f(\mu_{ih}) \right| \leqq c \gamma_h \gamma_h^*,$$

*b)* $$\left| f^m(\lambda) - \prod_{i=1}^{m} f(\mu_{ih}) \right| \leqq c \gamma_h \gamma_h^*,$$

*c)* $$\max_{i=1\ldots m} |\lambda - \mu_{ih}| \leqq c (\gamma_h \gamma_h^*)^{1/\alpha},$$

*d)* $$\min_{i=1\ldots m} |\lambda - \mu_{ih}| \leqq c (\gamma_h \gamma_h^*)^{\beta/m}.$$

REMARKS : 1) In his original work, in a difference context, Osborn [5] has obtained theorem 3 *a* for $f(z) = z$ and theorem 3 *c*; in another context also Chatelin [1] proves, theorem 3 *a* for $f(z) = z$.

2) For $f(z) = 1/z$, theorem 3 *a* gives an estimate of $1/\lambda$ by the arithmetic mean of the $1/\mu_{ih}$'s; the result has been already obtained by [2]; we are indebted to Chatelin who showed us that it can also be deduced by Osborn's method.

In order to illustrate this theorem, we consider the example developped in section 4 of part 1 of this paper [3]; one can prove by Rappaz' method of elimination used in [6] the existence of an infinite number of isolated eigenvalues of finite multiplicities; by supposing the coefficients $\alpha$, $\beta$, ... sufficiently smooth, one verifies that the corresponding eigensubspaces are subsets of $H^2 \times (H^1)^2$; consequently $\gamma_h = O(h)$, $\gamma_h^* = O(h)$ and the estimates of theorem 3 *a*, *b* are of order $h^2$.

We conclude this section by stating a very elementary result for the self-adjoint case. We suppose that the forms *a* and *b* are hermitian. Because of its coercivity, a is a scalar product for which $X$ is a Hilbert space with norm $\| x \|_a^2 = a(x, x)$; then $A$, $B_h$ and $\Pi_h$ become hermitian. Let $\nu$ be an eigenvalue of $A$, which is not supposed isolated or of finite multiplicity, and $G$ be the corresponding eigensubspace. For the distance $\delta(\nu, \sigma(B_h))$ from $\nu$ to the spectrum $\sigma(B_h)$ of $B_h$, one gets the estimate

$$\delta(\nu, \sigma(B_h)) = \inf_{\substack{y \in X \\ \|y\|=1}} \|(B_h - \nu) y\|_a \leqq \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(B_h - \nu) y\|_a$$
$$\leqq \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(A - \nu) y\|_a = \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(A - \nu)(y - x)\|_a, \quad \forall x \in G,$$

i. e., since the norms $\|.\|$ and $\|.\|_a$ are equivalent:

$$\delta(\nu, \sigma(B_h)) \leqq c \inf_{\substack{x \in G \\ \|x\|=1}} \delta(x, X_h), \quad c \text{ independent of } h \text{ and } \nu. \tag{1}$$

REMARKS: 1) We have obtained the estimate (1) without supposing P1 or P2.

2) Examples show that it is not possible to replace the right member of (1), by $c \{ \inf_{\substack{x \in G \\ \|x\|=1}} \delta(x, X_h) \}^{1+\varepsilon}$, $\varepsilon > 0$.

## 2. PROOFS

In this section we prove theorem 1 and 2. We use the definitions and notations of section 1 and we suppose hypotheses P1 and P2. $c$ will denote a "generic" constant.

We first recall a well-known result. Since a is continuous and coercive, the projectors $\Pi_h$ are bounded uniformly with respect to $h$ and there exists a constant $c$ such that $\| x - \Pi_h x \| \leqq c\, \delta(x, X_h)$, $\forall x \in X$; the $\Pi_h^*$'s possess the same properties.

Lemma 1 of section 2 of [3] shows that P1 implies the inequality $\sup_{\substack{x \in X_h \\ \|x\|=1}} \| R_z(A_h) x \| \leqq c$, $\forall z \in \Gamma$ for $h$ small enough, $c$ independent of $h$. We extend this result to $B_h$ and $B_h^*$.

LEMMA 1: *There exists $h_0 > 0$ and $c$ such that*

$$\| R_z(B_h) \| \leqq c, \quad h < h_0, \quad z \in \Gamma$$

and

$$\| R_z(B_h^*) \| \leqq c, \quad h < h_0, \quad z \in \bar{\Gamma}.$$

*Proof:* Since $R_{\bar{z}}(B_h^*) = (R_z(B_h))^*$ we need to prove only the first statement; since $B_h$ is compact it suffices to verify that $\| (z - B_h) x \| \geqq c \| x \|$, $\forall x \in X$, $z \in \Gamma$. Taking in account the fact that $0 \notin \Gamma$ one has

$$\begin{aligned} \| x \| &\leqq \| \Pi_h x \| + \| x - \Pi_h x \| \\ &\leqq c \| (z - B_h) \Pi_h x \| + | z |^{-1} \| (z - B_h)(x - \Pi_h x) \| \leqq c \| (z - B_h) x \|. \quad \blacksquare \end{aligned}$$

We note that we shall not use any more P1 explicitely. Consequently, in the proofs of lemma 3 and theorem 1, the statements for the adjoints operators are obtained in the same way as for the direct operators.

We omit the proof of the following trivial:

LEMMA 2: *Let $Y$ and $Z$ be two subspaces of $X$ with the same finite dimension; 1 et $P$: $Y \to Z$ be a linear operator such that* $\| P y - y \| \leqq 0.5 \| y \|$, $\forall y \in Y$.

*Then $P$ is bijective,*

$$\|P^{-1}z\| \leqq 2\|z\|, \quad \forall z \in Z$$

and

$$\sup_{\substack{z \in Z \\ \|z\|=1}} \|P^{-1}z - z\| \leqq 2 \sup_{\substack{y \in Y \\ \|y\|=1}} \|Py - y\|.$$

LEMMA 3:

$$\|(E-F_h)|_{E(X)}\| \leqq c\|(A-B_h)|_{E(X)}\| \leqq c\gamma_h,$$

$$\|(E^*-F_h^*)|_{E^*(X)}\| \leqq c\|(A^*-B_h^*)|_{E^*(X)}\| \leqq c\gamma_h^*.$$

*Proof:* For $h$ small enough, by lemma 1, one has

$$\|(E-F_h)|_{E(X)}\| \leqq (2\Pi)^{-1}\int_\Gamma \|R_z(B_h)\|.\|(A-B_h)R_z(A)|_{E(X)}\|.|dz|$$
$$\leqq c\|(A-B_h)|_{E(X)}\|;$$

$$\|(A-B_h)|_{E(X)}\| \leqq \|(I-\Pi_h)A|_{E(X)}\| + \|\Pi_h A(I-\Pi_h)|_{E(X)}\|$$
$$\leqq c\|(I-\Pi_h)|_{E(X)}\| \leqq c\gamma_h. \quad \blacksquare$$

*Proof of theorem* 1: Lemma 3 implies that $\delta(E(X), F_h(X)) \leqq c\gamma_h$. Set, as in Section 1, $\Lambda_h = F_h|_{E(X)} : E(X) \to F_h(X)$; for $h$ small enough $E(X)$ and $F_h(X)$ have the same dimension $m$; on the other side P2 implies $\lim_{h \to 0} \gamma_h = 0$; by lemma 2, $\Lambda_h^{-1}$ exists for $h$ small enough and is uniformly bounded with respect to $h$; furthermore $\sup_{\substack{x \in F_h(x) \\ \|x\|=1}} \|\Lambda_h^{-1}x - x\| \leqq c\gamma_h$, i. e. $\delta(F_h(X), E(X)) \leqq c\gamma_h$. $\blacksquare$

*Proof of theorem* 2: Let $S_h = \Lambda_h^{-1}F_h - I : X \to X$; $S_h$ is uniformly bounded with respect to $h$ (*see* proof of theorem 1); from the identity

$$(\hat{A}-\hat{B}_h)x = (A-B_h)x + S_h(A-B_h)x, \; x \in E(X),$$

one obtains for $x \in E(X)$, $y \in E^*(X)$, since $F_h S_h = 0$,

$$a((\hat{A}-\hat{B}_h)x, y) = a((A-B_h)x, y) + a(S_h(A-B_h)x, (I-F_h^*)y); \tag{2}$$

$$\|\hat{A}-\hat{B}_h\|_{E(X)} \leqq c \sup_{\substack{x \in E(X),\, y \in X \\ \|x\|=\|y\|=1}} a((\hat{A}-\hat{B}_h)x, y)$$
$$\leqq c \sup_{\substack{x \in E(X),\, y \in E^*(X) \\ \|x\|=\|y\|=1}} a((\hat{A}-\hat{B}_h)x, y); \tag{3}$$

for $x \in E(X)$, $y \in E^*(X)$, $\|x\| = \|y\| = 1$, one has (using lemma 3):

$$|a(S_h(A-B_h)x, (I-F_h^*)y)| \leqq c\|(A-B_h)x\|.\|(I-F_h^*)y\| \leqq c\gamma_h\gamma_h^*; \quad (4)$$

$$\begin{aligned} a((A-B_h)x, y) &= a((I-\Pi_h)Ax, (I-\Pi_h^*)y) \\ &+ a((I-\Pi_h)x, (I-\Pi_h^*)A^*y) \\ &+ a((I-\Pi_h)x, A^*(\Pi_h^*-I)y); \end{aligned}$$

$$|a((A-B_h)x, y)| \leqq c\gamma_h\gamma_h^*; \quad (5)$$

theorem 2 follows from (2), (3), (4) and (5). ■

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