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L_∞ -CONVERGENCE OF FINITE ELEMENT GALERKIN APPROXIMATIONS FOR PARABOLIC PROBLEMS (*)

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Abstract. — Using weighted norms L_∞ -error estimates of the Galerkin method for second order parabolic initial-boundary value problems are derived.

0. INTRODUCTION

Let the model problem

$$\left. \begin{aligned} \dot{u} - \Delta u &= f \quad \text{in } \Omega \times (0, T], \\ u &= 0 \quad \text{on } \partial\Omega \times (0, T], \\ u_{t=0} &= u_0 \quad \text{in } \Omega \end{aligned} \right\} \quad (1)$$

be given. With $S_h \subseteq \dot{H}_1$ being a finite dimensional space — we will consider only finite elements — the standard Galerkin approximation $u_h = u_h(t) \in S_h$ is defined by

$$(\dot{u}_h, \chi) + D(u_h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h \text{ and } t \in (0, T] \quad (2)$$

with

$$u_h(0) = Q_h u_0. \quad (2')$$

Here $(\cdot, \cdot)$ is the $L_2(\Omega)$ -scalar-product and $D(\cdot, \cdot)$ the Dirichlet integral. Q_h may be any computable projection onto S_h . Substitution of f by $\dot{u} - \Delta u$ gives for the error $e = u - u_h$ the defining relation

$$(\dot{e}, \chi) + D(e, \chi) = 0 \quad \text{for } \chi \in S_h. \quad (3)$$

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Because of the Hilbert-space setting error estimates in Sobolev-norms are available primarily. This part of the convergence analysis is solved now in a satisfactory way. In part (b) of the bibliography a number of papers dealing with this question is listed.

With the help of special techniques in one space dimension there are also results in the maximum-norm. We refer to Archer [1], Cavendish-Hall [4], Douglas-Dupont [6], Douglas-Dupont-Wheeler [7], Thomee [15], Wahlbin [16], and Wheeler [17]. Seemingly L_∞ -estimates for general space dimensions are only treated by Bramble-Schatz-Thomee-Wahlbin [3]. The idea is to write (3) in the form

$$e + T_h \dot{e} = (I - R_h) u. \quad (4)$$

Here to any f the element $U_h = R_h \Delta^{-1} f = T_h f \in S_h$ is the Ritz approximation on $-\Delta^{-1} f$ defined by

$$D(U_h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h. \quad (5)$$

In this way L_∞ -estimates for the elliptic problem give rise to corresponding estimates for the parabolic problem. Using Sobolev-type embedding theorems Bramble *et al.* derive L_∞ -estimates in terms of L_2 -estimates of time derivatives of sufficiently high order depending on the dimension of Ω .

The aim of this paper is to give estimates the type

$$\|e\|_{L_x(L_x)} \leq ch^m \{ \|u\|_{L_x(W_x^m)} + \|\dot{u}\|_{L_x(W_x^m)} + \|\ddot{u}\|_{L_2(W_x^m)} \}. \quad (6)$$

Here we consider only the case $u_h(0) = R_h u_0$. More general initial conditions and also the discretisation in time will be discussed in a forthcoming paper.

Similar to the elliptic case extensively we use weighted norms, *see* Natterer [8] and Nitsche [10] and [11]. The corresponding approximation properties of finite elements are derived in sections 2,3. A needed generalization of the boundedness of the L_2 -projection is given in section 4 and the main error analysis in 5-7.

1. NOTATIONS, FINITE ELEMENTS

In the following $\Omega \subseteq \mathbb{R}^N$ denotes a bounded domain with boundary $\partial\Omega$ sufficiently smooth. For any $\Omega' \subseteq \mathbb{R}^N$ let $W_p^k(\Omega')$ be the Sobolev space of functions having L_p -integrable generalized derivatives up to order k . In case $p=2$ we also adopt $H_k(\Omega') = W_2^k(\Omega')$. The norms are indicated by the corresponding subscripts. $\dot{H}_1(\Omega')$ is the closure in $H_1(\Omega')$ of the functions with compact support.

For $T > 0$ fixed the spaces $L_p(W_q^k(\Omega')) = L_p(0, T, W_q^k(\Omega'))$ consist of functions $u = u(t) \in W_q^k(\Omega')$ such that $\|u(t)\|_{W_q^k(\Omega')}$ is L_p -integrable (respective a. e. bounded for $p = \infty$) in $(0, T)$ with the norms

$$\|u\|_{L_p(W_q^k(\Omega'))} = \left\{ \int_0^T \|u(t)\|_{W_q^k(\Omega')}^p dt \right\}^{1/p}. \quad (1.1)$$

In case $\Omega' = \Omega$ we drop Ω , i. e. $H_k = H_k(\Omega)$, etc. If there is no confusion we will also simply write u instead of $u(t)$.

In addition we consider weighted semi-norms. Let $|\cdot|$ denotes the euclidian distance in R^N :

$$\mu = |x - x_0|^2 + \rho^2 \quad (1.2)$$

with $x_0 \in \Omega$ and $\rho > 0$. We define

$$\|\nabla^k v\|_{a, \Omega} = \left\{ \sum_{|\xi|=k} \int_{\Omega} \mu^{-a} |D^\xi v|^2 dx \right\}^{1/2}. \quad (1.3)$$

$(\cdot, \cdot)_{a, \Omega}$ is the corresponding bilinear form. According to above we drop Ω' in case $\Omega' = \Omega$. Furthermore, the $L_p(0, T)$ -norm of $\|u\|_a = \|u(t)\|_{a, \Omega}$ is denoted by $\|\cdot\|$ with subscript $L_p(a)$.

By Γ_h a subdivision of Ω into generalized simplices Δ_i is meant, i. e. Δ_i is a simplex if $\bar{\Delta}_i$ intersects $\partial\Omega$ in at most a finite number of points and otherwise one of the faces may be curved. Γ_h is called κ -regular if to any $\Delta_i \in \Gamma_h$ there are two spheres with radii $\kappa^{-1}h$ and κh such that Δ_i contains the one and is contained in the other.

The finite element spaces $S_h = S(\Gamma_h)$ have the following structure: Let the integer m be fixed. Any $\chi \in S_h$ is in $C^0(\Omega)$, i. e. continuous in Ω , and the restriction to $\Delta_i \in \Gamma_h$ is a polynomial of degree less than m . In the curved elements we use isoparametric modifications as discussed by Ciarlet-Raviart [5], Zlamal [18]. $\hat{S}_h$ is the intersection of S_h and $\hat{H}_1$.

By construction we have $S_h \subseteq H_1$ but in general $S_h \not\subseteq H_k$ for $k \geq 2$. It is useful to introduce the spaces $H'_k = H'_k(\Gamma_h)$ consisting of functions in L_2 the restriction of which to any $\Delta_i \in \Gamma_h$ is in $H_k(\Delta_i)$. Obviously $S_h \subseteq H'_k$ for all k . Parallel to (1.3) we use the "broken" semi-norms

$$\|\nabla^k v\|'_a = \left\{ \sum_{\Delta_i \in \Gamma_h} \|\nabla^k v\|_{a, \Delta_i}^2 \right\}^{1/2}. \quad (1.4)$$

In order to avoid difficulties we will use three different letters for the "constants" in the estimates: k , γ , and c with the following distinctions:

(i) $k_1, k_2, \dots$ denote numerical constants depending only on N and m (the space-dimension and the degree of the finite elements used);

(ii) the parameter ρ in (1.2) is independent of x but will change with h . Most of the lemmata and theorems are only valid if ρ is not too small compared with h . The corresponding conditions are formulated by "for $\gamma_i h \leq \rho$ " respective "let $\gamma_i h = \rho$ ". Of course the γ 's depend on N , m , the domain Ω and the regularity factor κ of Γ_h ;

(iii) numerical constant with the same dependence as the γ 's but entering directly the estimates are denoted by $c, c_1, c_2, \dots$. Normally just c is used, it may differ at different locations. In order not to loose control in section 5 the constants c are numbered.

The case $m=2$, i.e. linear finite elements, need special treatment. Then logarithmic terms of h will appear in the error bounds, see [12] for the elliptic problem. In order not to overburden the paper we assume

$$m \geq 3. \quad (1.5)$$

Furthermore we consider only regular subdivisions with some fixed κ . Finally we remark the powers of μ for the weights μ^a are always within the limit $|a| \leq 2N$.

2. APPROXIMATION PROPERTIES IN WEIGHTED NORMS

Let $\Delta_i \in \Gamma_h$ be any simplex as described in section 1. Then μ (1.2) does not change too fast if ρ is not too small compared with h :

LEMMA 1: Let $\gamma_1 h < \rho$ with $\gamma_1 = 2\kappa$. Then

$$\sup_{x \in \Delta_i} \mu(x) \leq 3 \inf_{x \in \Delta_i} \mu(x). \quad (2.1)$$

Proof: Let $\underline{x}, \bar{x} \in \bar{\Delta}_i$ be points with

$$\left. \begin{aligned} \underline{\mu} &= \mu(\underline{x}) = \inf \{ \mu(x) \mid x \in \Delta_i \}, \\ \bar{\mu} &= \mu(\bar{x}) = \sup \{ \mu(x) \mid x \in \Delta_i \}. \end{aligned} \right\} \quad (2.2)$$

Since in Δ_i :

$$|\nabla \mu| \leq 2 |x - x_0| \leq 2 \bar{\mu}^{1/2}, \quad (2.3)$$

we get

$$\bar{\mu} = \mu(\bar{x}) = \mu(\underline{x}) + (\bar{x} - \underline{x}) \cdot \nabla \mu \leq \underline{\mu} + |\bar{x} - \underline{x}| 2 \bar{\mu}^{1/2}. \quad (2.4)$$

We have $|\bar{x} - \underline{x}| \leq \kappa h$ and therefore

$$\bar{\mu} \leq \underline{\mu} + 2 \kappa h \bar{\mu}^{1/2}. \quad (2.5)$$

Schwartz's inequality in the form

$$2\kappa h \mu^{1/2} \leq \frac{1}{2} \mu + 2\kappa^2 h^2$$

gives

$$\underline{\mu} \leq 2\underline{\mu} + 4\kappa^2 h^2. \quad (2.6)$$

Now — independent of x :

$$\mu \geq \rho^2 \geq \gamma_1^2 h^2 \quad (2.7)$$

and therefore the lemma is shown.

The approximability property

$$\|\nabla^k(v - \chi)\|_{L_2(\Delta_i)}^2 \leq ch^{2(l-k)} \|\nabla^l v\|_{L_2(\Delta_i)}^2 \quad (0 \leq k \leq l \leq m) \quad (2.8)$$

with a proper interpolation resp. approximation $\chi \in S_h$ is well known. Because of lemma 1 we get from this

$$\|\nabla^k(v - \chi)\|_{a, \Delta_i}^2 \leq 3^a h^{2(l-k)} \|\nabla^l v\|_{a, \Delta_i}^2 \quad (2.9)$$

and after summation over $\Delta_i \in \Gamma_h$.

LEMMA 2: Let $\gamma_1 h \leq \rho$. To any $v \in H'_1$ there is a $\chi \in S_h$ according to

$$\|\nabla^k(v - \chi)\|'_a \leq ch^{l-k} \|\nabla^l v\|'_a \quad (0 \leq k \leq l \leq m). \quad (2.10)$$

REMARK: Since (2.8) holds also for $v \in \mathring{H}_1$ with a $\chi \in \mathring{S}_h$ lemma 2 remains valid if H'_1 , S_h is replaced by $H'_1 \cap \mathring{H}_1$ and $\mathring{S}_h$.

The proof of the next lemma follows the same lines and is omitted here.

LEMMA 3: Let $\gamma_1 h \leq \rho$. Then Bernstein-type inequalities hold: For any $\chi \in S_h$:

$$\|\nabla^l \chi\|'_a \leq ch^{k-l} \|\nabla^k \chi\|'_a \quad (0 \leq k \leq l < m). \quad (2.11)$$

Multiplication of a function in S_h resp. $\mathring{S}_h$ gives no longer a function in these spaces. But still a certain "super-approximability" property of such functions is valid (see Nitsche-Schatz [13]):

LEMMA 4: A function $\mu^{-b} \varphi$ with $\varphi \in S_h$ (resp. $\mathring{S}_h$) can be approximated by a $\chi \in S_h$ (resp. $\mathring{S}_h$) according to

$$\|\nabla^k(\mu^{-b} \varphi - \chi)\|'_a \leq c \{ h^{m-k} \|\varphi\|_{a+2b+m} + h^{2-k} \|\nabla \varphi\|_{a+2b+1} \}. \quad (2.12)$$

Before proving the lemma let us consider e. g. the case $a = -b$ and $k = 0$. Then (2.12) means

$$\|\mu^{-b} \varphi - \chi\|_{-b} \leq c \{ h^m \|\varphi\|_{b+m} + h^2 \|\nabla \varphi\|_{b+1} \}. \quad (2.13)$$

Now using (2.11) with $l=1$, $k=0$ and the obvious inequality

$$\|\varphi\|_{b+b'} \leq \rho^{-b'} \|\varphi\|_b \quad (2.14)$$

for $b' \geq 0$ we get

$$\|\mu^{-b} \varphi - \chi\|_{-b} \leq c(h/\rho) \|\varphi\|_b. \quad (2.15)$$

By choosing γ in $\gamma h \leq \rho$ sufficiently large the bound on the right hand side becomes as small as wanted.

In order to prove lemma 4 we apply lemma 2 with $l=m$ and get

$$\|\nabla^k (\mu^{-b} \varphi - \chi)\|'_a \leq c h^{m-k} \|\nabla^m (\mu^{-b} \varphi)\|'_a. \quad (2.16)$$

Since $\varphi \in S_h$ is piecewise a polynomial of degree $< m$ and because of

$$|D^\xi \mu^{-b}| \leq c \mu^{-b-|\xi|/2} \quad (2.17)$$

Leibniz's rule gives

$$\|\nabla^m (\mu^{-b} \varphi)\|'_a \leq c \sum_{n=0}^{m-1} \|\nabla^n \varphi\|'_{a+2b+m-n}. \quad (2.18)$$

The term with $n=0$ in connection with (2.16) leads to the first term of the right hand side in (2.12). Using lemma 3 and (2.14) we get for the rest

$$\begin{aligned} \sum_{n=1}^{m-1} \|\nabla^n \varphi\|'_{a+2b+m-n} &\leq c \sum_{n=1}^{m-1} h^{1-n} \|\nabla \varphi\|_{a+2b+m-n} \\ &\leq c h^{2-m} \|\nabla \varphi\|_{a+2b+1} \sum_{n=1}^{m-1} (h/\rho)^{m-1-n}. \end{aligned} \quad (2.19)$$

The last sum is bounded because of $h \leq \rho$, thus the lemma is proved.

3. SHIFT THEOREMS, "A PRIORI" ESTIMATES

Solutions of boundary value problems obey certain shift theorems. Assume $u \in \mathring{H}_1$ and $k \geq 0$. Then the norm of u in H_{k+2} is equivalent to that of Δu in H_k :

$$c^{-1} \|u\|_{H_{k+2}} \leq \|\Delta u\|_{H_k} \leq c \|u\|_{H_{k+2}}. \quad (3.1)$$

A direct consequence is :

LEMMA 5: Let $k \geq 2$ be an integer. Then for any $u \in \mathring{H}_1 \cap H_k$:

$$\|\nabla^k u\|_a \leq c \left\{ \sum_{n=0}^{k-2} \|\nabla^n \Delta u\|_{a+k-2-n} + \|\nabla u\|_{a+k-1} + \|u\|_{a+k} \right\}. \quad (3.2)$$

In order to prove the lemma the shift theorem (3.1) has to be applied to $\mu^{-b/2} u$ with $b = a$ resp. $a + 1, a + 2, \dots$ and k resp. $k - 1, k - 2, \dots$. The details are left.

There are some exceptions if a is an integer and one of the indices $a + k - 2 - n$ in the sum of (3.2) is zero. We will only need

LEMMA 5' : Let $w \in \dot{H}_1 \cap H_3$. Then

$$\|\nabla^3 w\|_{-1} \leq c \{ \|\nabla \Delta w\|_{-1} + \|\Delta w\| \}, \quad (3.3)$$

$$\|\nabla^3 w\|_{-2} \leq c \{ \|\nabla \Delta w\|_{-2} + \|\Delta w\|_{-1} + \|\nabla w\| \}. \quad (3.4)$$

We will only give the proof of (3.3). We have

$$\|\nabla^3 w\|_{-1}^2 = \rho^2 \|\nabla^3 w\|^2 + \sum_{i=1}^N \iint (x_i - x_{0/i})^2 |\nabla^3 w|^2. \quad (3.5)$$

The shift theorem gives for the first term

$$\begin{aligned} \rho \|\nabla^3 w\| &\leq c \rho \{ \|\nabla \Delta w\| + \|\Delta w\| \} \\ &\leq c \{ \|\nabla \Delta w\|_{-1} + \|\Delta w\|_{-1} \} \leq c \{ \|\nabla \Delta w\|_{-1} + \|\Delta w\| \}. \end{aligned} \quad (3.6)$$

For the other terms we apply (3.1) with $k=1$ and $u = (x_i - x_{0/i})w$. Since $\nabla^3 u$ differs from $(x_i - x_{0/i}) \nabla^3 w$ only by derivatives of w up to order 2 and the same is true for $\nabla \Delta u$ and $(x_i - x_{0/i}) \nabla \Delta w$ we get

$$\iint (x_i - x_{0/i})^2 |\nabla^3 w|^2 \leq c \iint \{ (x_i - x_{0/i})^2 |\nabla \Delta w|^2 + |\nabla^2 w|^2 + |\nabla w|^2 \}. \quad (3.7)$$

The first integrand is bounded by $\|\nabla \Delta w\|_{-1}^2$ whereas the rest is bounded by $\|\Delta w\|^2$.

In general in (3.2) the terms with u and ∇u are present. But depending on a and k they may be interchangeable resp. can be dropped.

LEMMA 6: Let $u \in \dot{H}_1 \cap H_2$. Then:

(i) for $b < 0$ the norms $\|\nabla u\|_b$ and $\|u\|_{b+1}$ are comparable modulo $\|\Delta u\|_{b-1}$, i. e.:

$$\begin{aligned} \|\nabla u\|_b &\leq k \{ \|u\|_{b+1} + \|\Delta u\|_{b-1} \}, \\ \|u\|_{b+1} &\leq k \{ \|\nabla u\|_b + \|\Delta u\|_{b-1} \}. \end{aligned} \quad (3.8)$$

(ii) for $0 < b < (N/2) - 1$ ($N > 2$) both terms are bounded by the last, i. e.:

$$\|u\|_{b+1} + \|\nabla u\|_b \leq k \|\Delta u\|_{b-1}. \quad (3.9)$$

(iii) the case $b = (N/2) - 1$ gives

$$N(N-2)\rho^2 \|u\|_{b+2}^2 + 2 \|\nabla u\|_b^2 = 2D(u, \mu^{-b}u). \quad (3.10)$$

(iv) for arbitrary b the term with ∇u is always bounded by the others

$$\|\nabla u\|_b \leq k(\|u\|_{b+1} + \|\Delta u\|_{b-1}). \quad (3.11)$$

The relation

$$\|\nabla u\|_b^2 = D(u, \mu^{-b}u) - \iint u \nabla u \nabla \mu^{-b} \quad (3.12)$$

is an identity which may be written also in the form

$$\|\nabla u\|_b^2 = D(u, \mu^{-b}u) + \frac{1}{2} \iint u^2 \Delta \mu^{-b} = (u, -\Delta u)_b + \frac{1}{2} \iint u^2 \Delta \mu^{-b}. \quad (3.13)$$

Now direct differentiation gives $-r = |x - x_0|$:

$$\Delta \mu^{-b} = -2b \mu^{-b-2} (N\rho^2 + (N-2b-2)r^2). \quad (3.14)$$

We prove only case (i) in detail, the other proofs follow the same lines. Now let $b < 0$. Then $\Delta \mu^{-b}$ is positive and $\mu^{b+1} \Delta \mu^{-b}$ is bounded and bounded away from zero (3.13) then gives

$$\|\nabla u\|_b^2 \leq (u, -\Delta u)_b + k \|u\|_{b+1}^2 \leq (u, -\Delta u)_b + k^{-1} \|u\|_{b+1}^2. \quad (3.15)$$

Now the assertions of the lemma, part (i) follow from this and the obvious generalization of Schwarz's inequality $-b'$ being arbitrary:

$$(u, v)_b \leq \|u\|_{b-b'} \|v\|_{b+b'}. \quad (3.16)$$

For the sake of completeness we note also

$$D(u, v) \leq \|\nabla u\|_{-b'} \|\nabla v\|_{b'}. \quad (3.17)$$

In section 5 we will introduce to $\Phi \in \mathring{S}_h$ an auxiliary function w defined by

$$\left. \begin{aligned} -\Delta w &= \mu^{-\alpha-1} \Phi \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right\} \quad (3.18)$$

Some of the needed estimates are handled here, the rest will be given in the appendix.

Because of $\mathring{S}_h \subseteq \mathring{H}_1$ we have the regularity $w \in \mathring{H}_1 \cap H_3$. We will need a bound for the $(-\alpha)$ -seminorm of the third derivatives. With the help of lemma 5 we get

$$\|\nabla^3 w\|_{-\alpha} \leq c \{ \|\nabla \Delta w\|_{-\alpha} + \|\Delta w\|_{-\alpha+1} + \|\nabla w\|_{-\alpha+2} + \|w\|_{-\alpha+3} \}. \quad (3.19)$$

First we have

$$\|\Delta w\|_{-\alpha+1} = \|\Phi\|_{\alpha+3}. \quad (3.20)$$

Next we get

$$\|\nabla(\Delta w)\|_{-\alpha} = \|\nabla(\mu^{-\alpha-1}\Phi)\|_{-\alpha} \leq c \{ \|\Phi\|_{\alpha+3} + \|\nabla\Phi\|_{\alpha+2} \}. \quad (3.21)$$

Lemma 3 and (2.14) give

$$\|\nabla\Delta w\|_{-\alpha} + \|\Delta w\|_{-\alpha+1} \leq ch^{-2}(h/\rho)\|\Phi\|_{\alpha+1}. \quad (3.22)$$

In this way we have shown.

LEMMA 7: Let w be defined by (3.18) with α arbitrary. Then

$$\|\nabla^3 w\|_{-\alpha} \leq c \{ h^{-2}(h/\rho)\|\Phi\|_{\alpha+1} + \|\nabla w\|_{-\alpha+2} + \|w\|_{-\alpha+3} \}. \quad (3.23)$$

In deriving this lemma we have applied lemma 5. According to lemma 5' there is the modification.

LEMMA 7': Let w be defined by (3.18). In case of the exceptional values $\alpha = 1, 2$ instead of (3.23) the estimates hold true

$$\left. \begin{aligned} \|\nabla^3 w\|_{-1} &\leq ch^{-2}(h/\rho)\|\Phi\|_2, \\ \|\nabla^3 w\|_{-2} &\leq ch^{-2}(h/\rho)\|\Phi\|_3 + \|\nabla w\|. \end{aligned} \right\} \quad (3.24)$$

4. L_2 -PROJECTIONS

To any v the approximations $\chi \in S_h$ guaranteed by lemma 2 may be replaced by $V_h := P_h v \in S_h$ with the L_2 -projector P_h defined by

$$(V_h, \chi) = (v, \chi) \quad \text{for } \chi \in S_h. \quad (4.1)$$

As a first result we mention :

THEOREM 1: P_h is bounded with respect to any weighted norm, i.e. for a fixed there is a $\gamma_2 \geq \gamma_1$ depending only on N, m, κ and a such that for $\gamma_2 h \leq \rho$:

$$\|P_h v\|_a \leq 2\|v\|_a. \quad (4.2)$$

This was presented at Second Conference on Finite Elements, Rennes 1975, and appeared in the proceedings of that conference, see [10]. But those were distributed only in a limited number. With the above preparations the proof is rather short and will be reproduced here. Let $\varphi = P_h v$ and $\chi \in S_h$ be arbitrary. Then with Schwarz's inequality (3.16):

$$\begin{aligned} \|\varphi\|_a^2 &= (\varphi, \mu^{-a}\varphi) = (\varphi - v, \mu^{-a}\varphi - \chi) + (v, \varphi)_a \\ &\leq \|\varphi - v\|_a \|\mu^{-a}\varphi - \chi\|_{-a} + \|v\|_a \|\varphi\|_a. \end{aligned} \quad (4.3)$$

The consequence (2.15) of lemma 4 gives

$$\|\varphi\|_a^2 \leq c(h/\rho) \|\varphi\|_a^2 + (1+c(h/\rho)) \|v\|_a \|\varphi\|_a. \quad (4.4)$$

Now we choose $\gamma_2 = \text{Max}(\gamma_1, 3c)$ and get in case of $\gamma_2 h \leq \rho$:

$$\|\varphi\|_a \leq \frac{1}{3} \|\varphi\|_a + \frac{4}{3} \|v\|_a. \quad (4.5)$$

A well-known consequence of theorem 1 is the "almost best" approximability

$$\|v - P_h v\|_a \leq 3 \inf \{ \|v - \chi\|_a \mid \chi \in S_h \}. \quad (4.6)$$

In addition we have the property of simultaneous approximability of $P_h v$ on v which we formulate only in the way needed below:

COROLLARY 1: *With the assumptions of theorem 1:*

$$\|v - P_h v\|_a + h \|\nabla(v - P_h v)\|_a \leq c \inf \{ \|v - \chi\|_a + h \|\nabla(v - \chi)\|_a \mid \chi \in S_h \}. \quad (4.7)$$

Proof: Let again $\varphi = P_h v$ for abbreviation and let $\chi \in S_h$ be arbitrary. Then in using lemma 3 applied to $\varphi - \chi \in S_h$ we get

$$\begin{aligned} h \|\nabla(v - \varphi)\|_a &\leq h \|\nabla(v - \chi)\|_a + h \|\nabla(\varphi - \chi)\|_a \\ &\leq h \|\nabla(v - \chi)\|_a + c \|\varphi - \chi\|_a \\ &\leq h \|\nabla(v - \chi)\|_a + c \|v - \varphi\|_a + c \|v - \chi\|_a \end{aligned} \quad (4.8)$$

and therefore with (4.6):

$$\|v - \varphi\|_a + h \|\nabla(v - \varphi)\|_a \leq 3(1+c) \{ \|v - \chi\|_a + h \|\nabla(v - \chi)\|_a \}. \quad (4.9)$$

Since $\chi \in S_h$ is arbitrary (4.9) is also correct with the infimum taken on the right hand side.

REMARK: All of the above statements hold true if S_h is replaced by $\hat{S}_h$.

REMARK: If $v \in H'_l$ resp. $v \in \hat{H}_1 \cap H'_l$ then according to lemma 2 the right hand side of (4.7) is bounded by $ch^l \|\nabla^l v\|'_a$. This gives the simultaneous error estimates

$$\|\nabla^k(v - P_h v)\|_a \leq ch^{l-k} \|\nabla^l v\|'_a \quad (k=0, 1). \quad (4.10)$$

For completeness we mention the result of Bramble-Scott [2] on simultaneous approximability which could be applied also here. But since the question of interpolation in weighted norms is not well-developed the direct proof is shorter. Another possibility would have been to apply the ideas of [9].

5. ESTIMATES IN WEIGHTED NORMS FOR FIXED TIME

In order to derive error estimates for the Galerkin method it is convenient to compare the Galerkin solution u_h with an appropriate approximation U_h on u in the subspace $\dot{S}_h$. We will take the Ritz approximation $U_h = R_h u \in \dot{S}_h$ defined by — see (5):

$$D(u - U_h, \chi) = 0 \quad \text{for } \chi \in \dot{S}_h. \quad (5.1)$$

The error

$$e = u - u_h \quad (5.2)$$

can be splitted

$$e = (u - U_h) - (u_h - U_h) = \varepsilon - \Phi \quad (5.3)$$

with the effect that now Φ is an element of $\dot{S}_h$. The defining relation for Φ is — see (3):

$$(\dot{\Phi}, \chi) + D(\Phi, \chi) = (\dot{\varepsilon}, \chi) \quad \text{for } \chi \in \dot{S}_h. \quad (5.4)$$

Since estimates for ε , i.e. the error of the Ritz method, are available it will be sufficient to bound Φ in terms of ε resp. $\dot{\varepsilon}$. The aim of this section is the proof of

THEOREM 2: *Let $\alpha = N/2$ with $N \neq 3$ and let $\gamma_3 h \leq \rho$ with γ_3 properly chosen. Then*

$$\|\Phi\|_{\alpha+1}^2 + \|\nabla \Phi\|_\alpha^2 \leq c_1 \rho^{-2} \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha-1}^2, \quad (5.5)$$

in case $N = 3$:

$$\|\Phi\|_2^2 + \|\nabla \Phi\|_1^2 \leq c_1 \rho^{-1} \|\dot{\varepsilon} - \dot{\Phi}\|^2. \quad (5.6)$$

Firstly we will give the proof of (5.5) which is divided into three steps. In order to control the constants in this section they are numbered. c denotes in this section an upper bound of the constants in the previous sections. In step 1 we show the validity of

$$\|\nabla \Phi\|_\alpha^2 \leq c_2 \{ \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha-1}^2 + \|\Phi\|_{\alpha+1}^2 \} \quad (5.7)$$

for α, N arbitrary. Using (5.4) and (3.13), (3.14) we get with $\chi \in \dot{S}_h$ arbitrary

$$\begin{aligned} \|\nabla \Phi\|_\alpha^2 &\leq D(\Phi, \mu^{-\alpha} \Phi) + c \|\Phi\|_{\alpha+1}^2, \\ &\leq D(\Phi, \mu^{-\alpha} \Phi - \chi) - (\dot{\varepsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi - \chi) + (\dot{\varepsilon} - \dot{\Phi}, \Phi)_\alpha + c \|\Phi\|_{\alpha+1}^2. \end{aligned} \quad (5.8)$$

Using Schwarz's inequality (3.16), (3.17) we derive

$$\begin{aligned} \|\nabla \Phi\|_\alpha^2 &\leq \frac{1}{4} \|\nabla \Phi\|_\alpha^2 + c_3 \{ \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha-1}^2 + \|\Phi\|_{\alpha+1}^2 \} \\ &\quad + \|\nabla(\mu^{-\alpha} \Phi - \chi)\|_{-\alpha}^2 + \|\mu^{-\alpha} \Phi - \chi\|_{-\alpha+1}^2. \end{aligned} \quad (5.9)$$

Now let χ be an appropriate approximation on u . Lemma 4 with $k=0$, $b=\alpha$, and $a=-\alpha+1$ gives

$$\|\mu^{-\alpha}\Phi - \chi\|_{-\alpha+1} \leq c_4 \{h^m \|\Phi\|_{\alpha+m+1} + h^2 \|\nabla\Phi\|_{\alpha+2}\} \quad (5.10)$$

and because of (2.14) and $h/\rho < 1$:

$$\|\mu^{-\alpha}\Phi - \chi\|_{-\alpha+1} \leq c_4(h/\rho) \{ \|\Phi\|_{\alpha+1} + \|\nabla\Phi\|_{\alpha} \}. \quad (5.11)$$

In the same way we come to

$$\|\nabla(\mu^{-\alpha}\Phi - \chi)\|_{-\alpha} \leq c_5(h/\rho) \{ \|\Phi\|_{\alpha+1} + \|\nabla\Phi\|_{\alpha} \}. \quad (5.12)$$

With the last two bounds (5.9) gives

$$\begin{aligned} \|\nabla\Phi\|_{\alpha}^2 \leq & \left\{ \frac{1}{4} + 2(c_4^2 + c_5^2)(h/\rho)^2 \right\} \|\nabla\Phi\|_{\alpha}^2 \\ & + c_6 \{ \|\dot{\epsilon} - \dot{\Phi}\|_{\alpha-1}^2 + \|\Phi\|_{\alpha+1}^2 \}. \end{aligned} \quad (5.13)$$

Now we choose $\gamma_3 = \text{Max}(\gamma_2, 4(c_4 + c_5))$. Then obviously the coefficient of $\|\nabla\Phi\|_{\alpha}^2$ on the right hand side is less than 1/2 and so (5.7) is shown.

In order to get an estimate for $\|\Phi\|_{\alpha+1}$ we introduce an auxiliary function w defined by

$$\left. \begin{aligned} -\Delta w &= \mu^{-\alpha-1}\Phi \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right\} \quad (5.14)$$

Then with any $\chi \in \dot{S}_h$ we have

$$\|\Phi\|_{\alpha+1}^2 = D(\Phi, w) = D(\Phi, w - \chi) - (\dot{\epsilon} - \dot{\Phi}, w - \chi) + (\dot{\epsilon} - \dot{\Phi}, w). \quad (5.15)$$

In step 2 of the proof of (5.5) we will show

$$\begin{aligned} \|\Phi\|_{\alpha+1}^2 \leq & c_7(h/\rho) \{ \|\Phi\|_{\alpha+1}^2 + \|\nabla\Phi\|_{\alpha}^2 \} \\ & + \delta \|w\|_{\alpha+1}^2 + c_8(1 + \delta^{-1}) \|\dot{\epsilon} - \dot{\Phi}\|_{\alpha-1}^2 \end{aligned} \quad (5.16)$$

with $\delta > 0$ arbitrary. The two terms with δ come from

$$(\dot{\epsilon} - \dot{\Phi}, w) \leq \|\dot{\epsilon} - \dot{\Phi}\|_{\alpha-1} \|w\|_{-\alpha+1} \leq \delta \|w\|_{\alpha+1}^2 + \frac{1}{4\delta} \|\dot{\epsilon} - \dot{\Phi}\|_{\alpha-1}^2. \quad (5.17)$$

With χ chosen properly next we have

$$D(\Phi, w - \chi) \leq \|\nabla\Phi\|_{\alpha} \|\nabla(w - \chi)\|_{-\alpha} \leq \|\nabla\Phi\|_{\alpha} ch^2 \|\nabla^3 w\|_{-\alpha}. \quad (5.18)$$

Firstly let us consider the case $N > 4$. Then we have to apply lemma 7. Since

then $-\alpha + 2 = -N/2 + 2$ is negative part (i) of lemma 6 can be used. In this way we get

$$D(\Phi, w - \chi) \leq c_9 \|\nabla \Phi\|_\alpha \{ (h/\rho) \|\Phi\|_{\alpha+1} + h^2 \|\nabla w\|_{-\alpha+2} \}. \quad (5.19)$$

An essential aid is the next lemma the proof of which is given in the appendix:

LEMMA 8: Let $N \geq 4$ and $\alpha = N/2$. For w defined by (5.14) the a priori estimate

$$\|\nabla w\|_{-\alpha+2}^2 \leq c_{10} \rho^{-4} \|\Phi\|_{\alpha+1}^2 \quad (5.20)$$

is valid.

Obviously the right hand side of (5.19) is bounded by that of (5.16).

For $N=4$ we have by lemma 7'—note $\alpha=2$ in this case:

$$h^2 \|\nabla^3 w\|_{-\alpha} \leq c(h/\rho) \|\Phi\|_{\alpha+1} + ch^2 \|\nabla w\|. \quad (5.21)$$

Applying lemma 8 also here shows that the term $D(\Phi, w - \chi)$ is bounded by the right hand side of (5.16). Finally for $N=2$ lemma 7' gives directly

$$D(\Phi, w - \chi) \leq \|\nabla \Phi\|_\alpha c(h/\rho) \|\Phi\|_{\alpha+1} \leq \frac{1}{2} c(h/\rho) \{ \|\nabla \Phi\|_\alpha^2 + \|\Phi\|_{\alpha+1}^2 \}. \quad (5.22)$$

It remains to bound the middle term in (5.15).

We have

$$(\dot{\epsilon} - \dot{\Phi}, w - \chi) \leq \|\dot{\epsilon} - \dot{\Phi}\|_{\alpha-1} \|w - \chi\|_{-\alpha+1} \quad (5.23)$$

and

$$\|w - \chi\|_{-\alpha+1} \leq ch^3 \|\nabla^3 w\|_{-\alpha+1} \leq ch^2 \|\nabla^3 w\|_{-\alpha}. \quad (5.24)$$

With the help of the bounds given above for $\|\nabla^3 w\|_{-\alpha}$ we see that this term is bounded in the same way by the right hand side of (5.16).

In step 3 of the proof of (5.5) we apply a lemma which also is proved in the appendix.

LEMMA 9: Let $N \geq 2$, $\alpha = N/2$. Then for any $w \in \dot{H}_1 \cap H_2$:

$$\|w\|_{-\alpha+1}^2 \leq c_{11} \rho^{-2} \|\Delta w\|_{-\alpha-1}^2. \quad (5.25)$$

For w defined by (5.14) this gives

$$\|w\|_{-\alpha+1}^2 \leq c_{11} \rho^{-2} \|\Phi\|_{\alpha+1}^2. \quad (5.25)$$

Therefore we may rewrite (5.16):

$$\begin{aligned} \|\Phi\|_{\alpha+1}^2 \leq \{ c_7(h/\rho) + c_{11} \delta \rho^{-2} \} \{ \|\Phi\|_{\alpha+1}^2 + \|\nabla \Phi\|_\alpha^2 \} \\ + c_8(1 + \delta^{-1}) \|\dot{\epsilon} - \dot{\Phi}\|_{\alpha-1}^2 \end{aligned} \quad (5.27)$$

and compare this with (5.7). If

$$\{c_7(h/\rho) + c_{11}\delta\rho^{-2}\}\{1 + c_2\} < 1 \quad (5.28)$$

then $\|\Phi\|_{\alpha+1}$ and $\|\nabla\Phi\|_\alpha$ are bounded by $\|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha-1}$. We may choose

$$\delta = \rho^2 \{4c_{11}(1 + c_2^2)\}^{-1} \quad (5.29)$$

and $\gamma_4 h \leq \rho$ with

$$\gamma_4 = \text{Max}(\gamma_3, 4c_7(1 + c_2)) \quad (5.30)$$

to guarantee this. In this way (5.5) is proved.

Now we turn over to (5.6) of theorem 2. We have already

$$\|\nabla\Phi\|_1^2 \leq c_2 \{ \|\dot{\varepsilon} - \dot{\Phi}\|^2 + \|\Phi\|_2^2 \} \quad (5.31)$$

since the power α was not restricted in step 1. Similar to above we define w by

$$\left. \begin{aligned} -\Delta w &= \mu^{-2}\Phi & \text{in } \Omega, \\ w &= 0 & \text{on } \partial\Omega, \end{aligned} \right\} \quad (5.32)$$

and get now – using lemma 7':

$$\begin{aligned} \|\Phi\|_2^2 &= D(\Phi, w - \chi) - (\dot{\varepsilon} - \dot{\Phi}, w - \chi) + (\dot{\varepsilon} - \dot{\Phi}, w) \\ &\leq ch^2 \|\nabla\Phi\|_1 \|\nabla^3 w\|_{-1} + \|\dot{\varepsilon} - \dot{\Phi}\| \{ \|w\| + h^3 \|\nabla^3 w\| \} \\ &\leq c_{12}(h/\rho) \|\nabla\Phi\|_1 \|\Phi\|_2 + c_{13} \|\dot{\varepsilon} - \dot{\Phi}\| \{ \|w\| + (h/\rho) \|\Phi\|_2 \}. \end{aligned} \quad (5.33)$$

In the analogue way (5.6) is then proved with the only difference that instead of (5.25) now the following lemma – see appendix – has to be applied.

LEMMA 9': Let $N=3$. Then for any $w \in \mathring{H}_1 \cap H_2$:

$$\|w\|^2 \leq c\rho^{-1} \|\Delta w\|_{-2}^2. \quad (5.34)$$

6. ERROR ESTIMATES IN WEIGHTED NORMS

Theorem 2 gives in case $N=2,3$:

$$\|\Phi\|_2 \leq c\rho^{-4+N} \{ \|\dot{\Phi}\|^2 + \|\dot{\varepsilon}\|^2 \}. \quad (6.1)$$

Since by differentiation of (5.5):

$$(\ddot{\Phi}, \chi) + D(\dot{\Phi}, \chi) = (\ddot{\varepsilon}, \chi) \quad \text{for } \chi \in \mathring{S}_h \quad (6.2)$$

we get putting $\chi = \dot{\Phi}$ and integrating

$$\|\dot{\Phi}(t)\|^2 \leq \|\dot{\Phi}(0)\|^2 + 2 \int_0^t \|\ddot{\epsilon}\| \|\dot{\Phi}\| d\tau \quad (6.3)$$

and therefore by Gronwall's lemma

$$\|\dot{\Phi}(t)\|^2 \leq c \left\{ \|\dot{\Phi}(0)\|^2 + \int_0^t \|\ddot{\epsilon}\|^2 d\tau \right\}. \quad (6.4)$$

Since our initial condition—see (5.3) and the remarks in the introduction regarding the choice of the initial value of u_h —is

$$\Phi(0) = 0 \quad (6.5)$$

$\chi = \dot{\Phi}(0)$ in (5.4) gives

$$\|\dot{\Phi}(0)\|^2 = (\dot{\epsilon}(0), \dot{\Phi}(0)) \leq \|\dot{\epsilon}(0)\|^2. \quad (6.6)$$

Therefore we can rewrite (6.4) in the form

$$\|\dot{\Phi}\|_{L_\infty(L_2)} \leq c \left\{ \|\ddot{\epsilon}\|_{L_\infty(L_2)} + \|\ddot{\epsilon}\|_{L_2(L_2)} \right\}. \quad (6.7)$$

In connection with (6.1) we have shown—note that $L_\infty(a)$ is the $L_\infty(0, T)$ norm of $\|\cdot\|_a$:

THEOREM 3': Let $N = 2, 3$. Then

$$\|\Phi\|_{L_\infty(2)}^2 \leq c \rho^{-4+N} \left\{ \|\dot{\epsilon}\|_{L_\infty(L_2)}^2 + \|\ddot{\epsilon}\|_{L_2(L_2)}^2 \right\}. \quad (6.8)$$

In the case $N \geq 4$ the $(\alpha - 1)$ -norm of $\dot{\Phi}$ in (5.5) still is a weighted norm which has to be discussed further. The structure of the defining relation of Φ and $\dot{\Phi}$ is the same. Therefore we will work with Φ firstly and show

THEOREM 4: Let $N \geq 4$ and $\beta = (N/2) - 1$. Then

$$\|\Phi(t)\|_\beta^2 \leq \|\Phi(0)\|_\beta^2 + c \int_0^t \|\dot{\epsilon}\|_\beta^2 d\tau. \quad (6.9)$$

Now we will apply this with $\Phi, \dot{\epsilon}$ replaced by $\dot{\Phi}, \ddot{\epsilon}$. Further we have—the proof is given below.

LEMMA 10: Let $\Phi(0) = 0$ and a, N arbitrary. Then

$$\|\dot{\Phi}(0)\|_a^2 \leq c \|\dot{\epsilon}(0)\|_a^2. \quad (6.10)$$

With the help of (6.9), (6.10) theorem 2 leads to the counterpart of theorem 3'.

THEOREM 3: Let $N \geq 4$, $\alpha = N/2$ and $\beta = \alpha - 1$. Then

$$\|\Phi\|_{L_\infty(\alpha+1)}^2 \leq c \rho^{-2} \{ \|\dot{\epsilon}\|_{L_\infty(\beta)}^2 + \|\ddot{\epsilon}\|_{L_2(\beta)}^2 \}. \quad (6.11)$$

Proof of lemma 10: We take

$$\chi = P_h(\mu^{-\alpha} \Phi(0)) \quad (6.12)$$

with P_h being the L_2 -projector in (5.4):

$$\|\dot{\Phi}(0)\|_a^2 = (\dot{\Phi}(0), \chi) = -D(\Phi(0), \chi) + (\dot{\epsilon}(0), \chi) \leq \|\dot{\epsilon}(0)\|_a \|\chi\|_{-a}. \quad (6.13)$$

Because of theorem 1 we get

$$\|\chi\|_{-a} \leq c \|\mu^{-\alpha} \dot{\Phi}\|_{-a} = c \|\dot{\Phi}(0)\|_a. \quad (6.14)$$

Proof of theorem 4: We start with the identity $-\chi \in \dot{S}_h$ is arbitrary

$$\begin{aligned} (\dot{\Phi}, \Phi)_\beta + D(\Phi, \mu^{-\beta} \Phi) &= (\dot{\Phi}, \mu^{-\beta} \Phi - \chi) + D(\Phi, \mu^{-\beta} \Phi - \chi) \\ &\quad - (\dot{\epsilon}, \mu^{-\beta} \Phi - \chi) + (\dot{\epsilon}, \Phi)_\beta. \end{aligned} \quad (6.15)$$

The choice $\chi = P_h(\mu^{-\beta} \Phi)$ causes that the first term on the right hand side disappears. Further in our case of β (3:10) gives

$$D(\Phi, \mu^{-\beta} \Phi) = \|\nabla \Phi\|_\beta^2 + N(N-2) \frac{1}{2} \rho^2 \|\Phi\|_{\beta+2}^2. \quad (6.16)$$

Therefore with the special χ :

$$\begin{aligned} (\dot{\Phi}, \Phi)_\beta + \|\nabla \Phi\|_\beta^2 + k \rho^2 \|\Phi\|_{\beta+2}^2 \\ = D(\Phi, \mu^{-\beta} \Phi - \chi) - (\dot{\epsilon}, \mu^{-\beta} \Phi - \chi) + (\dot{\epsilon}, \Phi)_\beta. \end{aligned} \quad (6.17)$$

Now lemma 4 with $b = -a = \beta$ and $k=0$ resp. $k=1$ in connection with theorem 1 gives

$$\begin{aligned} \|\nabla^k(\mu^{-\beta} \Phi - \chi)\|_{-\beta} &\leq c h^{-k} \{ h^m \|\Phi\|_{\beta+m} + h^2 \|\nabla \Phi\|_{\beta+1} \} \\ &\leq c h^{1-k} (h/\rho) \{ \rho \|\Phi\|_{\beta+2} + \|\nabla \Phi\|_\beta \}. \end{aligned} \quad (6.18)$$

In this way we get for the first two terms on the right hand side of (6.17):

$$\begin{aligned} D(\Phi, \mu^{-\beta} \Phi - \chi) + (\dot{\epsilon}, \mu^{-\beta} \Phi - \chi) \\ \leq c (h/\rho) \{ \|\nabla \Phi\|_\beta + h \|\dot{\epsilon}\|_\beta \} \{ \|\nabla \Phi\|_\beta + \rho \|\Phi\|_{\beta+2} \}. \end{aligned} \quad (6.19)$$

In the way analogue to the proof of theorem 2 – see especially (5.27) – we get with $\gamma_5 h \leq \rho$ and $\gamma_5 \geq \gamma_4$ chosen properly

$$(\Phi, \dot{\Phi})_\beta + \|\nabla \Phi\|_\beta^2 + \rho^2 \|\Phi\|_{\beta+2}^2 \leq c \{ \|\dot{\epsilon}\|_\beta^2 + \|\Phi\|_\beta^2 \} \quad (6.20)$$

respective

$$\frac{d}{dt} \|\Phi(t)\|_{\beta}^2 = 2(\Phi, \dot{\Phi})_{\beta} \leq c \left\{ \|\dot{\varepsilon}\|_{\beta}^2 + \|\Phi\|_{\beta}^2 \right\}. \quad (6.21)$$

Then Gronwall's lemma gives (6.9).

7. POINTWISE ERROR ESTIMATES

Up to now we had conditions on ρ of the type $\gamma_i h \leq \rho$. Now we fix $\rho = \gamma_5 h$.

Let $t \in [0, T]$ be fixed. There is an $\hat{x} = \hat{x}_t \in \Omega$ such that

$$\Phi(\hat{x}, t) = \pm \|\Phi(t)\|_{L_\infty}. \quad (7.1)$$

We identify x_0 entering μ (1.2) with this $\hat{x}$. Further let $\Delta \in \Gamma_h$ be the simplex (or one of the simplices) with $\hat{x} \in \bar{\Delta}$.

The function Φ restricted to Δ is a polynomial of degree less than m , i.e. an element of a finite dimensional space. Therefore any two norms are equivalent. Because of the κ -regularity of Δ there is a $k = k(N, m, \kappa)$ such that

$$\|\Phi\|_{L_\infty(\Delta)}^2 \leq k \left\{ h^{-N} \iint \Phi^2 dx \right\}. \quad (7.2)$$

Since $x_0 \in \bar{\Delta}$ we have in Δ :

$$\gamma_5^2 h^2 \leq \mu \leq (\gamma_5^2 + \kappa^2) h^2 \quad (7.3)$$

and therefore with $\alpha = N/2$:

$$h^{-N} \iint \Phi^2 dx \leq c \rho^2 \int_{\Delta} \int_{\Delta} \mu^{-\alpha-1} \Phi^2 dx \leq c \rho^2 \|\Phi\|_{\alpha+1}^2 \quad (7.4)$$

resp. combining (7.1), (7.2), (7.4):

$$\|\Phi(t)\|_{L_\infty} \leq c \rho \|\Phi(t)\|_{\alpha+1} \quad (7.5)$$

With the help of theorem 3 we deduce for $N \geq 4$ with $\beta = N/2 - 1$:

$$\|\Phi\|_{L_x(L_x)} \leq c \left\{ \|\dot{\varepsilon}\|_{L_x(\beta)} + \|\ddot{\varepsilon}\|_{L_2(\beta)} \right\}. \quad (7.6)$$

In case $N \leq 3$ the same arguments give — see (7.4):

$$h^{-N} \iint \Phi^2 dx \leq c \rho^{4-N} \int_{\Delta} \int_{\Delta} \mu^{-2} \Phi^2 dx \leq c \rho^{4-N} \|\Phi\|_2^2. \quad (7.7)$$

Because of theorem 3' (7.6) is valid for $N \leq 3$ with $\beta = 0$.

At the end the weighted norms may be replaced by L_p -norms. The factor $\mu^{-\beta}$ is L_q -integrable for $q < N/(N-2)$. Since then q' defined by $q^{-1} + q'^{-1} = 1$ is greater than $N/2$ for any $p > N$:

$$\|v\|_{\beta} \leq c_p \|v\|_{L_p}. \quad (7.6)$$

In this way we get

THEOREM 5: Let $p = 2$ for $N \leq 3$ and $p > N$ for $N \geq 4$. Then

$$\|\Phi\|_{L_x(L_x)} \leq c \{ \|\dot{\varepsilon}\|_{L_x(L_p)} + \|\ddot{\varepsilon}\|_{L_2(L_p)} \}. \quad (7.9)$$

Scott [14] and Nitsche [10] gave the error estimates for the Ritz-method

$$\|\varepsilon\|_{L_\infty} = \|u - R_h u\|_{L_\infty} \leq ch^k \|u\|_{W_x^k} \quad (7.10)$$

for $k \leq m$. Because of $e = u - u_h = \varepsilon - \Phi$ — see (5.3) — we have the final result:

THEOREM 6: Assume the regularity of the solution u of the initial-boundary value problem (1):

- (i) $u \in L_\infty(0, T, W_\infty^k(\Omega))$;
- (ii) $\dot{u} \in L_\infty(0, T, W_\infty^k(\Omega))$;
- (iii) $\ddot{u} \in L_2(0, T, W_\infty^k(\Omega))$.

Then the error $e = u - u_h$ between the exact solution u and the Galerkin approximation u_h defined by (2) is of order h^k with $k \leq m$ — the order of the finite elements used.

REMARK: For $N \leq 3$ the regularity assumptions on $\dot{u}$, $\ddot{u}$ can be lowered:

$$\dot{u} \in L_\infty(0, T, W_2^k(\Omega)), \quad \ddot{u} \in L_2(0, T, W_2^k(\Omega))$$

is sufficient.

REMARK: Having theorem 5 in mind one would expect assumptions of the type:

- (ii') $\dot{u} \in L_\infty(0, T, W_p^k(\Omega))$;
- (iii') $\ddot{u} \in L_2(0, T, W_p^k(\Omega))$,

instead of (ii), (iii) of theorem 6. As was pointed out by Scott the estimates (7.10) together with the L_2 -bounds

$$\|\varepsilon\|_{L_2} \leq ch^k \|u\|_{W_x^k} \quad (7.11)$$

do not imply

$$\|\varepsilon\|_{L_p} \leq ch^k \|u\|_{W_p^k}. \quad (7.12)$$

This is the reason for the formulation with L_∞ -norms in theorem 6.

The convergence rate up to h^m is optimal with respect to the power of h . But in order to get this bounds for the second time derivative are needed. We can get from (6.9) a reduced convergence result but without needing $\bar{\varepsilon}$. With $\Phi(0)=0$ we have

$$\|\Phi\|_{L_\infty(\beta)} \leq c \|\dot{\varepsilon}\|_{L_2(\beta)}. \quad (7.13)$$

For $\beta = N/2 - 1$ now $c \|\Phi\|_\beta$ is an upper bound of $h \|\Phi\|_{L_\infty}$ if x_0 (1.2) is chosen properly. This gives

THEOREM 7: Let $N \geq 3$ and $p > N$. Then

$$\|\Phi\|_{L_\infty(L_\infty)} \leq ch^{-1} \|\dot{\varepsilon}\|_{L_\infty(L_p)}. \quad (7.14)$$

The counterpart of theorem 6 is then

THEOREM 8: The error of the Galerkin approximation is of order h^{k-1} ($k \leq m$) provided the regularity assumptions

$$(i) \quad u \in L_\infty(0, T, W_\infty^{k-1}(\Omega));$$

$$(ii) \quad \dot{u} \in L_2(0, T, W_\infty^k(\Omega)),$$

hold.

8. APPENDIX : PROOF OF LEMMATA 8, 9

For bounded domains $\Omega' \subseteq R^N$ let

$$\lambda(\Omega') = \sup \left\{ \left\| \frac{\nabla w}{\Delta w} \right\|_{\frac{2}{-\alpha+2} \cdot \Omega'}^2 \mid w \in H_1(\Omega') \cap H_2(\Omega') \right\} \quad (8.1)$$

and

$$\Lambda(\Omega') = \sup \left\{ \left\| \frac{w}{\Delta w} \right\|_{\frac{2}{-\alpha+3} \cdot \Omega'}^2 \mid w \in \dot{H}_1(\Omega') \cap H_2(\Omega') \right\}. \quad (8.2)$$

Because of the definition of w (5.14) lemma 8 is proved if we can show $\lambda(\Omega) \leq c \rho^{-4}$. Firstly we consider the case $N > 4$. Then $-\alpha + 2$ is negative and lemma 6, (i) gives

$$\lambda(\Omega') \leq k \{ \Lambda(\Omega') + \rho^{-4} \}, \quad \Lambda(\Omega') \leq k \{ \lambda(\Omega') + \rho^{-4} \} \quad (8.3)$$

with k independent of Ω' . Obviously Λ is monotone in Ω' , i. e. $\Lambda(\Omega') \leq \Lambda(\Omega'')$ for $\Omega' \subseteq \Omega''$. Next let $K = K_R(x_0)$ be a sphere of radius $R = \text{diam}(\Omega)$ with center x_0 . Then $\Omega \subseteq K$ and hence $\Lambda(\Omega) \leq \Lambda(K)$. The supremum $\Lambda(K)$ is attained for a positive function w_K with $-\Delta w_K > 0$ because of the maximum principle, and w_K

solves the eigenvalue problem

$$\left. \begin{aligned} \Delta(\mu^{\alpha+1} \Delta w) &= \Lambda^{-1} \mu^{\alpha-3} w \quad \text{in } K, \\ w = \Delta w &= 0 \quad \text{on } \partial K. \end{aligned} \right\} \quad (8.4)$$

Without loss of generality we can assume $w_K = w_K(r)$ with $r = |x - x_0|$ since μ depends only on r , for otherwise the spherical average of w_K solves the same eigenvalue problem and is also positive. Therefore we can restrict the space of admissible functions without changing Λ :

$$\Lambda(K) = \sup \left\{ \frac{\|w\|_{-\alpha+3, K}^2}{\|\Delta w\|_{-\alpha-1, K}} \mid w \in V_K \right\} \quad (8.5)$$

with $V_K = \dot{H}_1(K) \cap H_2(K) \cap \{w \mid w = w(r)\}$. Now with lemma 6, (i) we get

$$\lambda(\Omega) \leq k \{ \rho^{-4} + \Lambda(K) \} \leq k \left\{ \rho^{-4} + \sup \left\{ \frac{\|\nabla w\|_{-\alpha+2, K}}{\|\Delta w\|_{-\alpha-1, K}} \mid w \in V_K \right\} \right\}. \quad (8.6)$$

Functions $w \in V_K$ have the representation ($w' = dw/dr$):

$$w' = r^{1-N} \int_0^r s^{N-1} \Delta w \, ds. \quad (8.7)$$

Schwarz's inequality gives

$$|w'|^2 \leq r^{2-2N} f(r) \int_0^r s^{N-1} \mu^{\alpha+1} |\Delta w|^2 \, ds \quad (8.8)$$

with

$$f(r) = \int_0^r s^{N-1} \mu^{-\alpha-1} \, ds \leq c \begin{cases} \rho^{-N-2} r^N & \text{for } r \leq \rho, \\ \rho^{-2} & \text{for } r \geq \rho, \end{cases} \quad (8.9)$$

because of $\alpha = N/2$.

Therefore

$$\begin{aligned} \|\nabla w\|_{-\alpha+2, K}^2 &= k \int_0^R r^{N-1} \mu^{\alpha-2} |w'|^2 \, dr \\ &\leq k \int_0^R r^{1-N} \mu^{\alpha-2} f(r) \, dr \int_0^r s^{N-1} \mu^{\alpha+1} |\Delta w|^2 \, ds \\ &= k \int_0^R s^{N-1} \mu^{\alpha+1} |\Delta w|^2 \, ds \int_s^R r^{1-N} \mu^{\alpha-2} f(r) \, dr \\ &\leq k \|\Delta w\|_{-\alpha-1, K}^2 \int_0^R r^{1-N} \mu^{\alpha-2} f(r) \, dr. \end{aligned} \quad (8.10)$$

The last integral is bounded by $c\rho^{-4}$. This completes the proof in case $N > 4$. For $N = 4$ without using lemma 6 we directly consider the supremum of $\|\nabla w\|^2 / \|\Delta w\|_{-3}^2$ and get the same result with the same arguments.

Proof of lemma 9: The proof follows the above lines. In the definition of λ , Λ we replace the indices of $\|\nabla w\|$ resp. $\|w\|$ by $-\alpha$ resp. $-\alpha + 1$. Then $-\alpha = N/2$ is negative. Up to formula (8.9) nothing is changed. But then

$$\|\nabla w\|_{-\alpha, K}^2 = \int_0^R r^{N-1} \mu^\alpha |w'|^2 dr \leq \|\Delta w\|_{-\alpha-1, K}^2 \int_0^R r^{1-N} \mu^\alpha f(r) dr \quad (8.11)$$

and the last integral is bounded by $c(1 + R^2 \rho^{-2}) \leq c' \rho^{-2}$.

The proof of lemma 9' is analogue to the preceding one and is omitted here.

There is an interesting remark to be added. In (8.1) resp. (8.2) the $(-\alpha + 2)$ -norm of the first derivatives resp. the $(-\alpha + 3)$ -norm of the function itself is compared with the $(-\alpha - 1)$ -norm of the second derivatives. Roughly speaking each differentiation in weighted norms may be considered as reducing the weight-power by one. Then $\|\nabla w\|_{-\alpha+2}$ and $\|w\|_{-\alpha+3}$ would be something like $\|\Delta w\|_{-\alpha+1}$. Since this is compared with $\|\Delta w\|_{-\alpha-1}$ the behavior λ , $\Lambda \approx \rho^{-4}$ is "understandable". Of course this "rule" is only valid for special α and has to be checked in each case. Just lemma 9 is an example that it may be violated.

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