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GRADIENT METHODS FOR THE CONSTRUCTION OF LJUSTERNIK-SCHNIRELMANN CRITICAL VALUES (*)

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Abstract. — In this paper gradient methods are proposed for the search for the Ljusternik-Schnirelmann critical values and the corresponding critical vectors of a functional g even with respect to the unit sphere. The paper describes a discretization of a continuous method proposed earlier by one of the authors.

Résumé. — Cet article propose les méthodes du gradient pour trouver les valeurs critiques et les vecteurs critiques correspondant de Ljusternik-Schnirelmann de la fonctionnelle g paire par rapport à la sphère unité. Le papier représente la discrétisation d'une méthode continue proposée par un des auteurs.

1. INTRODUCTION

Existence theorems for nonlinear eigenvalue problems in the form

$$\mu f'(x) - g'(x) = 0,$$

where f and g are functionals on a Hilbert space H , and $f'(x)$ and $g'(x)$ are the corresponding gradients, are considered in many papers (for an extensive list of references see S. Fučík, J. Nečas, J. Souček and V. Souček [2]). These existence theorems are based on the existence of a critical vector of $g(x)$ with respect to the manifold $M_r(f) = \{x \in H; f(x) = r\}$. Under suitable conditions it is proved that there exist at least one eigenvector, or an infinite number of eigenvectors, on the manifold $M_r(f)$.

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Iteration methods for the construction of an eigenvector and the corresponding eigenvalue are considered by M. A. Altman [1], J. Schröder [6], and W. Petry [5] where the Newton method, or the gradient method, is applied.

For all the Ljusternik-Schnirelmann critical values and critical vectors, a numerical approach was proposed in the paper by J. Nečas [4].

For the construction of the first Ljusternik-Schnirelmann critical value and the corresponding eigenvector and eigenvalue, the secant modulus method is used in the paper by the authors [3].

In this paper we shall consider, for the sake of simplicity, the eigenvalue problem

$$\mu x - g'(x) = 0,$$

in a Hilbert space H , where $g'(x)$ is the gradient of an even functional $g(x)$. For finding all the Ljusternik-Schnirelmann values of the functional $g(x)$ with respect to the sphere S , we shall construct some modifications of the method of steepest descent.

2. ITERATIVE CONSTRUCTION OF THE FIRST LJUSTERNIK-SCHNIRELMANN CRITICAL VALUE

Let H be a real Hilbert space with the inner product (\cdot, \cdot) and norm $\|\cdot\|$. Further we set $S = \{x \in H; \|x\| = 1\}$. Let g be an even functional (nonquadratic, generally) on H possessing the Fréchet differential $g'(x)$ at each $x \in H$. Let $g'(x)$ be strongly continuous on H , i. e., for each sequence $\{x_n\}_{n=1}^{\infty} \subset H$ weakly converging to $x_0 \in H$, the sequence $\{g'(x_n)\}_{n=1}^{\infty}$ converges to $g'(x_0)$.

Let M be a positive number. Suppose that for each $x, y \in H$, the following conditions are fulfilled:

$$(g'(x+h) - g'(x), h) \leq M \|h\|^2, \quad (2.1)$$

$$(g'(x+h) - g'(x), h) > 0 \quad \text{for } h \neq 0, \quad (2.2)$$

$$g(0) = 0, \quad (2.3)$$

$$g'(0) = 0. \quad (2.4)$$

THEOREM 2.1: Let the above assumptions be fulfilled. Let x_1 be an arbitrary initial approximation from S . If the sequence $\{x_n\}_{n=1}^\infty \subset S$ is defined by

$$x_{n+1} = \frac{x_n + (1/2 M) g'(x_n)}{\|x_n + (1/2 M) g'(x_n)\|}, \tag{2.5}$$

then each subsequence $\{x_{n_k}\}_{k=1}^\infty$ contains a subsequence $\{x_{n_{k_j}}\}_{j=1}^\infty$ such that

$$\begin{aligned} \lim_{j \rightarrow \infty} \|x_{n_{k_j}} - x_0\| &= 0, \\ \lim_{j \rightarrow \infty} \left\{ \left[\left\| x_{n_{k_j}} + \frac{1}{2 M} g'(x_{n_{k_j}}) \right\| - 1 \right] 2 M - \mu \right\} &= 0, \\ \lim_{n \rightarrow \infty} (g(x_n) - g(x_0)) &= 0, \end{aligned}$$

and

$$\mu x_0 - g'(x_0) = 0. \tag{2.6}$$

Proof: From (2.2) we get

$$\left\| x_n + \frac{1}{2 M} g'(x_n) \right\| > 1 \tag{2.7}$$

for an arbitrary integer n .

By a simple calculation we obtain

$$\begin{aligned} g(x_{n+1}) - g(x_n) &= (g'(x_n + \tau(x_{n+1} - x_n)), x_{n+1} - x_n) \\ &\geq (g'(x_n), x_{n+1} - x_n) - M \|x_{n+1} - x_n\|^2 \\ &= 2 M \left\{ \left\| x_n + \frac{1}{2 M} g'(x_n) \right\| \cdot \|x_n\| \right. \\ &\quad \left. - \left(x_n + \frac{1}{2 M} g'(x_n), x_n \right) \right\} \geq 0, \end{aligned}$$

in virtue of (2.1), (2.2), and (2.5) and thus

$$\left. \begin{aligned} g(x_n) &\leq g(x_{n+1}), \\ (g'(x_n), x_{n+1} - x_n) &\geq M \|x_{n+1} - x_n\|^2. \end{aligned} \right\} \tag{2.8}$$

From the last inequality,

$$M \|x_{n+1} - x_n\|^2 \leq (g'(x_n), x_{n+1} - x_n) < g(x_{n+1}) - g(x_n) \tag{2.9}$$

follows with respect to (2.2). The functional g is bounded and we thus obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0, \quad (2.10)$$

from (2.8) and (2.9).

The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded; thus there exists a subsequence (in the sequel we shall denote each subsequence as the original) converging weakly to some $x_0 \in H$. Therefore $g'(x_n)$ and $g(x_n)$ converge to $g'(x_0)$ and $g(x_0)$, respectively.

From (2.2), (2.3), (2.4), and (2.8) we get

$$0 < g(x_n) \leq g(x_0).$$

In virtue of (2.3) then

$$x_0 \neq 0. \quad (2.11)$$

There exists a subsequence of

$$\left\{ \left\| x_n + \frac{1}{2M} g'(x_n) \right\| \right\}_{n=1}^{\infty},$$

such that

$$\lim_{n \rightarrow \infty} \left\| x_n + \frac{1}{2M} g'(x_n) \right\| = c_0 \geq 1, \quad (2.12)$$

with respect to (2.7).

Suppose that $c_0 = 1$. Then

$$\lim_{n \rightarrow \infty} \left\| x_n + \frac{1}{2M} g'(x_n) \right\|^2 = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{M} (g'(x_n), x_n) + \frac{1}{4M^2} \|g'(x_n)\|^2 \right] = 1,$$

i. e.

$$\frac{1}{M} (g'(x_0), x_0) + \frac{1}{4M^2} \|g'(x_0)\|^2 = 0,$$

which contradicts (2.2) in virtue of (2.11). Thus

$$c_0 > 1. \quad (2.13)$$

From (2.5) we have

$$x_n = \frac{1}{\|x_n + (1/2 M)g'(x_n)\| - 1} \times \left[\frac{1}{2 M} g'(x_n) - \left\| x_n + \frac{1}{2 M} g'(x_n) \right\| (x_{n+1} - x_n) \right],$$

and thus in virtue of (2.10), (2.12), and (2.13) the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x_0 .

THEOREM 2.2: *Let the assumptions of theorem 2.1 be satisfied. Moreover let*

$$\sup_{x \in S} \|g'(x)\|^2 \leq 2 M^2. \quad (2.14)$$

Let x_1 be an arbitrary initial approximation from S . If the sequence $\{x_n\}_{n=1}^{\infty} \subset S$ is defined by

$$x_{n+1} = \lambda_n x_n + \frac{1}{2 M} g'(x_n), \quad (2.15)$$

with

$$\lambda_n = -\frac{1}{2 M} [(g'(x_n), x_n) - \sqrt{(g'(x_n), x_n)^2 - \|g'(x_n)\|^2 + 4 M^2}], \quad (2.16)$$

then each subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ contains a subsequence $\{x_{n_{k_j}}\}_{j=1}^{\infty}$ such that

$$\begin{aligned} \lim_{j \rightarrow \infty} \|x_{n_{k_j}} - x_0\| &= 0, \\ \lim_{j \rightarrow \infty} (\|g'(x_{n_{k_j}})\| - \mu) &= 0, \\ \lim_{n \rightarrow \infty} (g(x_n) - g(x_0)) &= 0, \end{aligned}$$

and

$$\mu x_0 - g'(x_0) = 0. \quad (2.6)$$

Proof: It is easy to see that

$$\left\| \lambda_n x_n + \frac{1}{2 M} g'(x_n) \right\|^2 = 1,$$

for $x_n \in S$ with respect to (2.14), (2.15), and (2.16), i. e. $x_{n+1} \in S$.

From (2.2), (2.4), and (2.14), $\lambda_n > 0$ follows. Analogously as in theorem 2.1 we get

$$\begin{aligned} g(x_{n+1}) - g(x_n) &\geq (g'(x_n), x_{n+1} - x_n) - M \|x_{n+1} - x_n\|^2 \\ &= -M(\lambda_n - 1)^2 + \frac{1}{4M} \|g'(x_n)\|^2, \end{aligned}$$

from (2.1), (2.2), and (2.15). We wish to show that

$$g(x_{n+1}) \geq g(x_n).$$

It follows from (2.2) and (2.4) that

$$2M(\lambda_n - 1) < 0;$$

thus according to the above inequality it is sufficient to show that

$$\|g'(x_n)\| \geq 2M(1 - \lambda_n).$$

This inequality is equivalent to

$$[(g'(x_n), x_n) + 2M - \|g'(x_n)\|]^2 \leq (g'(x_n), x_n)^2 - \|g'(x_n)\|^2 + 4M^2,$$

in virtue of (2.2), (2.4), (2.14), and (2.16). According to (2.14),

$$\begin{aligned} (g'(x_n), x_n)^2 + 4M^2 + \|g'(x_n)\|^2 + 4M(g'(x_n), x_n) \\ - 2(g'(x_n), x_n)\|g'(x_n)\| - 4M\|g'(x_n)\| - (g'(x_n), x_n)^2 \\ + \|g'(x_n)\|^2 - 4M^2 \\ = 2[\|g'(x_n)\| - (g'(x_n), x_n)][\|g'(x_n)\| - 2M] \leq 0, \end{aligned}$$

and thus

$$g(x_{n+1}) - g(x_n) \geq 0.$$

The rest of the proof now follows as in theorem 2.1.

COROLLARY 2.1: *If, in addition to the assumptions of theorems 2.1 or 2.2, we assume that (2.6) has only isolated solutions on S , then the whole sequence $\{x_n\}_{n=1}^{\infty}$ converges to an element x_0 satisfying (2.6), moreover the whole sequences*

$$\left\{ \left[\left\| x_n + \frac{1}{2M} g'(x_n) \right\| - 1 \right] 2M \right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{ \|g'(x_n)\| \right\}_{n=1}^{\infty},$$

respectively, converge to a number μ satisfying (2.6).

Proof: The assertion follows analogously as in the paper [3] by the authors.

COROLLARY 2.2: *In addition to the assumptions of theorems 2.1 or 2.2, we assume that γ_1 is the first critical value of the functional g with respect to the sphere S . Furthermore, let there exist a constant $\varepsilon > 0$ such that there is no critical value in the interval $(\gamma_1 - \varepsilon, \gamma_1)$. Let $x_1 \in H$, $g(x_1) > \gamma_1 - \varepsilon$.*

Then for each limit point x_0 of the sequence $\{x_n\}_{n=1}^\infty$ defined by (2.5) or (2.15), respectively, we have

$$g(x_0) = \text{Max}_{x \in S} g(x) = \gamma_1.$$

Proof: The assertion follows from theorems 2.1, or 2.2, respectively.

3. ITERATIVE CONSTRUCTION OF THE LJUSTERNIK-SCHNIRELMANN CRITICAL VALUES

For the convenience of the reader we shall briefly recall principal definitions and results concerning the Ljusternik-Schnirelmann theory in a Hilbert space.

Let K be a symmetric closed set in H . We say that $\text{ord } K = 0$ if K is empty; that $\text{ord } K = 1$ if $K = K_1 \cup K_2$, where the K_i are closed subsets of K and neither K_1 nor K_2 contains antipodal points; that $\text{ord } K = n$ if $K = \bigcup_{i=1}^{n+1} K_i$, where the K_i are closed subsets of K not containing antipodal points and n is the least possible number; and that $\text{ord } K = \infty$ if no such n exists.

Let $V_n = \{K; K \subset S \text{ is a symmetric compact set and } \text{ord } K \geq n\}$. Let

$$\gamma_k = \sup_{K \in V_k} \min_{x \in K} g(x).$$

The fundamental theorem of Ljusternik and Schnirelmann is the following:

THEOREM 3.1: *Under the assumptions of theorem 2.1 there exist $x_k \in S$, $k = 1, 2, \dots$ such that*

$$\begin{aligned} g'(x_k) - \mu_k x_k &= 0, \\ g(x_k) &= \gamma_k, \quad \gamma_k \searrow 0, \quad x_k \rightarrow 0 \text{ (weakly)}. \end{aligned}$$

The proof, which is in a very easy version given in the paper [4] by J. Nečas, is based on the Ljusternik-Schnirelmann principle of critical values which, roughly speaking, means that for every γ_k , there exists a saddle point x_k such that $\gamma_k = g(x_k)$ and

$$g'(x_k) - \mu_k x_k = 0.$$

The proof of the Ljusternik-Schnirelmann principle can be done as in paper [4] by deformations of sets of prescribed order along the trajectories of the solutions of differential equations on S ,

$$\dot{x} = g'(x) - x(x, g'(x)), \quad x(0) = x_0 \in S,$$

(for this equation, see also M. M. Vajnberg [7], theorem 14.1). For details, see e. g. S. Fučík, J. Nečas, J. Souček and V. Souček [2].

Let the assumptions of theorem 2.1 hold for a functional g . Let γ_1 and γ_2 be the first and second Ljusternik-Schnirelmann critical values of the functional g with respect to the sphere S , $\gamma_1 > \gamma_2$. Furthermore, let there exist a constant $\varepsilon > 0$ such that there is no critical value in the interval $(\gamma_2 - \varepsilon, \gamma_2)$. Let K_1 be a compact symmetric subset of S , $\text{ord } K_1 \geq 2$ (e. g. $K_1 = L \cap S$, L is a subspace of H , and $\dim L = 2$),

$$\gamma_2 - \varepsilon < \underset{x \in K_1}{\text{Min}} g(x) < \gamma_2. \quad (3.1)$$

For $x \in K_1$, put

$$x_{n+1}(x) = \frac{x_n(x) + (1/2M) g'(x_n(x))}{\|x_n(x) + (1/2M) g'(x_n(x))\|}, \quad (3.2)$$

where $x_1(x) = x$.

Let $x_n^{(0)}$ be a vector from K_1 such that

$$\underset{x \in K_1}{\text{Min}} g(x_n(x)) = g(x_n(x_n^{(0)})) \quad (3.3)$$

for an arbitrary integer n .

THEOREM 3.1: *Let the above assumptions be fulfilled. Then the following assertions hold:*

$$(i) \quad \lim_{n \rightarrow \infty} g(x_n(x_n^{(0)})) = \gamma_2;$$

(ii) there exists $x^{(0)} \in K_1$ such that

$$\lim_{n \rightarrow \infty} g(x_n(x^{(0)})) = \gamma_2;$$

(iii) each subsequence $\{x_{n_k}^{(0)}\}_{k=1}^\infty$ contains a subsequence $\{x_{n_{k_j}}^{(0)}\}_{j=1}^\infty$ such that

$$\lim_{j \rightarrow \infty} \|x_{n_{k_j}}^{(0)} - x^{(0)}\| = 0,$$

and $x^{(0)}$ satisfies (ii);

(iv) for each $x^{(0)}$ satisfying (ii), each subsequence $\{x_{n_k}(x^{(0)})\}_{k=1}^\infty$ contains a subsequence $\{x_{n_{k_j}}(x^{(0)})\}_{j=1}^\infty$ such that

$$\lim_{j \rightarrow \infty} \|x_{n_{k_j}}(x^{(0)}) - x_0\| = 0,$$

$$\lim_{j \rightarrow \infty} \left[\left(\left\| x_{n_{k_j}}(x^{(0)}) + \frac{1}{2M} g'(x_{n_{k_j}}(x^{(0)})) \right\| - 1 \right) 2M - \mu \right] = 0,$$

and

$$\mu x_0 - g'(x_0) = 0. \tag{2.6}$$

Proof: From theorem 2.1 we have

$$g(x_n(x)) \leq g(x_{n+1}(x)), \tag{3.4}$$

for each $x \in K_1$.

Put

$$\varphi(x) = \frac{x + (1/2M)g'(x)}{\|x + (1/2M)g'(x)\|},$$

for $x \in S$.

Then φ is an odd continuous operator from S into S and thus

$$\text{ord } K_{n+1} = \text{ord } \varphi(K_n) \geq \dots \geq \text{ord } K_1 = 2,$$

where

$$K_{n+1} = \left\{ x_{n+1} \in H; x_{n+1} = \frac{x_n + (1/2M)g'(x_n)}{\|x_n + (1/2M)g'(x_n)\|}, x_n \in K_n \right\}.$$

From this we immediately get

$$\lim_{n \rightarrow \infty} \operatorname{Min}_{x \in K_1} g(x_n(x)) \leq \gamma_2, \quad (3.5)$$

in virtue of the definition of γ_2 . This limit exists because of (3.4).

Put

$$\lim_{n \rightarrow \infty} \operatorname{Min}_{x \in K_1} g(x_n(x)) = \kappa. \quad (3.6)$$

Furthermore, there exist a subsequence of $\{x_n^{(0)}\}_{n=1}^{\infty}$ (we use the same notation for it as for the previous one) and $x^{(0)} \in K_1$ such that

$$\lim_{n \rightarrow \infty} \|x_n^{(0)} - x^{(0)}\| = 0 \quad (3.7)$$

and, with respect to (3.3) and (3.6),

$$\lim_{n \rightarrow \infty} g(x_n(x_n^{(0)})) = \kappa. \quad (3.8)$$

According to theorem 2.1,

$$\lim_{n \rightarrow \infty} g(x_n(x^{(0)})),$$

is a critical value of the functional g with respect to S , thus in virtue of (3.1), (3.4) and the assumption that there is no critical value in the interval $(\gamma_2 - \varepsilon, \gamma_2)$, we obtain

$$\lim_{n \rightarrow \infty} g(x_n(x^{(0)})) \geq \gamma_2.$$

Hence with respect to (3.7) there exist integers n_0 and n_1 such that

$$g(x_{n_0}(x_{n_0}^{(0)})) \geq \gamma_2 - \eta, \quad (3.9)$$

for each $\eta > 0$ and each $n \geq n_1$.

According to (3.4) and (3.9) this implies that there exists an integer $n_1 \geq n_0$ such that

$$g(x_n(x_n^{(0)})) \geq g(x_{n_0}(x_{n_0}^{(0)})) \geq \gamma_2 - \eta, \quad (3.10)$$

for each integer $n \geq n_1$.

From (3.8) and (3.10) we obtain

$$\kappa = \lim_{n \rightarrow \infty} g(x_n(x_n^{(0)})) \geq \gamma_2 - \eta$$

for each $\eta > 0$ and thus

$$\lim_{n \rightarrow \infty} g(x_n(x_n^{(0)})) = \gamma_2$$

in virtue of (3.3) and (3.5).

The rest of the proof now follows as in theorem 2.1.

COROLLARY 3.1: *Let the assumptions of theorem 2.1 hold for a functional g . Let*

$$\gamma_1 \geq \dots \geq \gamma_k > \gamma_{k+1} = \dots = \gamma_{k+l} > \gamma_{k+l+1},$$

be the Ljusternik-Schnirelmann critical values of the functional g with respect to the sphere S .

Let there exist a constant $\varepsilon > 0$ such that there is no critical value in the interval $(\gamma_{k+l} - \varepsilon, \gamma_{k+l})$. Let K_1 be a compact symmetric subset of S ,

$$\text{ord } K_1 \geq k+1,$$

$$\gamma_{k+l} - \varepsilon < \underset{x \in K_1}{\text{Min}} g(x) < \gamma_{k+l}.$$

For $x \in K$, let the sequences $\{x_n(x)\}_{n=1}^{\infty}$ and $\{x_n^{(0)}\}_{n=1}^{\infty}$ be defined by (3.2) and (3.3), respectively.

Then

$$\lim_{n \rightarrow \infty} g(x_n(x_n^{(0)})) = \gamma_{k+l},$$

and there exists $x^{(0)} \in K_1$ such that

$$\lim_{n \rightarrow \infty} g(x_n(x^{(0)})) = \gamma_{k+l}.$$

Moreover, the assertions (iii) and (iv) of theorem 3.1 hold.

Proof: The proof is analogous to the proof of theorem 3.1.

COROLLARY 3.2: *If, in addition to the assumptions of theorem 3.1 or corollary 3.1 we assume that (2.6) has only isolated solutions on S , then the whole sequence $\{x_n(x^{(0)})\}_{n=1}^{\infty}$ converges to a vector $x^{(0)}$ satisfying (ii) and, moreover,*

$$\lim_{n \rightarrow \infty} \left[\left(\left\| x_n(x^{(0)}) + \frac{1}{2M} g'(x_n(x^{(0)})) \right\| - 1 \right) 2M - \mu \right] = 0,$$

where μ is a number satisfying (2.6).

REFERENCES

- 1 M A ALTMAN, *A Generalized Gradient Method of Minimizing a Functional on a Nonlinear Surface with Application to Nonlinear Programming*, *Mathematica (Cluj)*, Vol 11, No 34 1969 pp 13-27
- 2 S FUČIK, J NEČAS J SOUCEK and V SOUČEK, *Spectral Analysis of Nonlinear Operators* In *Lecture Notes in Mathematics* Springer-Verlag, 1973
- 3 A KRATOCHVIL and J NEČAS *Secant Modulus Method for the Construction of a Solution of Nonlinear Eigenvalue Problems*, *Bollessino U M I* , Vol 16-B, No 5, 1979, pp 694-710
- 4 J NEČAS, *An Approximate Method for Finding Critical Points of Even Functionals* (in Russian), *Trudy Matem Inst A N S S S R* , Vol 134, 1975, pp 235-239
- 5 W PETRY, *Iterative Construction of a Solution of Nonlinear Eigenvalue Problems*, *Mathematica (Cluj)*, Vol 14, No 37, 2, 1972, pp 317-337
- 6 J SCHRODER, *Storungsrechnung bei Eigenwertaufgaben und Verzweigungsaufgaben*, *Arch Rat Mech Anal* Vol 1, 1957/1958, pp 436-468
- 7 M M VAJNBERG, *Variational Methods for the Study of Nonlinear Operators*, G I T T L , Moscow, 1956, English, transl , Holden-Day, San Francisco, Calif , 1964