Vladimiro Valerio

On the partitioned matrix \( O\begin{bmatrix} A & Q \end{bmatrix} \) and its associated system \( AX = T, A^*Y + QX = Z \)


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ON THE PARTITIONED MATRIX $\begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$
AND ITS ASSOCIATED SYSTEM $AX = T, A^* Y + QX = Z$ (*)

by Vladimiro Valerio (1) (**)

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Abstract — Inverses of the partitioned matrix $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$, where $Q$ is possibly nonnegative definite, and solutions of its associated system $AX = T, A^* Y + QX = Z$ are the object of this note. Some results in an earlier paper are extended. Finally, condition for inverting the square regular matrix $N$, when $Q$ is also singular, and a different construction of the inverse $N^{-1}$ are given using a particular $g$-inverse of $Q$.

Résumé — L’objet de cet article est l’étude des inverses de matrices partitionnées sous la forme $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$, où $Q$ peut être semi-définie positive, ainsi que l’étude des solutions du système associé $AX = T, A^* Y + QX = Z$. On généralise les résultats d’un article antérieur. Enfin, utilisant un $g$-inverse particulier de $Q$, on donne des conditions pour inverser la matrice carrée inversible $N$ quand $Q$ est singulière, ainsi qu’une construction différente de l’inverse $N^{-1}$.

LIST OF SYMBOLS

- $\alpha$ lower case greek alfa
- $\beta$ lower case greek beta
- $*$ star
- $\Rightarrow$ arrow
- $\oplus$ circle with plus inside

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(**) The author worked on the same subject when he was on a visiting appointment at the Delhi Campus of the Indian Statistical Institute (Sept 1977-Jan 1978)
(1) Istituto di Matematica, Facoltà di Architettura, Napoli, Italia

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1. INTRODUCTION

An increasing number of papers has been appeared in the last ten years on the generalized inverses of a partitioned matrix. One of the approaches depends on the Schur-complement \( M/A = D - CA^{-1}B \) defined for a square regular matrix \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where \( A \) is also regular. Its generalization to rectangular and singular matrices under any partition has also been studied in [6, 7, 11, 14] and [15]. Partitioned matrices are given in [3] and [10] which give conditions on the rank and the range of the partition in order to define their generalized inverses; [8] considers the Moore-Penrose inverse of \( M \). Some particular aspects, useful for correcting least squares estimates, are found in [9, 10, 12, 16] and [18], where the matrix is in the form \((A : a)\) and \( a \) is a vector. In [5] we have partitioned matrices like \( A = [U, V] \) in which conditions for the existence of the Moore-Penrose inverse are given. A more detailed discussion on the latter is in [2].

In the present note we consider the partitioned matrix \( N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix} \) where \( Q \) is ndd, if it is not otherwise stated, and the associated system \( AX = T, A^*Y + QX = Z \). A matrix partitioned like \( N \) could be found in [19] and [20].

The above system arises in many problems of applied Mechanics, where \( Q \) is also symmetric and pd, and in calculating space structures (trusses) or continuous structures finding a discrete structure which matches the continuous one. We refer to an earlier paper [21] and give additional results. Theorem 1 gives a particular set of solution to the considered system if we observe that \( X \) and \( Y \) are possibly two different kind of unknowns [22]. Finally, conditions for inverting the square regular matrix \( N \) when \( Q \) is singular and a different construction of the regular inverse \( N^{-1} \) are given using a particular \( g \)-inverse of \( Q \).

2. DEFINITIONS AND NOTATIONS

We denote by \( C^{m\times n} \) the vector space of all \( m \times n \) matrices defined over the complex number field. For a given matrix \( A r(A) \) is its rank, \( R(A) \) is the range or the space spanned by the columns of \( A \), \( A^* \) is the conjugate transpose of \( A \). \( A^- \) is any \( g \)-inverse of \( A \) satisfying \( AA^-A = A \) and \( A^- \) is a reflexive \( g \)-inverse satisfying also \( A^-AA^- = A^- \). In general we use the notations of [19].

Let \( A \in C^{m\times n} \) and \( X \in C^{n\times p} \), we consider the system

\[
\begin{pmatrix}
AX = T \\
A^*Y + QX = Z
\end{pmatrix}
\]

(1)

R.A.I.R.O. Analyse numérique/Numerical Analysis
We have $Q \in \mathbb{C}^{n \times m}$, $Y \in \mathbb{C}^{m \times p}$, $T \in \mathbb{C}^{m \times p}$ and $Z \in \mathbb{C}^{n \times p}$. System (1) can be constrained in the form $NU = W$, where $N \in \mathbb{C}^{n+m \times n+m}$, $U \in \mathbb{C}^{n+m \times m}$ and $W \in \mathbb{C}^{n+m \times p}$.

In particular
\[
N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}, \quad U = \begin{pmatrix} Y \\ X \end{pmatrix}, \quad W = \begin{pmatrix} T \\ Z \end{pmatrix}.
\]

3. MAIN RESULTS

We use the following lemmas.

**Lemma 1**: A necessary and sufficient condition that $AX = T$ is consistent is that $AA^* T = T$.

**Lemma 2**: Let $G = \begin{bmatrix} -H^* & H^- AK^- \\ K^* A^* H^- & K^- - K^- A^* H^- AK^- \end{bmatrix}$ be a partitioned matrix in which $K = Q + A^* A$ and $H = AK^* A^*$. Then:

(a) $G$ is a $g$-inverse of $N$;

(b) if $R(A^*) \subset R(Q)$, $G$ is a $g$-inverse of $N$ replacing the expression of $K$ by $Q$.

A proof of lemma 1 and lemma 2 is in [19]. But for lemma 2(b) we can give the following alternative proof. The generalized Schur-complement (1) of $Q$ reduces to $N/Q = AQ^- A^*$, thus according to [14] and [15], $G$ is a $g$-inverse of $N$ iff the rank is additive on the Schur-complement; that's true if

\[
R(A^*) \subset R(Q)
\]

in view of [14, corollary 19.1].

**Theorem 1**: If system (1) is consistent $R(Z - QA^- T) \subset R(A^*)$ is n.s. for $\forall X/AX = T \Leftrightarrow X \in U$.

**Proof**: If $AX = T$ and $X \in U$, there exists a solution of $A^* Y + QA^- T = Z$ for any $Z$ and $QA^- T$. Thus in view of lemma 1 : $R(Z - QA^- T) \subset R(A^*)$, and vice versa. ■

By straightforward multiplication we obtain :

**Corollary 1**: If $K^*$ and $H^*$ (respectively $Q^*$ and $H^*$) in the expression for $G$ in lemma 2(a) (lemma 2(b)) are replaced by $K^*_r$ and $H^*_r$ ($Q^*_r$ and $H^*_r$), $G$ is a reflexive $g$-inverse of $N$ no further conditions being required.

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(1) For the Schur-complement and other references see [11].

vol. 15, no 2, 1981
Lemma 3: The set of all solutions of system (1) is given by

\[ Y = H^{-1} AK^{-1} Z - H^{-1} T, \]
\[ X = K^{-1} A^* H^{-1} T + (I - K^{-1} A^* H^{-1} A) K^{-1} Z; \]

where \( H \) and \( K \) are defined as in lemma 2.

As far as the uniqueness of solution of system (1) is concerned we state the following.

Lemma 4: System (1) has a unique solution only if \( r(A) = m \) and \( r(Q) \geq n - m \).

Theorem 2: (a) A necessary and sufficient condition that system (1) has a unique solution is that:

(i) \( r(A) = m \) and \( R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = C^n \),

or what is the same

(ii) \( r(A) = m \), \( r(Q) \geq n - m \) and \( A \) and \( Q \) are virtually disjoint, or

(iii) \( K = (Q + A^* A) \) has full rank.

(b) \( r(A) = m \) and \( r(Q) = n \) are n.s. that system (1) has a unique solution iff \( R(A^*) \subset R(Q) \).

Proof of (a): The matrix \( N \) is not singular, so its rows are linearly independent hence \( r(A) = m \) and \( R(A^*) \oplus R(Q) = C^n \). The same for its columns, thus \( R(A^*) \oplus R(Q) = C^n \). This condition is obviously equivalent to (ii). (iii) follows from Lemma 3, and if (iii) holds then (i) holds.

Proof of (b): The matrix \( G \) as defined in lemma 2(b) is the regular inverse of \( N \) with \( R(A^*) \subset R(Q) \), hence \( H^{-1} \) and \( Q^{-1} \) exist, so that \( r(A) = m \) and \( r(Q) = n \). For the only if part we consider that if \( r(A) = m \) and \( r(Q) = n \) then \( R(A^*) \subset R(Q) \) since \( m \leq n \) and both \( A \) and \( Q \) have full rank.

An alternative proof of Theorem 2(b) is in [7, theorem 1].

We point out that Theorem 2(a) provides a general statement for the uniqueness of solution of system (1). A particular case of (a), when \( r(Q) = n - m \) is stated in [19, p. 19] when the matrix is \( \begin{pmatrix} A & U \\ V^* & O \end{pmatrix} \), and \( U \) and \( V \) have the same dimension. Theorem 2 emphasizes that the inverse of a matrix partitioned like in \( N \) (2) can be constructed even if \( Q \) is not of full rank (for \( Q \) with full rank see [13, p. 107]), but only \( r(Q) \geq n - m \). Theorem 2 holds for any \( Q \).

On the other hand, it is natural to expect some g-inverse of \( Q \) gets involved in computing the regular inverse of \( N \) whenever \( Q \) is singular just as the regular inverse plays when \( Q \) is not singular. The following lemma clears up this

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(2) This result can be extended to the general form \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \).

R.A.I.R.O. Analyse numérique/Numerical Analysis
apparent contradiction by showing how a particular $g$-inverse of $Q$ arises from the formula of lemma 2 under the conditions of theorem 2(a).

**Lemma 5:** Let $A \in C^{m \times n}$ and $Q \in C^{n \times n}$, if $r(A) = m$, $(Q + A^* A)^{-1}$ exists and is one choice of $Q^{-}$ with maximum rank iff $A$ and $Q$ are virtually disjoint, $R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = C^n$.

We do not prove this lemma since it follows easily from [19, theorem 2.7.1],

**Lemma 6(a):** Under the conditions of theorem 2(a)

$$G = \begin{bmatrix} 0 & A_{Q0}^* \\ A_{Q0}^{-} & \tilde{Q}^{-} - A_{Q0}^{-} A\tilde{Q}^{-} \end{bmatrix}$$

is the regular inverse of $N$, where $A_{Q0}^{-} = \tilde{Q}^{-} A^* H^{-}$ is a $g$-inverse of $A$, $H = A\tilde{Q}^{-} A^*$ and $\tilde{Q}^{-}$ is a selected $g$-inverse of $Q$ with maximum rank as defined in lemma 5.

The solution of system (1) is

$$Y = A_{Q0}^{-} Z, \quad X = A_{Q0}^{-} T + (I - A_{Q0}^{-} A) \tilde{Q}^{-} Z.$$ 

(b) If theorem 2(b) holds then

$$G = \begin{bmatrix} -H^{-1} & A_{Q0}^{-} \\ A_{Q0}^{-1} & Q^{-1} - A_{Q0}^{-1} A Q^{-1} \end{bmatrix}$$

is the regular inverse of $N$, where $A_{Q0}^{-1} = Q^{-1} A^* H^{-1}$ is the $g$-inverse of $A$ as defined by [4] and $H$ is defined in lemma 2(b). The solution of system (1) is

$$Y = A_{Q0}^{-1} Z - H^{-1} T, \quad X = A_{Q0}^{-1} T + (I - A_{Q0}^{-1} A) Q^{-1} Z.$$ 

Examples

$$N = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad r(A) = 2, \quad r(Q) = 1.$$
It easy to verify that \( R(A^*) \not\subset R(Q) \) and
\[
R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = R^3,
\]
thus \( A \) and \( Q \) are disjoint. The conditions of theorem 2(a) are fulfilled and \( G \) as defined in lemma 6(a) is the regular inverse of \( N \). Thus \( \tilde{Q}^{-1} = (Q + A^* A)^{-1} \),
\[
H = AQ^{-1} A^*, A_{Q0} = Q^{-1} A^* H^{-1}
\]
and by easy computation
\[
N^{-1} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/2
\end{bmatrix},
\]
\[
N = \begin{bmatrix}
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}; \quad A = (1 \ 0); \quad Q = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix};
\]
\[
r(A) = 1, \quad r(Q) = 2.
\]
In this case \( R(A^*) \subset R(Q) \) and theorem 2(b) holds. Then by lemma 6(b) \( H = AQ^{-1} A^* \) and
\[
N^{-1} = \begin{bmatrix}
-1 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}.
\]

4. OTHER INVERSES OF \( N \)

As stated in lemma 4 system (1) does not have a unique solution whenever \( A \in C^{m\times n} \) and \( m > n \). However we can find other particular solutions when system (1) is possibly inconsistent. A set of equivalent conditions is stated in [18] in order to obtain a \( g \)-inverse minimum norm, least squares or both them for the system \( AX = T \). We denote these by \( A_m^-, A_1^-, A^+ \) : the last one is the Moore-Penrose inverse of \( A \). Thus we have the following :

**Theorem 3** : Let \( G \) be a partitioned matrix as defined in lemma 2(b),
(a) \( G \) is a minimum norm inverse of \( N \) if \( (I - H^* H) A = 0 \), \( Q^{-1} \) is replaced by \( Q_m^- \) and \( R(A^*) \subset R(Q^*) \).
(b) \( G \) is a least squares inverse of \( N \) if \( Q^{-1} \) is replaced by \( Q_1^- \) and
\[
A^*(I - HH^{-1}) = 0.
\]
(c) \( G \) is the Moore-Penrose inverse of \( N \) if \( Q^- \) and \( H^- \) are replaced by \( Q^+ \) and \( H^+ \) and \( R(A^*) \subseteq R(Q^+) \), \( R(AQ^+) \subseteq R(H) \) and \( R((Q^+ A^*)^*) \subseteq R(H^*) \).

Remark If \( Q \) is Hermitian, then \( G \) is the Moore-Penrose inverse of \( N \) if \( Q^- \) and \( H^- \) are replaced by \( Q^+ \) and \( H^+ \) and \( R(AQ^+) \subseteq R(H) \) only.

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