RAIRO
ANALYSE NUMÉRIQUE

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RAIRO – Analyse numérique, tome 15, n^o 2 (1981), p. 177-184.
<http://www.numdam.org/item?id=M2AN_1981__15_2_177_0>

R A I R O Analyse numerique/Numerical Analysis
(vol 15, n° 2, 1981, p 177 a 184)

ON THE PARTITIONED MATRIX $\begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$ AND ITS ASSOCIATED SYSTEM $AX = T, A^* Y + QX = Z$ (*)

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Communiqué par P G CIARLET

Abstract — *Inverses of the partitioned matrix* $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$, *where* Q *is possibly nonnegative definite, and solutions of its associated system* $AX = T$, $A^* Y + QX = Z$ *are the object of this note Some results in an earlier paper are extended Finally, condition for inverting the square regular matrix* N, *when* Q *is also singular, and a different construction of the inverse* N^{-1} *are given using a particular g-inverse of* Q.

Résumé — *L'objet de cet article est l'étude des inverses de matrices partitionnées sous la forme* $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$, *où* Q *peut être semi-définie positive, ainsi que l'étude des solutions du système associé* $AX = T$, $A^* Y + QX = Z$ *On généralise les résultats d'un article antérieur Enfin, utilisant un g-inverse particulier de* Q, *on donne des conditions pour inverser la matrice carrée inversible* N *quand* Q *est singulière, ainsi qu'une construction différente de l'inverse* N^{-1}

LIST OF SYMBOLS

α lower case greek alfa
β lower case greek beta
$*$ star
$\Rightarrow$ arrow
$\oplus$ circle with plus inside

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1. INTRODUCTION

An increasing number of papers has been appeared in the last ten years on the generalized inverses of a partitioned matrix. One of the approaches depends on the Schur-complement $M/A = D - CA^{-1} B$ defined for a square regular matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A is also regular. Its generalization to rectangular and singular matrices under any partition has also been studied in [6, 7, 11, 14] and [15]. Partitioned matrices are given in [3] and [10] which give conditions on the rank and the range of the partition in order to define their generalized inverses; [8] considers the Moore-Penrose inverse of M. Some particular aspects, useful for correcting least squares estimates, are found in [9, 10, 12, 16] and [18], where the matrix is in the form $(A : a)$ and a is a vector. In [5] we have partitioned matrices like $A = [U, V]$ in which conditions for the existence of the Moore-Penrose inverse are given. A more detailed discussion on the latter is in [2].

In the present note we consider the partitioned matrix $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$ where Q is nnd, if it is not otherwise stated, and the associated system $AX = T$, $A^* Y + QX = Z$. A matrix partitioned like N could be found in [19] and [20].

The above system arises in many problems of applied Mechanics, where Q is also symmetric and *pd*, and in calculating space structures (trusses) or continuous structures finding a discrete structure which matches the continuous one. We refer to an earlier paper [21] and give additional results. Theorem 1 gives a particular set of solution to the considered system if we observe that X and Y are possibly two different kind of unknowns [22]. Finally, conditions for inverting the square regular matrix N when Q is singular and a different construction of the regular inverse N^{-1} are given using a particular g-inverse of Q.

2. DEFINITIONS AND NOTATIONS

We denote by $C^{m,n}$ the vector space of all $m \times n$ matrices defined over the complex number field. For a given matrix A $r(A)$ is its rank, $R(A)$ is the range or the space spanned by the columns of A, A^* is the conjugate transpose of A. A^- is any g-inverse of A satisfying $AA^- A = A$ and A_r is a reflexive g-inverse satisfying also $A^- AA^- = A^-$. In general we use the notations of [19].

Let $A \in C^{m,n}$ and $X \in C^{n,p}$, we consider the system

$$\begin{pmatrix} AX = T \\ A^* Y + QX = Z \end{pmatrix} \tag{1}$$

We have $Q \in C^{n,m}$, $Y \in C^{m,p}$, $T \in C^{m,p}$ and $Z \in C^{n,p}$. System (1) can be constrained in the form $NU = W$, where $N \in C^{n+m,n+m}$, $U \in C^{n+m,p}$ *and* $W \in C^{n+m,p}$. In particular

$$N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}, \quad U = \begin{pmatrix} Y \\ X \end{pmatrix}, \quad W = \begin{pmatrix} T \\ Z \end{pmatrix}.$$

3. MAIN RESULTS

We use the following lemmas.

LEMMA 1 : *A necessary and sufficient condition that $AX = T$ is consistent is that $AA^- T = T$.*

LEMMA 2 : *Let* $G = \begin{bmatrix} -H^- & H^- AK^- \\ K^- A^* H^- & K^- - K^- A^* H^- AK^- \end{bmatrix}$ *be a partitioned matrix in which $K = Q + A^* A$ and $H = AK^- A^*$. Then :*

(α) *G is a g-inverse of N ;*

(β) *if $R(A^*) \subset R(Q)$, G is a g-inverse of N replacing the expression of K by Q.*

A proof of lemma 1 and lemma 2 is in [19]. But for lemma 2(β) we can give the following alternative proof. The generalized Schur-complement ([1]) of Q reduces to $N/Q = AQ^- A^*$, thus according to [14] and [15], G is a g-inverse of N iff the rank is additive on the Schur-complement ; that's true if

$$R(A^*) \subset R(Q)$$

in view of [14, corollary 19.1].

THEOREM 1 : *If system* (1) *is consistent $R(Z - QA^- T) \subset R(A^*)$ is n.s. for $\forall X/AX = T \Leftrightarrow X \in U$.*

Proof : If $AX = T$ and $X \in U$, there exists a solution of $A^* Y + QA^- T = Z$ for any Z and $QA^- T$. Thus in view of lemma 1 : $R(Z - QA^- T) \subset R(A^*)$, and vice versa. ■

By straightforward multiplication we obtain :

COROLLARY 1 : *If K^- and H^-* (respectively Q^- and H^-) *in the expression for G in lemma* 2(α) (*lemma* 2(β)) *are replaced by K_r^- and H_r^- (Q_r^- and H_r^-), G is a reflexive g-inverse of N no further conditions being required.*

([1]) For the Schur-complement and other references see [11].

LEMMA 3 : *The set of all solutions of system* (1) *is given by*

$$Y = H^- AK^- Z - H^- T,$$

$$X = K^- A^* H^- T + (I - K^- A^* H^- A) K^- Z;$$

where H and K are defined as in lemma 2.

As far as the uniqueness of solution of system (1) is concerned we state the following.

LEMMA 4 : *System* (1) *has a unique solution only if* $r(A) = m$ *and* $r(Q) \geqslant n - m$.

THEOREM 2 : (*a*) *A necessary and sufficient condition that system* (1) *has a unique solution is that* : (i) $r(A) = m$ *and* $R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = C^n$, *or what is the same* (ii) $r(A) = m$, $r(Q) \geqslant n - m$ *and A and Q are virtually disjoint, or* (iii) $K = (Q + A^* A)$ *has full rank.*

(*b*) $r(A) = m$ *and* $r(Q) = n$ *are n.s. that system* (1) *has a unique solution iff* $R(A^*) \subset R(Q)$.

Proof of (*a*) : The matrix N is not singular, so its rows are linearly independent hence $r(A) = m$ and $R(A) \oplus R(Q^*) = C^n$. The same for its columns, thus $R(A^*) \oplus R(Q) = C^n$. This condition is obviously equivalent to (ii). (iii) follows from lemma 3, and if (iii) holds then (i) holds.

Proof of (*b*) : The matrix G as defined in lemma 2(β) is the regular inverse of N with $R(A^*) \subset R(Q)$, hence H^{-1} and Q^{-1} exist, so that $r(A) = m$ and $r(Q) = n$. For the only if part we consider that if $r(A) = m$ and $r(Q) = n$ then $R(A^*) \subset R(Q)$ since $m \leqslant n$ and both A and Q have full rank. ∎

An alternative proof of theorem 2(*b*) is in [7, theorem 1].

We point out that theorem 2(*a*) provides a general statement for the uniqueness of solution of system (1). A particular case of (*a*), when $r(Q) = n - m$ is stated in [19, p. 19] when the matrix is $\begin{pmatrix} A & U \\ V^* & O \end{pmatrix}$, and U and V have the same dimension. Theorem 2 emphasizes that the inverse of a matrix partitioned like in N (²) can be constructed even if Q is not of full rank (for Q with full rank see [13, p. 107]), but only $r(Q) \geqslant n - m$. Theorem 2 holds for any Q.

On the other hand, it is natural to expect some g-inverse of Q gets involved in computing the regular inverse of N whenever Q is singular just as the regular inverse plays when Q is not singular. The following lemma clears up this

(²) This result can be extended to the general form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

apparent contradiction by showing how a particular g-inverse of Q arises from the formula of lemma 2 under the conditions of theorem 2(a).

LEMMA 5 : *Let* $A \in C^{m,n}$ *and* $Q \in C^{n,n}$, *if* $r(A) = m$, $(Q + A^* A)^{-1}$ *exists and is one choice of* Q^- *with maximum rank iff A and Q are virtually disjoint,* $R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = C^n$.

We do not prove this lemma since it follows easily from [19, theorem 2.7.1],

LEMMA 6(a) : *Under the conditions of theorem* 2(a)

$$G = \begin{bmatrix} O & A_{QO}^{*-} \\ A_{QO}^{-} & \tilde{Q}^- - A_{QO}^{-} A \tilde{Q}^- \end{bmatrix}$$

is the regular inverse of N, *where* $A_{QO}^{-} = \tilde{Q}^- A^* H^-$ *is a* g-*inverse of* A, $H = A\tilde{Q}^- A^*$ *and* $\tilde{Q}^-$ *is a selected* g-*inverse of* Q *with maximum rank as defined in lemma* 5.

The solution of system (1) *is*

$$Y = A_{QO}^{*-} Z,$$
$$X = A_{QO}^{-} T + (I - A_{QO}^{-} A) \tilde{Q}^- Z.$$

(b) *If theorem* 2(b) *holds then*

$$G = \begin{bmatrix} -H^{-1} & A_{QO}^{*-1} \\ A_{QO}^{-1} & Q^{-1} - A_{QO}^{-1} A Q^{-1} \end{bmatrix}$$

is the regular inverse of N, *where* $A_{QO}^{-1} = Q^{-1} A^* H^{-1}$ *is the* g-*inverse of* A *as defined by* [4] *and* H *is defined in lemma* 2(b). *The solution of system* (1) *is*

$$Y = A_{QO}^{*-1} Z - H^{-1} T,$$
$$X = A_{QO}^{-1} T + (I - A_{QO}^{-1} A) Q^{-1} Z.$$

Examples

$$N = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}; \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad r(A) = 2, \quad r(Q) = 1.$$

It easy to verify that $R(A^*) \not\subset R(Q)$ and

$$R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = R^3,$$

thus A and Q are disjoint. The conditions of theorem 2(*a*) are fulfilled and G as defined in lemma 6(*a*) is the regular inverse of N. Thus $\tilde{Q}^- = (Q + A^* A)^{-1}$, $H = A\tilde{Q}^- A^*$, $A_{QO}^- = \tilde{Q}^- A^* H^{-1}$ and by easy computation

$$N^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}; \qquad A = (1 \quad 0); \quad Q = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix};$$

$$r(A) = 1, \qquad r(Q) = 2.$$

In this case $R(A^*) \subset R(Q)$ and theorem 2(*b*) holds. Then by lemma 6(*b*) $H = AQ^{-1} A^*$ and

$$N^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

4. OTHER INVERSES OF N

As stated in lemma 4 system (1) does not have a unique solution whenever $A \in C^{m,n}$ and $m > n$. However we can find other particular solutions when system (1) is possibly inconsistent. A set of equivalent conditions is stated in [18] in order to obtain a g-inverse minimum norm, least squares or both them for the system $AX = T$. We denote these by A_m^-, A_1^-, A^+ : the last one is the Moore-Penrose inverse of A. Thus we have the following :

THEOREM 3 : *Let G be a partitioned matrix as defined in lemma 2(b),*

(*a*) *G is a minimum norm inverse of N if $(I - H^- H) A = 0$, Q^- is replaced by Q_m^- and $R(A^*) \subset R(Q^*)$.*

(*b*) *G is a least squares inverse of N if Q^- is replaced by Q_1^- and*

$$A^*(I - HH^-) = 0.$$

(c) G is the Moore-Penrose inverse of N if Q^- *and* H^- *are replaced by* Q^+ *and* H^+ *and* $R(A^*) \subset R(Q^*)$, $R(AQ^+) \subset R(H)$ *and* $R((Q^+ A^*)^*) \subset R(H^*)$

Remark If Q is Hermitian, then G is the Moore-Penrose inverse of N if Q^- and H^- are replaced by Q^+ and H^+ and $R(AQ^+) \subset R(H)$ only

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