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THE STEEPEST-ASCENT METHOD FOR THE LINEAR PROGRAMMING PROBLEM (*)

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Communiqué par P. J. LAURENT

Abstract — This paper deals with a finite projection method (called the steepest-ascent method) proposed in 1974 by one of the authors for maximizing a linear function on a polyhedron. In the particular case of the maximization of a piecewise-linear concave function the method simply gives a recently published algorithm stated in the framework of the nondifferentiable convex optimization.

Résumé — Ce papier traite d'une méthode de projection (appelée méthode de la plus forte pente) proposée en 1974 par l'un des auteurs pour maximiser une fonction linéaire sur un polyèdre. Dans le cas particulier de la maximisation d'une fonction concave linéaire par morceaux la méthode donne simplement un algorithme récemment publié dans le cadre de l'optimisation convexe non différentiable.

1. INTRODUCTION

Solving a linear program is classically done by the simplex method (Ref. 3) or by any nonlinear programming method (for instance, projected gradient method (Ref. 7), reduced gradient method (Refs. 6, 8)). We propose in section 2 a finite method using the steepest ascent direction given by the projection of the gradient onto the cone of tangents at the current point, which is different from Rosen's method (Ref. 7) where the projection is done on a linear variety. This finite method has certainly been well-known for a long time but apparently did not lead to a publication to our knowledge.

In section 3, this method is applied in an obvious manner to the maximization of a piecewise-linear concave function by treating the problem in the hypograph

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space. It is shown that it is identical with a recently published algorithm (Ref. 2) given in the framework of nondifferentiable convex optimization. This latter paper lead us to publish some details about this method.

2. LINEAR OPTIMIZATION BY THE STEEPEST ASCENT METHOD

Let us consider the following problem

$$(P) : \begin{cases} \max f, x \\ \text{st} : x \in P = \{ x \in \mathbb{R}^n \mid Ax \geq a \} \end{cases}$$

where $A, a \sqsubset (m, n)$ matrix whose rows are denoted by $A_i (i = 1, 2, \dots, m)$, is such that $P \neq \emptyset$.

Let us denote for a given point $x \in P$:

- $I(x) = \{ i \in \{ 1, \dots, m \} \mid A_i x = a_i \}$,
- $T(P, x)$ the cone of tangents of P at x i.e.

$$T(P, x) = \{ y \in \mathbb{R}^n \mid A_i y \geq 0, i \in I(x) \},$$

- $f'(x)$ the projection of f onto $T(P, x)$,
- $D(x) = \{ z \in \mathbb{R}^n \mid z = x + \lambda f'(x), \lambda \geq 0 \}$

$D(x)$ is locally the steepest ascent direction in P (at x) for f . (See iii) of the following lemma for justification of this definition.)

- $[t, u] = \{ z \in \mathbb{R}^n \mid z = \lambda t + (1 - \lambda) u, 0 \leq \lambda \leq 1 \}$ for any t, u belonging to \mathbb{R}^n .

2.1. Description of the method

Starting point : $x_0 \in P$.

Step k : let x_k be given :

- determine $f'(x_k)$, if $f'(x_k) = 0$: stop,
- if $D(x_k) \cap P = D(x_k)$: stop

otherwise construct x_{k+1} such that

$$[x_k, x_{k+1}] = D(x_k) \cap P.$$

2.2. Finite convergence

The next theorem proves that while solving (P) by the steepest ascent method one constructs a finite sequence $\{ x_k \in P \mid 0 \leq k \leq k^* \}$ such that, either x_{k^*} is an optimal solution of (P) or $D(x_k)$ is an objective-increasing half-line.

This result was proposed in 1974 for an exam at the University of Lille (Ref. 1). Its proof uses the following elementary lemma.

LEMMA 2.1 : For $x \in P$, the following properties hold :

- i) if $f'(x) = 0$ then x is optimal for (P) ,
- ii) if $f'(x) \neq 0$ and $D(x) \cap P \neq D(x)$ then there exists $x' \in P$, $x' \neq x$ such that $[x, x'] = D(x) \cap P$,
- iii) if $[x, x'] = D(x) \cap P$ then

$$\forall x'' \in P, x'' \notin [x, x'] : f \cdot \frac{(x' - x)}{\|x' - x\|} > f \cdot \frac{(x'' - x)}{\|x'' - x\|}.$$

THEOREM 2.1 : The algorithm, described in 2.1, converges in a finite number of steps.

Proof : The number of cones of tangents of P is finite ; this implies that the number of directions $f'(\cdot)$ that can be used by the algorithm is finite. Thus, it suffices to prove only that the consecutive directions used have strictly decreasing slopes. This is obvious by using lemma 2.1 iii).

3. PARTICULARIZATION : MAXIMIZING A PIECEWISE, LINEAR CONCAVE FUNCTION

3.1. Notations

Let us consider the following piecewise-linear concave function :

$$\theta : \mathbb{R}^n \rightarrow \mathbb{R}, \theta(x) = \min \{ \beta_i x + \alpha_i \mid 1 \leq i \leq m \}.$$

Let us denote by $E(\theta)$ the hypograph of θ , i.e.

$$E(\theta) = \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leq \theta(x) \}$$

$E(\theta)$ defines a polyedron in \mathbb{R}^{n+1} whose boundary points $(x, \theta(x))$ belong to one of the following subsets F_J :

$$F_J = \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} \mid \beta_j x + \alpha_j = z \forall j \in J, \beta_j x + \alpha_j \neq z \forall j \notin J \}$$

with J an arbitrary subset of $\{ 1, \dots, m \}$.

This partition of the faces of $E(\theta)$ into a finite number of subsets $F_J \cap E(\theta)$ leads to a partition of \mathbb{R}^n (by projection onto \mathbb{R}^n) which is identical with the one given in reference 2.

If (P_1) and (P_2) are the following mathematical programs :

$$(P_1) : \max \{ \theta(x) \mid x \in \mathbb{R}^n \}$$

$$(P_2) : \max \{ \mu \mid (x, \mu) \in E(\theta) \}$$

it is known that (P_1) and (P_2) are equivalent in the sense that if (x^*, μ^*) is optimal for (P_2) then x^* is optimal for (P_1) and conversely, \bar{x} being a solution of (P_1) , then $(\bar{x}, \theta(\bar{x}))$ is optimal for (P_2) .

3.2. Connexions between the steepest ascent direction for θ in \mathbb{R}^n (at x) and the steepest ascent direction for the objective function $\tilde{f}(x, \mu) = 0 \cdot x + 1 \cdot \mu$ in $E(\theta)$ at $(x, \theta(x))$

In reference 2, the authors proposed for solving (P_1) an algorithm that uses (at $x \in \mathbb{R}^n$) the steepest ascent direction for the function θ . This direction is classically defined by the shortest subgradient $g(x)$ given by the projection

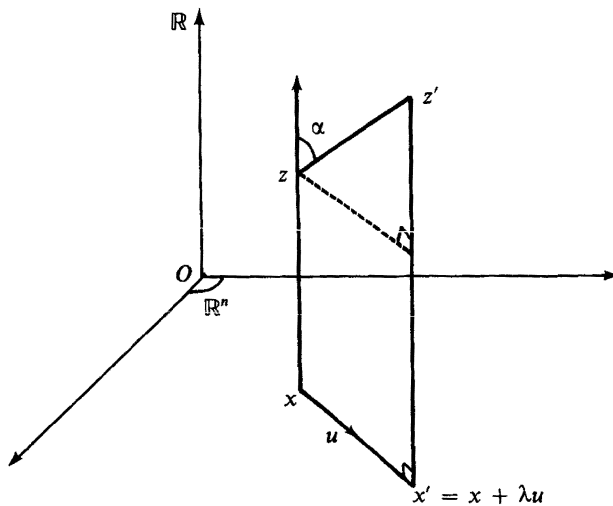


Figure.

$$\left\{ \begin{array}{l} z = (x, \theta(x)) \in E(\theta) \\ z' = (x', \theta(x')) \in E(\theta) \\ u \in \mathbb{R}^n, \| u \| = 1 \\ P_u = \lim_{\lambda \rightarrow 0^+} \frac{\theta(x + \lambda u) - \theta(x)}{\lambda} = \theta'(x; u) \\ \text{for sufficiently small } \lambda, \\ P_u = \frac{\theta(x + \lambda u) - \theta(x)}{\lambda} = \frac{1}{\text{tg } \alpha} \end{array} \right.$$

of the origin on $\partial\theta(x)$. Recall that $g(x)$ is colinear with the unique solution of the problem : $\max \{ \theta'(x; u) \mid u \in \mathbb{R}^n, \| u \| = 1 \}$ where $\theta'(x; u)$ is the one-sided directional derivative at x with respect to u .

For a given $u \in \mathbb{R}^n$ ($\| u \| = 1$), we have $\theta'(x; u) = 1/\text{tg } \alpha$ (see the two dimensional figure and its foot note). Thus the steepest ascent direction (for θ in \mathbb{R}^n) corresponds to an α minimum which is obtained by the projection of the " vertical axis " of \mathbb{R}^{n+1} onto the cone of tangents of $E(\theta)$ at $(x, \theta(x))$. Indeed this projection defines the steepest ascent direction on $E(\theta)$ according to the objective $0 \cdot x + 1 \cdot \mu$.

As a consequence, the stepsize walk of the algorithm defined in reference 2 for solving (P_1) is identical with the projection onto \mathbb{R}^n of the piecewise-linear path on $E(\theta)$ constructed by the steepest ascent method for solving the linear program (P_2) . The construction of one or the other of these paths (in \mathbb{R}^n or \mathbb{R}^{n+1}) involves the same computations. Recall that generally, the steepest ascent method differs from the simplex method (Ref. 3) or from Rosen's gradient projection method (Ref. 7).

3.3. Maximizing a piecewise-linear concave function subject to linear constraints (Ref. 5)

The problem is

$$(P'_1) \quad \max \{ \theta(x) \mid Dx \geq d \}$$

with θ defined as in 3.1

D is a (p, n) matrix, $d \in \mathbb{R}^p$

(P'_1) is equivalent to

$$(P'_2) \quad \max \{ \rho \mid \beta_i x + \alpha_i \geq \rho, i = 1, 2, \dots, m, Dx \geq d \} .$$

In this case, if x (resp. $(x, \theta(x))$) is not an optimal solution, the projection on \mathbb{R}^n of the steepest ascent direction in the (x, ρ) space as given in section 2 is colinear to the unique solution of the quadratic problem :

$$(q_1) \quad \max_{\substack{\|t\| \leq 1 \\ D_j t \geq 0, j \in I(x, \theta(x))}} \min_{h \in \partial\theta(x)} t \cdot h$$

where

$$I(x, \theta(x)) = \{ i \mid i = 1, 2, \dots, p, D_i x = d_i \}$$

$$\partial\theta(x) = \text{co} \{ \beta_i \mid \beta_i x + \alpha_i = \theta(x) \} .$$

In order to solve (P'_1) it would be possible to define in \mathbb{R}^n an algorithm following reference 2. The moving direction is then the unique solution of (q_1)

which is called the feasible steepest ascent direction. The step-size criteria need to take into account the constraints $D_i x \geq d_i$.

The similitude between (P'_1) and (P'_2) and the results of section 2 prove the finite convergence of such a method.

Remark 3.1 : It is to be noted that the finite convergence is essentially connected with the linear character of the problem. In nonlinear maximin problems even convergence alone cannot be guaranteed without substantial modifications (Ref. 4).

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