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On the sensitivity of the matrix exponential problem


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ON THE SENSITIVITY
OF THE MATRIX EXPONENTIAL PROBLEM (*)

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Resume — On discute le probleme de comparer les applications \( \text{Exp}(At) \) et \( \text{Exp}((A + B)t) \) ou la matrice \( B \) est consideree comme une perturbation de \( A \)
On montre que ce probleme est fortement lie a la multiplicité des valeurs propres de \( A \) et \( A + B \)
En conclusion, on montre que l'application \( \text{Exp}(At) \) est moins affectee par les perturbations de \( A \), si le spectre de \( A \) est simple

Abstract — We discuss the problem of comparing the mapping \( \text{Exp}(At) \) and \( \text{Exp}((A + B)t) \) where the square matrix \( B \) is considered as a perturbation of \( A \)
We show that this problem is strongly related to the multiplicity of eigenvalues of \( A \) and \( A + B \)
In conclusion, we set that the matrices \( A \), for which \( \text{Exp}(At) \) is less sensitive to perturbations, are those which have a simple spectrum

I. INTRODUCTION

Many models of physical, biological and economic processes involve systems of linear, constant coefficient ordinary differential equations

\[
\dot{X}(t) = AX(t) \quad (1)
\]

where \( A \) is a fixed square matrix, of dimension \( n \)
The solution is given by \( X(t) = \text{Exp}(At) \), where \( \text{Exp}(At) \) can be formally defined by

\[
\text{Exp}(At) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}, t \geq 0, A^0 = I
\]

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The subject of this paper concerns the sensitivity of the quantity $\exp(At)$ with respect to a perturbation of $A$.

Van Loan [4] has suggested that the problem under consideration is related to the behaviour of the function:

$$\theta(t) = \frac{\| \exp((A + B)t) - \exp(At) \|}{\| \exp(At) \|}$$

as $t$ tends to infinity.

We are going to show that $\theta(t)$ is a quantity related not only to the structure of $A$, but also to the structure of $B$.

It follows that it is not possible to characterize those $A$ for which $\exp(At)$ is very sensitive to changes in $A$.

Then we study the quantity:

$$\phi(t) = \frac{\| \exp(At) - \exp(Dt) \|}{\max \{ \| \exp(Dt) \|, \| \exp(At) \| \}}$$

when $t$ tends to infinity.

A characterization of $\phi(t)$ is given as a function of the structure of $A$ and $D$.

II. NOTATIONS AND SOME PRELIMINARY LEMMAS

Let us note $\sigma(A)$ the spectrum of $A$,

$$\rho(A) = \mathbb{C} - \sigma(A), \quad (2.1)$$

$$\alpha(A) = \max \{ \text{Re}(\lambda) : \lambda \in \sigma(A) \}, \quad (2.2)$$

$$A^* = (\bar{a}_{ij}) \quad (2.3)$$

$\text{Det}(A)$ the determinant of $A$.

We shall work exclusively with the 2-norms:

$$\| x \| = \left[ \sum_{i=1}^{n} |x_i|^2 \right]^{1/2}, \quad \| A \| = \max_{\| x \| = 1} \| Ax \| . \quad (2.4)$$

**Lemma 1:** Let $A$ be a matrix $n \times n$ and $\sigma(A)$ its spectrum. Let $\Gamma$ be a closed Jordan curve in $\mathbb{C}$ around $\sigma(A)$ which contains no point of $\sigma(A)$. Then

$$\exp(At) = \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1} e^{zt} \, dz . \quad (2.5)$$

**Proof [2].**
LEMMA 2 [Souriau’s form] : Let $A$ be a matrix of dimension $n$. If:

\[
\text{Det}(z) = \text{determinant of } (zI - A)
\]

$A_0 = I$ the identity of dimension $n$,

\[
c_{n-k} = -\text{trace } (A_{k-1} * A)
\]

$A_k = A_{k-1} * A - c_{n-k} I$; $k = 1, ..., n - 1$.

Then the resolvent

\[
(zI - A)^{-1} = \sum_{k=0}^{n-1} \frac{z^{n-k-1}}{\text{Det}(z)} A_k.
\]

Proof [1].

LEMMA 3 : Let $f : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be the function defined by

\[
f(z, t) = -\frac{z^t e^{zt}}{\prod_{i=1}^{n} (z - \lambda_i)} \ ; \lambda_i \in \mathbb{C}.
\]

Then $\frac{d^k}{dz^k} f(z, t) = e^{zt} p(z, t)$, where $p(z, t)$ is a polynomial of degree $k$ in $t$, with coefficient of $t^k$ equal to $z^t \prod_{i=1}^{n} (z - \lambda_i)$.

Proof [3].

III. THE ANALYSIS OF $\theta(t)$

Van Loan [4] has concluded that the bounds of $\theta(t)$ for normal matrices are as small as it can be expected. Furthermore, when $A$ is normal the $\text{Exp}(At)$ problem is « well conditioned ».

We are going to give an example of a normal matrix such that for different choices of $B$, $\theta(t)$ behaves as a constant or an exponential when $t$ tends to infinity.

Let $A$ be a square normal matrix.

Let $\sigma(A) = \{ \lambda_i \}$ and

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n.
\]

Let $B$ be a square matrix such that $\sigma(A + B) = \{ \mu_i \}$ is real and simple.

By lemmas 1, 2, 3 we have

\[
\theta(t) = \frac{1}{e^{\lambda t}} \left| \sum_{k=0}^{n-1} D_k \sum_{p=1}^{n} \mu_{p}^{n-k-1} e^{\mu_p t} - \sum_{k=1}^{n-1} A_k \sum_{p=1}^{n} \frac{\lambda^{n-k-1} e^{\lambda_p t}}{\prod_{i \neq p} (\lambda_p - \lambda_i)} \right| (3.1)
\]

where $D_k = (A + B)_k$ in the Souriau’s form.
It is easy to show that:
\[
\lim_{t \to \infty} \left( \sum_{k=1}^{n-1} A_k \sum_{p=1}^{n} \lambda_p^{n-p} \frac{e^{(\lambda_p - \lambda_1)t}}{\prod_{i \neq p} \left( \lambda_i - \lambda_p \right)} \right)
\]
converges to:
\[
\lim_{t \to \infty} \left( \sum_{k=0}^{n-1} \frac{\lambda^{n-k-1} A_k}{\prod_{i \neq p} \left( \lambda_i - \lambda_p \right)} \right), \quad \text{as } t \text{ tends to infinity.}
\]

If \(\lambda_n < \mu_n\) then
\[
\lim_{t \to \infty} \left( \sum_{k=0}^{n-1} D_k \sum_{p=1}^{n} \mu_p^{n-p} \frac{e^{(\mu_p - \lambda_1)t}}{\prod_{i \neq p} \left( \mu_i - \mu_p \right)} \right)
\]
tends to the infinity like \(e^{(\mu_n - \lambda_n)t}\) as \(t\) tends to the infinity. If \(0 < \mu_i < \lambda_i\); \(i = 1, \ldots, n\) then (3.4) tends to
\[
\sum_{k=0}^{n-1} D_k \sum_{p=1}^{n} \frac{\mu_p^{n-p} \prod_{i \neq p} \left( \mu_i - \mu_p \right)}{\prod_{i \neq p} \left( \mu_i - \mu_p \right)}, \quad \text{as } t \text{ tends to infinity.}
\]

Then according to the structure of \(B\), \(\theta(t)\) may converge to infinity as \(e^{ct}\), \(c > 0\), or to a constant.

This example shows that the structure of \(A\) is not enough to characterize the behaviour of \(\theta(t)\).

**IV. THE MAIN THEOREM**

In this section we introduce a function \(\phi(t)\) which enables us to study the sensitivity of the problem \(\text{Exp}(At)\). This function is symmetrical with respect to \(A\) and \(A + B\).

If we note \(D = A + B\):
\[
\phi(t) = \frac{\| \text{Exp}(Dt) - \text{Exp}(At) \|}{\text{Max} \{ \| \text{Exp}(Dt) \|, \| \text{Exp}(At) \| \}}; \quad t \geq 0.
\]

The main theorem is the following:

**Theorem**: Let \(A\) and \(D\) be two square matrices of dimension \(n\) and \(\{ \lambda_1, \ldots, \lambda_r \}\) equals \(\sigma(A) \cup \sigma(D)\).
If \( \lambda_i \in \sigma(A) \cup \sigma(D) - \sigma(A) \cap \sigma(D) \) let \( m_i \) be the corresponding multiplicity of \( \lambda_i \).

If \( \lambda_i \in \sigma(A) \cap \sigma(D) \), let \( m_i \) be the sum of the multiplicity of \( \lambda_i \) as eigenvalue of \( A \) plus the multiplicity of \( \lambda_i \) as eigenvalue of \( D \).

If \( m = \max_{1 \leq i \leq r} (m_i) \), then \( \phi(t) \equiv \| D - A \| p(t) \) where \( p(t) \) is a polynomial in \( t \) of degree less than \( m \).

The proof of the theorem: By lemma 1 we have

\[
\text{Exp}(At) - \text{Exp}(Dt) = \frac{1}{2\pi i} \int_{\Gamma} ((zI - A)^{-1} - (zI - D)^{-1}) e^{zt} \, dz, \quad (4.2)
\]

where \( \Gamma \) is a closed Jordan curve in \( \mathbb{C} \) around \( \sigma(A) \cup \sigma(D) \) which contains no point of \( \sigma(A) \cup \sigma(D) \).

It follows that

\[
\text{Exp}(At) - \text{Exp}(Dt) = \frac{1}{2\pi i} \int_{\Gamma} (zI - D)^{-1} (D - A) (zI - A)^{-1} e^{zt} \, dz. \quad (4.3)
\]

If we set \( c_1(z) = \text{Det}(zI - A) \) and \( c_2(z) = \text{Det}(zI - D) \) then by lemma 2

\[
(zI - A)^{-1} = \sum_{l=0}^{n-1} \frac{z^{n-l-1}}{c_1(z)} A_l, \quad (4.4)
\]

and

\[
(zI - D)^{-1} = \sum_{k=0}^{n-1} \frac{z^{n-k-1}}{c_2(z)} D_k.
\]

This yields

\[
\text{Exp}(At) - \text{Exp}(Dt) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} D_k (D - A) A_l \sum_{p=1}^{r} \text{Res} \left( \frac{z^{2n-l-k-2} e^{zt}}{\prod_{i=1}^{r} (z - \lambda_i)^{m_i}}, \lambda_p \right). \quad (4.6)
\]

If \( \lambda_p \) is of multiplicity \( m_p \)

\[
\text{Res} \left( \frac{z^{2n-l-k-2}}{\prod_{i=1}^{r} (z - \lambda_i)^{m_i}}, \lambda_p \right) = \frac{1}{(m_p - 1)!} \frac{d^{m_p - 1}}{dz^{m_p - 1}} \left( \frac{z^{2n-l-k-2} e^{zt}}{\prod_{i=1}^{r} (z - \lambda_i)^{m_i}} \right)_{z=\lambda_p}. \quad (4.7)
\]
Then by lemma 3 we have:

$$\text{Exp}(At) - \text{Exp}(Dt) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} D_k(D - A) A_l \sum_{p=1}^{r} \frac{e^{\lambda_p t}}{(m_p - 1)!} p_{l,k}(\lambda_p, t), \quad (4.8)$$

where $p_{l,k}(\lambda_p, t)$ is a polynomial of degree less than or equal to $m_p - 1$, and the coefficient of $(m_p - 1)$th power of $t$ is:

$$\frac{\lambda_p^{2n-k-l-2}}{\prod_{i=1}^{r} (\lambda_p - \lambda_i)^{m_i}}. \quad (4.9)$$

If we note $p_{l,k}(\lambda_p, t) = \sum_{i=0}^{s} n_i t^i$, we write

$$q_{l,k}(\lambda_p, t) = \sum_{i=0}^{s} n_i | t^i. \quad (4.10)$$

Then if $t \geq 0$,

$$| p_{l,k}(\lambda_p, t) | \leq q_{l,k}(\lambda_p, t). \quad (4.11)$$

It follows that

$$\| \text{Exp}(At) - \text{Exp}(Dt) \| \leq \| D - A \| \sum_{p=1}^{r} \frac{\| e^{\lambda_p t} \|}{(m_p - 1)!} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \| D_k \| \| A_l \| q_{l,k}(\lambda_p, t). \quad (4.12)$$

But $| e^{\lambda_p t} | \leq \max \{ \| \text{Exp}(Dt) \|, \| \text{Exp}(At) \| \}$, then

$$\phi(t) \leq \| D - A \| \sum_{p=1}^{r} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \| D_k \| \| A_l \| q_{l,k}(\lambda_p, t). \quad (4.13)$$

If we set

$$p(t) = \sum_{p=1}^{r} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \| D_k \| \| A_l \| q_{l,k}(\lambda_p, t). \quad (4.14)$$

Then

$$\phi(t) \leq \| D - A \| p(t) \quad (4.15)$$

where $p(t)$ is a polynomial of degree at most $m - 1$. 

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We can remark that if a non zero eigenvalue of $A$ with multiplicity $m$, exist then $p(t)$ is a polynomial of degree exactly $m - 1$.

If all the eigenvalues of $A$ and $D$ are simple and $\sigma(A) \cap \sigma(D) = \emptyset$ then $\phi(t)$ is bounded by a constant.

V. CONCLUSION

We have shown that the function $\theta(t)$ is insufficient to characterize the matrices for which the mapping $\text{Exp}(At)$ is sensitive to changes in $A$.

We have introduced a function $\phi(t)$ which measures the relative distance between $\text{Exp}(At)$ and $\text{Exp}((A + B)t)$. In the main theorem we show that the behaviour of the bound of $\phi(t)$ depends on the multiplicity of the eigenvalues of $A$ and $A + B$. Another factor is the distance between two different eigenvalues, but it's a secondary factor as it modifies the coefficients of $p(t)$ but not the degree.

This fact agrees with the conclusion obtained in section 3 by a formal development of $\text{Exp}(At)$.

The analysis of the (4.15) bound of $\phi(t)$ lead us to conclude that if $A$ in a matrix with a simple spectrum, the mapping $\text{Exp}(At)$ is less sensitive to change on $A$, because the degree of $p(t)$ may be at most $n$.

REFERENCES