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# ON THE SENSITIVITY OF THE MATRIX EXPONENTIAL PROBLEM (*)

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Communique par F ROBERT

Resume — *On discute le probleme de comparer les applications* Exp$(At)$ *et* Exp$((A + B)\,t)$ *ou la matrice $B$ est consideree comme une perturbation de $A$*
*On montre que ce probleme est fortement lie a la multiplicite des valeurs propres de $A$ et $A + B$*
*En conclusion, on montre que l'application* Exp$(At)$ *est moins affectee par les perturbations de $A$, si le spectre de $A$ est simple*

Abstract — *We discuss the problem of comparing the mapping* Exp$(At)$ *and* Exp$((A + B)\,t)$ *where the square matrix $B$ is considered as a perturbation of $A$*
*We show that this problem is strongly related to the multiplicity of eigenvalues of $A$ and $A + B$*
*In conclusion, we set that the matrices $A$, for which* Exp$(At)$ *is less sensitive to perturbations, are those which have a simple spectrum*

## I. INTRODUCTION

Many models of physical, biological and economic processes involve systems of linear, constant coefficient ordinary differential equations

$$\begin{aligned} \overset{\circ}{X}(t) &= AX(t) \\ X(0) &= I_{n \times n}, t \geqslant 0 \end{aligned} \tag{1}$$

where $A$ is a fixed square matrix, of dimension $n$

The solution is given by $X(t) = \operatorname{Exp}(At)$, where $\operatorname{Exp}(At)$ can be formally defined by

$$\operatorname{Exp}(At) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k\,!}, t \geqslant 0, A^0 = I$$

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The subject of this paper concerns the sensitivity of the quantity $\mathrm{Exp}(At)$ with respect to a perturbation of $A$.

Van Loan [4] has suggested that the problem under consideration is related to the behaviour of the function :

$$\theta(t) = \frac{\| \mathrm{Exp}((A + B)\, t) - \mathrm{Exp}(At) \|}{\| \mathrm{Exp}(At) \|}$$

as $t$ tends to infinity.

We are going to show that $\theta(t)$ is a quantity related not only to the structure of $A$, but also to the structure of $B$.

It follows that it is not possible to characterize those $A$ for which $\mathrm{Exp}(At)$ is very sensitive to changes in $A$.

Then we study the quantity :

$$\phi(t) = \frac{\| \mathrm{Exp}(At) - \mathrm{Exp}(Dt) \|}{\mathrm{Max}\, \{ \| \mathrm{Exp}(Dt) \|, \| \mathrm{Exp}(At) \| \}}$$

when $t$ tends to infinity.

A characterization of $\phi(t)$ is given as a function of the structure of $A$ and $D$.

## II. NOTATIONS AND SOME PRELIMINARY LEMMAS

Let us note $\sigma(A)$ the spectrum of $A$,

$$\rho(A) = \mathbb{C} - \sigma(A), \tag{2.1}$$

$$\alpha(A) = \mathrm{Max}\, \{ \mathrm{Re}\,(\lambda)/\lambda \in \sigma(A) \}, \tag{2.2}$$

$$A^* = (\overline{a}_{ij}) \tag{2.3}$$

Det $(A)$ the determinant of $A$.

We shall work exclusively with the 2-norms :

$$\| x \| = \left[ \sum_{i=1}^{n} | x_i |^2 \right]^{1/2}, \| A \| = \max_{\|x\| = 1} \| Ax \| . \tag{2.4}$$

LEMMA 1 : *Let $A$ be a matrix $n \times n$ and $\sigma(A)$ its spectrum. Let $\Gamma$ be a closed jordan curve in $\mathbb{C}$ arround $\sigma(A)$ which contains no point of $\sigma(A)$. Then*

$$\mathrm{Exp}(At) = \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1}\, e^{zt}\, dz . \tag{2.5}$$

*Proof* [2].

LEMMA 2 [Souriau's form] : *Let $A$ be a matrix of dimension $n$. If :*

$$\mathrm{Det}\,(z) = \textit{determinant of}\,(zI - A)$$

$A_0 = I$ *the identity of dimension $n$,*
$c_{n-k} = -\ \mathrm{trace}\,(A_{k-1} * A)$
$A_k = A_{k-1} * A - c_{n-k}\, I\,;\ k = 1, \dots, n-1$.
*Then the resolvent*

$$(zI - A)^{-1} = \sum_{k=0}^{n-1} \frac{z^{n-k-1}}{\mathrm{Det}\,(z)} A_k\,. \tag{2.6}$$

*Proof* [1].

LEMMA 3 : *Let $f: \mathbb{C} \times R \to \mathbb{C}$ be the function defined by*

$$f(z, t) = \frac{z^l\, e^{zt}}{\prod_{i=1}^{n} (z - \lambda_i)}\,;\ \lambda_i \in \mathbb{C}\,. \tag{2.7}$$

*Then* $\dfrac{d^k}{dz^k} f(z, t) = e^{zt}\, p(z, t)$, *where $p(z, t)$ is a polynomial of degree $k$ in $t$, with coefficient of $t^k$ equal to* $z^l \Big/ \prod_{i=1}^{n} (z - \lambda_i)$.

*Proof* [3].

### III. THE ANALYSIS OF $\theta(t)$

Van Loan [4] has concluded that the bounds of $\theta(t)$ for normal matrices are as small as it can be expected. Furthermore, when $A$ is normal the $\mathrm{Exp}(At)$ problem is « well conditioned ».

We are going to give an example of a normal matrix such that for different choices of $B$, $\theta(t)$ behaves as a constant or an exponential when $t$ tends to infinity.

Let $A$ be a square normal matrix.

Let $\sigma(A) = \{\lambda_i\}$ and

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n\,.$$

Let $B$ be a square matrix such that $\sigma(A + B) = \{\mu_i\}$ is real and simple. By lemmas 1, 2, 3 we have

$$\theta(t) = \frac{1}{e^{\lambda_N t}} \left\| \sum_{k=0}^{n-1} D_k \sum_{p=1}^{n} \frac{\mu_p^{n-k-1}\ e^{\mu_p t}}{\prod_{i \neq p}^{n} (u_p - \mu_i)} - \sum_{k=1}^{n-1} A_k \sum_{p=1}^{n} \frac{\lambda^{n-k-1}\ e^{\lambda_p t}}{\prod_{i \neq p}^{n} (\lambda_p - \lambda_i)} \right\| \tag{3.1}$$

where $D_k = (A + B)_k$ in the Souriau's form.

It is easy to show that :

$$\left\| \sum_{k=1}^{n-1} A_k \sum_{p=1}^{n} \lambda_p^{n-k-1} \frac{e^{(\lambda_p - \lambda_n)t}}{\prod_{i \neq p}^{n} (\lambda_p - \lambda_i)} \right\| \tag{3.2}$$

converges to :

$$\left\| \sum_{k=0}^{n-1} \frac{\lambda^{n-k-1} A_k}{\prod_{i \neq p}^{n} (\lambda_p - \lambda_i)} \right\|, \text{ as } t \text{ tends to infinity.} \tag{3.3}$$

If $\lambda_n < \mu_n$ then

$$\left\| \sum_{k=0}^{n-1} D_k \sum_{p=1}^{n} \mu_p^{n-k-1} \frac{e^{(\mu_p - \lambda_n)t}}{\prod_{i \neq p}^{n} (\mu_p - \mu_i)} \right\| \tag{3.4}$$

tends to the infinity like $e^{(\mu_n - \lambda_n)t}$ as $t$ tends to the infinity. If $0 < \mu_i < \lambda_i$; $i = 1, \dots, n$ then (3.4) tends to

$$\sum_{k=0}^{n-1} D_k \sum_{p=1}^{n} \frac{\mu_p^{n-k-1}}{\prod_{i \neq p}^{n} (\mu_p - \mu_i)} \quad , \quad \text{as } t \text{ tends to infinity}. \tag{3.5}$$

Then according to the structure of $B$, $\theta(t)$ may converge to infinity as $e^{ct}$, $c > 0$, or to a constant.

This exemple shows that the structure of $A$ is not enough to characterize the behaviour of $\theta(t)$.

## IV. THE MAIN THEOREM

In this section we introduce a function $\phi(t)$ which enables us to study the sensitivity of the problem $\operatorname{Exp}(At)$. This function is symmetrical with respect to $A$ and $A + B$.

If we note $D = A + B$ :

$$\phi(t) = \frac{\| \operatorname{Exp}(Dt) - \operatorname{Exp}(At) \|}{\operatorname{Max} \{ \| \operatorname{Exp}(Dt) \|, \| \operatorname{Exp}(At) \| \}}; \quad t \geqslant 0 . \tag{4.1}$$

The main theorem is the following :

THEOREM : *Let $A$ and $D$ be two square matrices of dimension $n$ and $\{ \lambda_1, \dots, \lambda_r \}$ equals $\sigma(A) \cup \sigma(D)$.*

*If* $\lambda_i \in \sigma(A) \cup \sigma(D) - \sigma(A) \cap \sigma(D)$ *let* $m_i$ *be the corresponding multiplicity of* $\lambda_i$.

*If* $\lambda_i \in \sigma(A) \cap \sigma(D)$, *let* $m_i$ *be the sum of the multiplicity of* $\lambda_i$ *as eigenvalue of* $A$ *plus the multiplicity of* $\lambda_i$ *as eigenvalue of* $D$.

*If* $m = \max\limits_{1 \leqslant i \leqslant r} (m_i)$, *then* $\phi(t) \leqslant \| D - A \| \, p(t)$ *where* $p(t)$ *is a polynomial in* $t$ *of degree less than* $m$.

*The proof of the theorem* : By lemma 1 we have

$$\operatorname{Exp}(At) - \operatorname{Exp}(Dt) = \frac{1}{2\,\Pi i} \int_\Gamma ((zI - A)^{-1} - (zI - D)^{-1})\, e^{zt}\, dz\,, \quad (4.2)$$

where $\Gamma$ is a closed Jordan curve in $\mathbb{C}$ arround $\sigma(A) \cup \sigma(D)$ which contains no point of $\sigma(A) \cup \sigma(D)$.

It follows that

$$\operatorname{Exp}(At) - \operatorname{Exp}(Dt) = \frac{1}{2\,\Pi i} \int_\Gamma (zI - D)^{-1} (D - A)(zI - A)^{-1}\, e^{zt}\, dz\,. \quad (4.3)$$

If we set $c_1(z) = \operatorname{Det}(zI - A)$ and $c_2(z) = \operatorname{Det}(zI - D)$ then by lemma 2

$$(zI - A)^{-1} = \sum_{l=0}^{n-1} \frac{z^{n-l-1}}{c_1(z)} A_l\,, \quad (4.4)$$

and

$$(zI - D)^{-1} = \sum_{k=0}^{n-1} \frac{z^{n-k-1}}{c_2(z)} D_k\,.$$

This yields

$$\operatorname{Exp}(At) - \operatorname{Exp}(Dt) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} D_k(D - A) A_l \sum_{p=1}^{r} \operatorname{Res}\left( \frac{z^{2n-l-k-2}\, e^{zt}}{\prod\limits_{i=1}^{r} (z - \lambda_i)^{m_i}}, \lambda_p \right). \quad (4.6)$$

If $\lambda_p$ is of multiplicity $m_p$

$$\operatorname{Res}\left( \frac{z^{2n-l-k-2}}{\prod\limits_{i=1}^{r} (z - \lambda_i)^{m_i}}, \lambda_p \right) = \frac{1}{(m_p - 1)!} \frac{d^{m_p-1}}{dz^{m_p-1}} \left( \frac{z^{2n-l-k-2}\, e^{zt} (z - \lambda_p)^{m_p}}{\prod\limits_{i=1}^{r} (z - \lambda_i)^{m_i}} \right)_{z=\lambda_p}. \quad (4.7)$$

Then by lemma 3 we have :

$$\operatorname{Exp}(At) - \operatorname{Exp}(Dt) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} D_k(D - A) A_l \sum_{p=1}^{r} \frac{e^{\lambda_p t}}{(m_p - 1)!} p_{l,k}(\lambda_p, t), \quad (4.8)$$

where $p_{l,k}(\lambda_p, t)$ is a polynomial of degree less than or equal to $m_p - 1$, and the coefficient of $(m_p - 1)$th power of $t$ is :

$$\frac{\lambda_p^{2n-k-l-2}}{\prod_{\substack{\iota=1 \\ \iota \neq p}}^{r} (\lambda_p - \lambda_\iota)^{m_\iota}}. \quad (4.9)$$

If we note $p_{l,k}(\lambda_p, t) = \sum_{\iota=0}^{s} n_\iota t^\iota$, we write

$$q_{l,k}(\lambda_p, t) = \sum_{\iota=0}^{s} | n_\iota | t^\iota . \quad (4.10)$$

Then if $t \geqslant 0$,

$$| p_{l,k}(\lambda_p, t) | \leqslant q_{l,k}(\lambda_p, t) . \quad (4.11)$$

It follows that

$$\| \operatorname{Exp}(At) - \operatorname{Exp}(Dt) \| \leqslant \| D - A \| \sum_{p=1}^{r} \frac{| e^{\lambda_p t} |}{(m_p - 1)!} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \| D_k \| \; \| A_l \| q_{l,k}(\lambda_p, t). \quad (4.12)$$

But $| e^{\lambda_p t} | \leqslant \operatorname{Max} \{ \| \operatorname{Exp}(Dt) \|, \| \operatorname{Exp}(At) \| \}$, then

$$\phi(t) \leqslant \| D - A \| \sum_{p=1}^{r} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \| D_k \| \; \| A_l \| q_{l,k}(\lambda_p, t) . \quad (4.13)$$

If we set

$$p(t) = \sum_{p=1}^{r} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \| D_k \| \; \| A_l \| q_{l,k}(\lambda_p, t) . \quad (4.14)$$

Then

$$\phi(t) \leqslant \| D - A \| p(t) \quad (4.15)$$

where $p(t)$ is a polynomial of degree at most $m - 1$.

We can remark that if a non zero eigenvalue of $A$ with multiplicity $m$, exist then $p(t)$ is a polynomial of degree exactly $m - 1$.

If all the eigenvalues of $A$ and $D$ are simple and $\sigma(A) \cap \sigma(D) = \varnothing$ then $\phi(t)$ is bounded by a constant.

## V. CONCLUSION

We have shown that the function $\theta(t)$ is insufficient to characterize the matrices for which the mapping $\mathrm{Exp}(At)$ is sensitive to changes in $A$.

We have introduced a function $\phi(t)$ which measures the relative distance between $\mathrm{Exp}(At)$ and $\mathrm{Exp}((A + B)\,t)$. In the main theorem we show that the behaviour of the bound of $\phi(t)$ depends on the multiplicity of the eigenvalues of $A$ and $A + B$. Another factor is the distance between two different eigenvalues, but it's a secondary factor as it modifies the coefficients of $p(t)$ but not the degree.

This fact agrees with the conclusion obtained in section 3 by a formal developement of $\mathrm{Exp}(At)$.

The analysis of the (4.15) bound of $\phi(t)$ lead us to conclude that if $A$ in a matrix with a simple spectrum, the mapping $\mathrm{Exp}(At)$ is less sensitive to change on $A$, because the degree of $p(t)$ may be at most $n$.

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