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EBERHARD SCHOCK

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THREE REMARKS ON THE USE OF ČEBYŠEV POLYNOMIALS FOR SOLVING EQUATIONS OF THE SECOND KIND (*)

by Eberhard SCHOCK ⁽¹⁾

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Abstract — Three methods are considered the Čebyšev-Euler method, the Čebyšev semi-iterative method and a refinement of the projection method for the approximation of the quasi inverse for self-adjoint operators A such that unity does not belong to the spectrum of A

Résumé — On considère trois méthodes la méthode de Čebyšev-Euler, la méthode semi-itérative de Čebyšev, et une amélioration de la méthode de projection, appliquées à l'approximation du quasi-inverse pour des opérateurs auto-adjoints A pour lesquels l'unité n'appartient pas au spectre de A

In this communication we consider three methods for the approximate solution of equations of the second kind

$$x - Ax = y$$

in a (complex) Hilbert space with a selfadjoint bounded linear operator A . We only assume that unity does not belong to the spectrum of A .

We give a new proof and an error estimate for the Čebyšev-Euler method, we discuss the Čebyšev-semi-iterative method (cf. Varga [7]) and we consider a refinement of the projection method of type Q_{ν} , introduced in [6].

1. INTRODUCTION

Let A be a bounded linear selfadjoint operator in a complex Hilbert space H . Let (E_{λ}) be its spectral decomposition, σ an interval containing the spectrum $\sigma(A)$ of A . Then

$$A = \int_{\sigma} \lambda dE$$

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(¹) Fachbereich Mathematik der Universität Kaiserslautern, D-6750 Kaiserslautern

and for each continuous function $p : \sigma \rightarrow \mathbb{R}$

$$p(A) = \int_{\sigma} p(\lambda) dE_{\lambda}$$

and

$$\| p(A) \| \leq \sup_{\lambda \in \sigma} | p(\lambda) | .$$

Especially, if $1 \notin \sigma$,

$$\| (1 - A)^{-1} - p(A) \| \leq \sup_{\lambda \in \sigma} \left| \frac{1}{1 - \lambda} - p(\lambda) \right| .$$

If p is an arbitrary polynomial of degree n , then $\| (1 - A)^{-1} - p(A) \|$ is minimal, if p is the proximum of r (with $r(\lambda) = (1 - \lambda)^{-1}$) in the space of all polynomials of degree n on the spectrum of A with respect of the sup-norm.

We call a method for the approximate solution of $x - Ax = y$ polynomial, if the approximate solution \hat{x} is of the form $\hat{x} = p(A) y$, where $p(A)$ is an operator polynomial.

2. THE ČEBYŠEV-EULER METHOD

If $\sigma = [a, b]$, $b < 1$ then the Čebyšev-Euler method consists of determining the proximum p_n to r in $[a, b]$ by polynomials degree n . Then

$$x_n = p_n(A) y \tag{1}$$

is the Čebyšev-Euler approximation of the solution x of $x - Ax = y$.

This approximation is easy to calculate : it is known Čebyšev [3], Bernstein [2], Meinardus [4], that in the interval $[- 1, 1]$ the proximum of

$$s_{\alpha}(\lambda) = \frac{1}{\lambda - \alpha} \quad \alpha > 1$$

is given by the polynomials q_n of degree n which fulfill

$$\begin{aligned} \frac{1}{\lambda - \alpha} - q_n(\lambda) &= \gamma_n \cos (n\varphi + \delta) \\ \gamma_n &= \frac{(\alpha - \sqrt{\alpha^2 - 1})^n}{\alpha^2 - 1}, \quad \lambda = \cos \varphi, \quad \frac{\alpha\lambda - 1}{\lambda - 1} = \cos \delta . \end{aligned} \tag{2}$$

Using the Čebyšev polynomials t_n and v_n of first resp. second kind, (2) is equivalent to

$$(\alpha - \lambda) q_n(\lambda) = 1 - \gamma_n(\alpha\lambda - 1) t_n(\lambda) + \gamma_n \sqrt{\alpha^2 - 1} (1 - \lambda^2) v_{n-1}(\lambda) .$$

The recursion formulas for t_n and v_n lead to the recursion formula for q_n

$$q_{n+1}(\lambda) = 2 \gamma \lambda q_n(\lambda) - \gamma^2 q_{n-1}(\lambda) - 2 \gamma$$

$$\gamma = \alpha - \sqrt{\alpha^2 - 1}, \quad q_0(\lambda) = \frac{\alpha}{1 - \alpha^2}, \quad q_1(\lambda) = \frac{\lambda + \sqrt{\alpha^2 - 1}}{\alpha^2 - 1}.$$

A linear transformation of the interval $[-1, 1]$ onto $[a, b]$ gives the polynomials p_n of best approximation of r by

$$p_n(\lambda) = -\frac{b-a}{2} q_n\left(\frac{2\lambda - b - a}{b-a}\right)$$

which leads to the recursion formula

$$p_{n+1}(\lambda) = -\frac{2}{b-a} \left[\frac{2\gamma}{b-a} (2\lambda - b - a) p_n(\lambda) - \gamma^2 p_{n-1}(\lambda) - 2\gamma \right]$$

$$p_0(\lambda) = \frac{b-a}{2} \frac{\alpha}{\alpha^2 - 1}$$

$$p_1(\lambda) = \frac{1}{\alpha^2 - 1} \left(-\lambda + \frac{b+a}{2} - \frac{b-a}{2} \sqrt{\alpha^2 - 1} \right)$$

$$\alpha = \frac{2 - b - a}{b - a}$$

$$\gamma = \alpha - \sqrt{\alpha^2 - 1}.$$

The error estimate is

$$\max_{\lambda \in [a,b]} \left| \frac{1}{1-\lambda} - p_n(\lambda) \right| = \max_{\lambda \in [-1,1]} \frac{b-a}{2} \left| \frac{1}{\lambda-\alpha} - q_n(\lambda) \right|$$

$$= \frac{b-a}{2} \frac{\gamma^n}{\alpha^2 - 1} \max |\cos(n\varphi + \delta)| \leq \frac{b-a}{2(\alpha^2 - 1)} \cdot \gamma^n.$$

If we replace λ by A , we obtain the following result :

If $\sigma(A) \subset [a, b]$, $b < 1$, then the best polynomial approximation method of $x - Ax = y$ is given by the following semi-iterative method.

$$x_{n+1} = -\frac{2}{b-a} \left[\frac{2\gamma}{b-a} (2Ax_n - (b+a)x_n) - \gamma^2 x_{n-1} - 2\gamma y \right]$$

$$x_0 = \frac{b-a}{2} \cdot \frac{\alpha}{\alpha^2 - 1} y, \quad x_1 = \frac{1}{\alpha^2 - 1} \left[-Ay + \left(\frac{b+a}{2} - \frac{b-a}{2} \sqrt{\alpha^2 - 1} \right) y \right]$$

$$\alpha = \frac{2 - b - a}{b - a}, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1}$$

with the error estimate

$$\|x - x_n\| \leq \frac{b-a}{2(\alpha^2-1)} \gamma^n.$$

This method also can be obtained by using methods of summability theory (cf. Niethammer [5]).

3. THE ČEBYŠEV SEMI-ITERATIVE METHOD

Let A be a linear selfadjoint bounded operator with $1 \notin \sigma(A)$ and (x_n) the Picard iteration sequence

$$x_{n+1} = Ax_n + y, \quad x_0 = y.$$

This method calculates a linear combination

$$\tilde{x}_n = \sum_{j=0}^n \gamma_j x_j$$

such that $x - \tilde{x}_n$ has a small norm.

Since for

$$\tilde{w}_n = \tilde{x}_n - x = \sum_{j=0}^n \gamma_j (x_j - x) + \sum_{j=0}^n (\gamma_j - 1) x$$

it is

$$\tilde{w}_n = \sum_{j=0}^n \gamma_j A^j (x_0 - x) = p(A) (x_0 - x)$$

with the condition $p(1) = \sum_{j=0}^n \gamma_j = 1$, so

$$\tilde{w}_n = p(A) (y - (I - A)^{-1} y) = \int_{\sigma} p(\lambda) \frac{-\lambda}{1-\lambda} dE_{\lambda} y$$

and $\|\tilde{w}_n\|$ is minimal, if p is a polynomial of degree n with $p(1) = 1$ and

$$\max_{\lambda \in \sigma(A)} \left| p(\lambda) \frac{\lambda}{1-\lambda} \right| \leq \max_{\lambda \in \sigma(A)} \left| q(\lambda) \frac{\lambda}{1-\lambda} \right|,$$

where q is an arbitrary polynomial of degree n with $q(1) = 1$. If both 1 and 0 do not belong to the spectrum of A , then p is up to a constant the same polynomial as the polynomial q with $q(1) = 1$ and $\max_{\lambda \in \sigma(A)} |q(\lambda)|$ is minimal.

In each case, this minimal polynomial does not lead to an easy semi-iterative method, so the usual minimization condition is to determine the polynomial p of degree n with $p(1) = 1$ and minimal norm.

It is well known that the transformed Čebyšev polynomials have this property that their norm on an interval is minimal. So one has to consider three cases

- 1° $\sigma(A) \subset [a, b], b < 1$
- 2° $\sigma(A) \subset [a, b], a > 1$
- 3° $\sigma(A) \subset [a_1, b_1] \cup [a_2, b_2]$.

In the first and second case

$$p_n(\lambda) = \frac{t_n\left(\frac{2\lambda - b - a}{b - a}\right)}{t_n\left(\frac{2 - b - a}{b - a}\right)}$$

is the minimal polynomial with

$$\|p_n\| = \max_{\lambda \in [a, b]} |p_n(\lambda)| = \left| \frac{1}{t_n\left(\frac{2 - b - a}{b - a}\right)} \right|.$$

Using the recurring formula for the Čebyšev polynomials we obtain for

$$\rho_n = t_n\left(\frac{2 - b - a}{b - a}\right)^{-1}$$

$$\rho_{n+1}^{-1} = 2 \frac{2 - b - a}{b - a} \rho_n^{-1} - \rho_{n-1}^{-1}, \quad \rho_0 = 1, \quad \rho_1^{-1} = \frac{2 - b - a}{b - a}$$

and

$$p_{n+1}(\lambda) = \frac{2\rho_{n+1}}{\rho_n} \frac{2\lambda - b - a}{b - a} p_n(\lambda) - \frac{\rho_{n+1}}{\rho_{n-1}} p_{n-1}(\lambda)$$

$$p_0(\lambda) = 1, \quad p_1(\lambda) = \frac{2\lambda - b - a}{2 - b - a}.$$

This gives after a short calculation, using the recursion formulas and

$$\tilde{w}_{n+1} = \tilde{x}_{n+1} - x = p_n(A) w_0$$

the semi-iterative method

$$\tilde{x}_{n+1} = \frac{4 \rho_{n+1}}{\rho_n(b-a)} \left(A\tilde{x}_n + y - \frac{b+a}{2} \tilde{x}_n \right) - \frac{\rho_{n+1}}{\rho_{n-1}} \tilde{x}_{n-1}$$

$$\tilde{x}_0 = y, \quad \tilde{x}_1 = \frac{2}{2-b-a} Ay + y$$

and the error estimate

$$\|x - \tilde{x}_n\| \leq \|p_n(A)(x - y)\| \leq |\rho_n| \|x - y\|.$$

In the third case we assume that there is known a number η such that

$$\sigma(A) \subset [-\rho, 1 - \eta] \cup [1 + \eta, \rho].$$

Since the polynomials

$$q_{2n}(\lambda) = t_n \left(\frac{2\lambda^2 - 1 - \alpha^2}{1 - \alpha^2} \right)$$

are the polynomials of minimal norm on the intervals $[-1, -\alpha] \cup [\alpha, 1]$ of degree $2n$ with $q_{2n}(1) = 1$ (cf. Achieser [1], p. 287) a linear transformation of $[-1, -\alpha] \cup [\alpha, 1]$ onto $[-\rho, 1 - \eta] \cup [1 + \eta, \rho + 2]$ (resp. $[2 - \rho, 1 - \eta] \cup [1 + \eta, \eta]$ if more convenient) and the substitution of λ by A leads to the semi-iterative method

$$x_{n+1} = \frac{4 \tau_{n+1}}{\tau_n((\rho + 1)^2 - \eta^2)} \left[(A^2 x_n - 2Ax_n + x_n - Ay - y) - \frac{1}{2}((\rho + 1)^2 + \eta^2) x_n \right] - \frac{\tau_{n+1}}{\tau_{n-1}} x_{n-1}$$

$$x_0 = y$$

$$x_1 = -\frac{2}{(\rho + 1)^2 + \eta^2} (A^2 y - Ay) + y$$

$$\tau_{n+1}^{-1} = t_n \left(-\frac{(\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2} \right)$$

$$\tau_{n+1}^{-1} = -2 \frac{(\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2} \tau_n^{-1} - \tau_{n-1}^{-1}$$

$$\tau_0 = 1$$

$$\tau_1^{-1} = -\frac{(\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2}.$$

The order of convergence of this method is

$$\|x_n - x\| \leq \|p(A)(y - x)\| = O(\tau_n^{-1}).$$

4. A ČEBYŠEV PROJECTION METHOD

Let A be again a bounded linear selfadjoint operator in a Hilbert space H with spectrum in $[a, b]$, $b < 1$.

Let $p_n : [a, b] \rightarrow \mathbb{R}$ be the polynomials from section 2, which are the proxima of $(1 - \lambda)^{-1}$ of degree n in $[a, b]$. Then for linear independent elements z_1, \dots, z_k of H we determine

$$z = \sum_{j=1}^n \gamma_j z_j$$

from the system of linear equations

$$\langle z - Az - y + (1 - A)p_n(A)y, z_j \rangle = 0$$

for $j = 1, 2, \dots, k$. Then

$$\hat{x}_n = p_n(A)y + z$$

is an approximation for the solution x of $x - Ax = y$.

If $p_n = 0$, then this method is the usual Ritz-Galerkin method, if

$$p_n(\lambda) = \sum_{j=0}^n \lambda^j,$$

then this method is the projection method of type Q_{n+1} introduced in [6].

As usual in the theory of the Ritz-Galerkin method, the optimal rate of convergence for compact A is obtained, if z_1, \dots, z_k are eigenvectors of A . In this case we get with

$$x = p_n(A)y + ((1 - A)^{-1} - p_n(A))y$$

$$\hat{x}_n = p_n(A)y + z$$

and a simple Hilbert space calculation

$$z = \sum_{j=1}^k \left(\frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right) \langle y, z_j \rangle z_j$$

$$\|x - x_n\|^2 = \sum_{j=k+1}^{\infty} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right|^2 |\langle y, z_j \rangle|^2$$

so

$$\|x - \hat{x}_n\| \leq \sup_{j \geq k+1} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right| \leq \frac{b-a}{2(\alpha^2 - 1)} \gamma^n$$

where

$$\alpha = \frac{2-b-a}{b-a}, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1}$$

as in a section 2. Also as in section 2 is shown $p_n(A)$ y can be calculated by a semi-iterative method.

5. CONCLUDING REMARKS

Niethammer [5] has shown that the order of convergence of the Čebyšev semi-iterative method tends to the order of convergence of the Čebyšev-Euler method. In [8] M. Wolf has demonstrated that the Čebyšev projection method in general gives quite better approximations than the usual Ritz-Galerkin method.

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