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$L_\infty$-error estimates for variational inequalities with Hölder continuous obstacle


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$L^\infty$-ERROR ESTIMATES FOR VARIATIONAL INEQUALITIES WITH HÖLDER CONTINUOUS OBSTACLE (*)

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Abstract — An error estimate is derived, using a linear finite element method, for the $L^\infty$-approximation of the solution of variational inequalities with Hölder continuous obstacle. If the obstacle is in $C^{0, \alpha}(\Omega) (0 < \alpha \leq 1)$, then the $L^\infty$-error for the linear element solution is in the order of $h^{\alpha-1}$ ($\forall \epsilon > 0$).

Resume. — On démontre que l'erreur d'approximation dans la norme $L^\infty$ de la solution d'une inéquation variationnelle, avec obstacle $\alpha$-holdérien ($0 < \alpha \leq 1$), par la méthode des éléments finis linéaires, est de l'ordre $h^{\alpha-1}$, pour tout $\epsilon > 0$.

1. INTRODUCTION

The interest for the study of variational inequalities (V.I.) with « irregular » obstacles has recently increased. Regularity properties of solutions have been proved for V.I. with Hölder continuous ([4], [7], [8], [12]), continuous [12], or one-sided Hölder continuous [13] obstacles.

The importance of such results lies in particular in their application to the theory of quasi-variational inequalities (Q.V.I.), namely V.I. with the obstacle depending on the solution itself. Such an implicit obstacle, in fact, is in general “fairly irregular” (see [3] for some examples connected to stochastic control theory).

From a numerical point of view, some recent results are known concerning the approximation of solutions of Q.V.I. connected to some stochastic impulse control problems (see [11], [15]), by means of finite element methods.

The aim of this paper is to show an error estimate in the $L^\infty$ norm, for the approximation, by means of linear finite elements, of the solution of variational

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inequalities with Hölder continuous obstacle. If the obstacle is in \( C^{0,\alpha}(\Omega) \), \( 0 < \alpha \leq 1 \) (so that, according to the mentioned regularity results, the solution itself is in \( C^{0,\alpha}(\Omega) \)), then, under reasonable hypotheses on the triangulation, the \( L^\infty \)-error of such an approximation is in the order of \( h^{\alpha - \epsilon} \) (for each \( \epsilon > 0 \)), that is the expected order of convergence.

In § 2 we introduce some notations and we recall the regularity of solutions. In § 3 the discretization is studied, and we state our principal result (Theorem 3.2) together with some remarks and corollaries. In § 4 we indicate some useful results which are needed, in § 5, to prove Theorem 3.2.

2. FORMULATION OF THE PROBLEM

Let \( \Omega \) be a convex bounded domain of \( \mathbb{R}^N \), with sufficiently smooth boundary \( \Gamma \) (we suppose for example \( \Gamma \in C^2 \)).

With classical notations, \( C^{0,\alpha}(\Omega) \), \( 0 < \alpha < 1 \) [\( \alpha = 1 \)], is the space of all the Hölder [Lipschitz] continuous functions of exponent \( \alpha \) over \( \Omega \), with the semi-norm

\[
[v]_\alpha = \sup_{x,y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x-y|^\alpha}.
\]

For \( p \geq 1 \), we let \( L^p(\Omega) \) denote the classical Banach space consisting of measurable functions on \( \Omega \) that are \( p \)-integrable, with the norm

\[
\| v \|_p = \left( \int_\Omega |v|^p \, dx \right)^{1/p} \quad \text{if} \quad 1 \leq p < +\infty,
\]

\[
\| v \|_\infty = \text{ess. sup}_{\Omega} |v| \quad \text{if} \quad p = \infty.
\]

Then for \( p \geq 1 \), \( m \in \mathbb{N} \), \( W^{m,p}(\Omega) \) is the classical Sobolev space defined by

\[
W^{m,p}(\Omega) = \{ v : D^\gamma v \in L^p(\Omega), \text{ for all } |\gamma| \leq m \} ;
\]

in \( W^{m,p}(\Omega) \) we introduce the norm

\[
\| v \|_{m,p} = \sum_{|\gamma| \leq m} \| D^\gamma v \|_p,
\]

and we set \( H^m(\Omega) = W^{m,2}(\Omega) \); then \( H^1_0(\Omega) \) is the closure, in the norm of \( W^{1,2}(\Omega) \), of \( C^0_0(\Omega) \), the space of all continuous functions with compact support in \( \Omega \), having all first derivatives continuous in \( \Omega \).

In the following \( c \) will be the notation for positive constants involved in calculation, and the terms on which \( c \) depends will be clarified each time.
Let $A$ be the second order linear elliptic operator defined by

$$A = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i} + c_0(x),$$

with the following assumptions:

i) $a_{ij} \in C^1(\bar{\Omega}), b_i, c_0 \in L^\infty(\Omega), i, j = 1, 2, ..., N$;

ii) There is a constant $\nu > 0$ such that (uniform ellipticity):

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2, \text{ a.e. in } \Omega, \forall \xi \in \mathbb{R}^N - \{ 0 \};$$

iii) $c_0(x) \geq \bar{c} > 0, \forall x \in \Omega$, with $\bar{c}$ sufficiently large (such that $A$ is a coercive operator on the space $H^1_0(\Omega)$).

Let $a(\cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ be the continuous and coercive bilinear form on $H^1_0(\Omega)$ associated with the operator $A$, namely, $\forall u, v \in H^1_0(\Omega)$,

$$a(u, v) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \sum_{i=1}^{N} \int_{\Omega} b_i(x) \frac{\partial u}{\partial x_i} v \, dx + \int_{\Omega} c_0(x) uv \, dx.$$

Let us now consider an "obstacle problem" for the operator $A$, i.e. the following V.I with homogeneous boundary conditions:

$$a(u, v - u) \geq (f, v - u), \quad \forall v \in K, \quad u \in K \quad (2.1)$$

where $K = \{ v \in H^1_0(\Omega) : v \geq \psi \text{ in } \Omega \}$ is a closed convex subset of $H^1_0(\Omega)$, and

$$f \in L^\infty(\Omega), \quad \psi \in C^{0,\alpha}(\bar{\Omega}), \quad 0 < \alpha \leq 1, \quad (2.2)$$

are two given functions. We assume $\psi|_{\Gamma} \leq 0$, in order to avoid $K$ being empty. Then the following regularity result is known:

**Theorem 2.1:** Under the assumptions (2.2) and (2.3), the unique solution $u$ of problem (2.1) is in $C^{\alpha}(\Omega)$.

The proof in the interior of $\Omega$ can be deduced for example from Caffarelli-Kinderlehrer [7], where it is shown that the solution of problem (2.1) has the same modulus of continuity of the obstacle. For a general proof we refer to Frehse [12], where the nonlinear case has been considered. For the case $\alpha = 1$, see also Chipot [8]. Lastly we mention the result of Biroli [4] : $u \in C^{0,\alpha'}(\bar{\Omega}), \alpha' < \alpha$, if more general boundary conditions are involved.
3. DISCRETIZATION AND PRINCIPAL RESULT

Let $\Omega_h$ denote a polyhedral domain inscribed in $\Omega$, such that the diameter of every "face" of $\Gamma_h = \partial \Omega_h$ has length less than $h$. Let us consider that over $\Omega_h$ a "triangulation" $\mathcal{T}_h$ is defined (in the usual way, see [9]), regular, in the sense that, setting $\forall T \in \mathcal{T}_h$:

$$h_T = \text{diam} (T),$$

$$\rho_T = \sup \{ \text{diam} (B) : B \subset T \text{ is a ball in } \mathbb{R}^N \},$$

then:

1. there is a constant $\sigma$ such that, $\forall T \in \mathcal{T}_h$, $\frac{h_T}{\rho_T} \leq \sigma$;
2. $h \geq \max_{T \in \mathcal{T}_h} h_T$.

A piecewise linear subspace $V_h$ can be defined on $\Omega_h$ in the following way:

$$V_h = \{ v \in C^0(\overline{\Omega}) : v_{|T} \text{ is a linear function, } \forall T \in \mathcal{T}_h; v \equiv 0 \text{ in } \overline{\Omega} - \Omega_h \}.$$

Let us denote by $\{ P_i \}_{i=1}^{r(h)}$ the internal nodes of $\mathcal{T}_h$. Then the functions $\{ \phi_i \}_{i=1}^{r(h)}$ of $V_h$ such that

$$\phi_i(P_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, r(h),$$

form a basis of $V_h$; in particular for every $v \in C^0(\overline{\Omega}) \cap H^1_0(\Omega)$ the function

$$v_i(x) = \sum_{i=1}^{r(h)} v(P_i) \phi_i(x) \quad (3.1)$$

represents the interpolate of $v$ over $\mathcal{T}_h$.

Furthermore, from the definition of $\mathcal{T}_h$,

$$P_i \in \partial T \Rightarrow T \subset B(P_i, h), \quad i = 1, 2, \ldots, r(h), \quad \forall T \in \mathcal{T}_h,$$

where $B(P_i, h)$ is the ball of $\mathbb{R}^N$ with its center in $P_i$ and radius $h$; then

$$\text{supp } \phi_i \subset \overline{B(P_i, h)}, \quad i = 1, 2, \ldots, r(h). \quad (3.2)$$

Now let us consider the discrete problem associated with (2.1):

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in K_h, \quad u_h \in K_h \quad (3.3)$$

where $K_h = \{ v_h \in V_h : v_h \geq \psi_h \}$, and $\psi_h$ is the piecewise linear function on $\Omega$.
equal to $\psi$ at the nodes of $\mathcal{C}_h$ (and defined on every connected component of $\Omega - \Omega_h$ by a constant extension in directions normal to $\Gamma_h$, see [6]).

**Remark 3.1:** Such a choice of $K_h$ means that the constraint $u_h \geq \psi$ is only imposed over the internal nodes of $\mathcal{C}_h$. It could in fact be defined in an equivalent way:

$$K_h = \{ v_h \in V_h : v_h(P_i) \geq \psi(P_i), i = 1, 2, \ldots, r(h) \} .$$

Let $M_h = (m_{ij})$ be the matrix of problem (3.3), i.e. the real $r(h) \times r(h)$ matrix whose generic term is

$$m_{ij} = a(\phi_j, \phi_i), \quad i, j = 1, 2, \ldots, r(h) .$$

The following assumption is needed:

$$m_{ij} \leq 0 \quad \text{if} \quad i \neq j, \quad i, j = 1, 2, \ldots, r(h); \quad (3.4)$$

then, by the hypotheses on the coefficients of $A$, $M_h$ is an $M$-matrix, and the discrete problem (3.3) satisfies a discrete maximum principle, in the sense of [10] (where conditions of essentially geometric type on the triangulation $\mathcal{C}_h$ are given, under which (3.4) holds).

The solution $u_h$ of (3.3) represents the approximation of the solution $u$ of (2.1) in the linear finite element discretization. Under the previous assumptions we are able to obtain an error estimate, in $L^\infty$ norm, for such an approximation.

Namely, our principal result is:

**Theorem 3.2:** If (2.2), (2.3), (3.4) hold, then $\forall p > 1$

$$\| u - u_h \|_\infty \leq c h^{a-N/p} | \log h | , \quad (3.5)$$

where $c$ depends on $\Omega$, $\psi$, $p$, and $\alpha$, not on $h$.

Estimate (3.5) is quasi-optimal. In fact the interpolation error in $L^\infty$ for Hölder continuous functions in $C^{0,\alpha}(\Omega)$ is a $O(h^\alpha)$. Here this result is shown under the hypotheses:

$$u |_\Gamma = 0 ; \quad (3.6)$$

$$\text{dist} (\Gamma, \Gamma_h) \leq c h^2 . \quad (3.7)$$

Condition (3.6) can be easily eliminated. It should also be noted that, under the assumptions made on $\Omega$ (convex, with $\Gamma \in C^2$), it is always possible to construct $\Omega_h$ such that (3.7) holds. (We remark that, in the non-convex case, assuming condition (3.7) as an hypothesis, we still obtain an estimate such as (3.5).)
Lemma 3.3: If \( u \in C^{0,\alpha}(\overline{\Omega}) \), \( 0 < \alpha \leq 1 \), and conditions (3.6), (3.7) are satisfied, then
\[
\| u - u_f \|_\infty \leq ch^2,
\]
where \( c \) depends only on \( u, \alpha \) and \( \Omega \).

Proof. — From the definition (3.1) (since \( \sum_{i=1}^{r(h)} \phi_i(x) \leq 1, \forall x \in \overline{\Omega} \)):
\[
| u(x) - u_f(x) | \leq \left(1 - \sum_{i=1}^{r(h)} \phi_i(x)\right) | u(x) | + \sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) | ;
\]
(3.8)
the first term in the right hand side of (3.8) is either equal to zero (when \( x \) belongs to the convex envelope of the internal nodes, \( \sum_{i=1}^{r(h)} \phi_i(x) = 1 \)), or, in the other case, it is less than \( ch^{2\alpha} \) (from (3.7)). For the second term we have
\[
\sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) | \leq [u]_\alpha \sum_{i=1}^{r(h)} \phi_i(x) | x - P_i |^\alpha
\]
\[
\leq [u]_\alpha h^\alpha,
\]
since, from (3.2), \( \phi_i(x) \neq 0 \) implies \( | x - P_i | < h \). □

As a corollary of theorem 3.2 we have an approximation result for the set \( D = \{ x \in \Omega : u(x) > \psi(x) \} \), where the solution does not touch the obstacle. The boundary of \( D \) is the so-called free boundary, and it is in many cases the real unknown of problems such as (2.1). Usually the convergence of \( u_h \) to \( u \) is not enough to ensure the convergence to \( D \) (in set theoretical sense) of sets \( D_h = \{ x \in \Omega : u_h(x) \geq \psi(x) \} \). However, theorem 3.2 implies:

Corollary 3.4: Under the same assumptions of theorem 3.2, the sequence \( \{ D_{h,\varepsilon} \} \), where
\[
D_{h,\varepsilon} = \{ x \in \Omega : u_h(x) > \psi(x) + h^{\alpha-\varepsilon} \} ,
\]
"converges from the interior" to \( D \), \( \forall \varepsilon > 0 \), in the sense that:

a) \( \lim_{h \to 0^+} D_{h,\varepsilon} = D \) (in set theoretical sense);

b) \( D_{h,\varepsilon} \subseteq D \), if \( h \) is sufficiently small.

(See [2] for the proof.)
4. PRELIMINARY RESULTS

Let us state some useful results in order to prove theorem 3.2.

— A priori estimates

The following relation between solutions and obstacles of two different V.I. is well known (see [5]):

**Lemma 4.1:** Let \( u \) [resp. \( w \) ] \( \in H_0^1(\Omega) \) be the unique solution of a V.I. such as (2.1), with obstacle \( \psi \) [resp. \( \varphi \) ] \( \in L^\infty(\Omega) \); then

\[
\| u - w \|_\infty \leq \| \psi - \varphi \|_\infty.
\]

The discrete analogue of lemma 4.1 is also valid (see [11]):

**Lemma 4.2:** Let \( u_h \) [resp. \( w_h \) ] denote the approximation of \( u \) [resp. \( w \) ] given by problem (3.3); if \( M_h \) satisfies (3.4), then

\[
\| u_h - w_h \|_\infty \leq \| \psi_h - \varphi_h \|_\infty.
\]

— V.I. with \( W^{2,p} \)-obstacle

Let us consider a V.I. such as (2.1), with the assumption (2.2), but now let \( \psi \in W^{2,p}(\Omega) \). Then it is well known [14] that the solution \( u \) is in \( W^{2,p}(\Omega) \). Baiocchi [1] and Nitsche [17] have already studied the approximation for the solution of this problem. In particular we have:

**Theorem 4.3:** Let \( f \in L^p(\Omega) \), \( \psi \in W^{2,p}(\Omega) \), \( \forall p < + \infty \); if (3.4) holds, then

\[
\| u - u_h \|_\infty \leq Ch^{1-N/p} \log h \{ \| u \|_{2,p} + \| \psi \|_{2,p} \}, \quad \forall p < + \infty,
\]

\( c \) independent of \( h \).

Proof of theorem 4.3 can be easily derived from [1], by means of the interpolation theory (see [9]), and of error estimates in \( L^\infty \) for solutions of equations. Estimates such as (4.1) hold in fact for equations with solutions in \( W^{2,p}(\Omega) \); they can be stated using Nitsche's techniques of weighted norms; when \( A = -\Delta \), see also [18], where a quasi-optimality result in \( L^\infty \) is given for the \( H_0^1 \)-projection into finite element spaces.
5. PROOF OF THEOREM 3.2

Without loss of generality, let us consider $\psi \mid _{\Gamma} = 0$ (such that in problem (3.3) now $\psi_h = \psi$); it can be shown in fact that solution $u$ of (2.1) is equal to solution $\hat{u}$ of

$$a(\hat{u}, z - \hat{u}) \geq (f, z - \hat{u}), \quad \forall z \in H^{1}_0(\Omega), \quad z \geq \hat{\psi}$$

$$\hat{u} \in H^{1}_0(\Omega), \quad \hat{u} \geq \hat{\psi}$$

where $\hat{\psi} = \psi \vee u_0$, and $u_0$ is the solution of the related equation

$$a(u_0, v) = (f, v), \quad v \in H^{1}_0(\Omega)$$

$$u_0 \in H^{1}_0(\Omega).$$

We have $u_0 \in W^{2,p}(\Omega), \forall p < + \infty$: hence $\hat{\psi} \in C^{0,\alpha}(\overline{\Omega})$, with the same $\alpha$ of $\psi$.

The proof of theorem 3.2 is based on a regularization procedure, consisting in the "approximation" of the initial problem by means of "more regular" V.I. (namely with $W^{2,p}$-obstacle, $\forall p < + \infty$), for which we can apply theorem 4.3. We then conveniently "go back" to problem (2.1), through continuity results. This procedure can be divided into four steps.

Step 1: Regularization by convolution.

**Lemma 5.1:** There is a sequence $\{\psi^n\}$ converging to $\psi$ in $L^{\infty}$, such that, $\forall n$,

$$\psi^n \in C^1(\overline{\Omega}), \quad \psi^n \mid _{\Gamma} = 0 , \quad (5.1)$$

$$\|\psi^n - \psi\|_{\infty} \leq cn^{-\alpha} , \quad (5.2)$$

$$\|\psi^n\|_{C^1(\overline{\Omega})} \leq cn^{1-\alpha}, \quad (5.3)$$

where $c$ depends on $\psi, \alpha, \Omega$, but not on $n$.

**Proof:** See [4]; (5.1) can be shown using convolutions of $\psi$ with suitable mollifiers and cut-off functions. $\blacksquare$

Let us call $u^n$ the solution of the V.I. (2.1) with obstacle $\psi^n$, and $u^n_0$ the solution of the corresponding discrete problem (where now the obstacle is $\psi^n$).

Step 2: Elliptic regularization.

**Lemma 5.2:** For every fixed $n$, there is a sequence $\{\psi^{n,m}\}$ converging, for $m \to + \infty$, to $\psi^n$ in $L^{\infty}$, such that $\forall m, \psi^{n,m}$ is the solution of

$$\begin{cases}
  m^{-1} A\psi^{n,m} + \psi^{n,m} = \psi^n \\
  \psi^{n,m} \mid _{\Gamma} = 0
\end{cases}$$

R.A.I.R.O. Analyse numérique/Numerical Analysis
and

\[ \psi^{n, m} \in W^{2, p}(\Omega), \quad \forall p < + \infty; \]
\[ \| \psi^{n, m} - \psi^n \|_\infty \leq cm^{-1/2} \| \psi^n \|_{1, p}, \quad \forall p < + \infty, \quad (5.4) \]
\[ \| A\psi^{n, m} \|_\infty \leq cm^{1/2} \| \psi^n \|_{1, p}, \quad \forall p < + \infty, \quad (5.5) \]

where \( c \) does not depend on \( m \) and \( n \).

(For the proof see [4] again.)

As we did in Step 1, let us call \( u^{n, m} \) the solution of the V.I. (2.1) with obstacle \( \psi^{n, m} \), and \( u_h^{n, m} \) the solution of the corresponding discrete problem. Of course \( u^{n, m} \in H^1_0(\Omega) \cap W^{2, p}(\Omega), \forall p < + \infty \); it follows

\[ \| u^{n, m} \|_{2, p} \leq c \| A\psi^{n, m} \|_p \leq c \| A\psi^{n, m} \|_\infty . \]

Furthermore the following inequality of Lewy-Stampacchia's type holds (see e.g. [16]) :

\[ f \leq A\psi^{n, m} \leq (A\psi^n) \lor f ; \]

this yields, recalling (5.5),

\[ \| u^{n, m} \|_{2, p} \leq c \| A\psi^{n, m} \|_\infty \leq cm^{1/2} \| \psi^n \|_{1, p}, \quad \forall p < + \infty . \]

Likewise,

\[ \| \psi^{n, m} \|_{2, p} \leq cm^{1/2} \| \psi^n \|_{1, p}, \quad \forall p < + \infty . \]

Applying theorem 4.3, then

\[ \| u^{n, m} - u_h^{n, m} \|_\infty \leq cm^{1/2} h^{2-\epsilon(p)} \| \psi^n \|_{1, p}, \quad \forall p < + \infty, \quad (5.6) \]

where for shortness we have set : \( h^{2-\epsilon(p)} = h^{2-N/p} | \log h | \).

Step 3 : Inversion of Step 2.

**Lemma 5.3 :** The following estimate holds :

\[ \| u^n - u_h^n \|_\infty \leq c h^{1-\epsilon(p)} \| \psi^n \|_{1, p}, \quad \forall n \in \mathbb{N}, \quad \forall p < + \infty . \quad (5.7) \]

**Proof :** For every choice of index \( m \), we have

\[ \| u^n - u_h^n \|_\infty \leq \| u^n - u^{n, m}_h \|_\infty + \| u^{n, m} - u_h^{n, m} \|_\infty + \| u_h^{n, m} - u_h^n \|_\infty , \]

and, by lemma 4.1 and (5.4), \( \forall p \),

\[ \| u^n - u^{n, m}_h \|_\infty \leq cm^{-1/2} \| \psi^n \|_{1, p} . \]
Likewise, using lemma 4.2,
\[ \| u_h^{n,m} - u_h^n \|_\infty \leq \| \psi_f^{n,m} - \psi_f^n \|_\infty \leq cm^{-1/2} \| \psi^n \|_{1,p}; \]
then, from (5.6), we obtain
\[ \| u^n - u_h^n \|_\infty \leq c(m^{-1/2} + m^{1/2} h^{2-\varepsilon(p)}) \| \psi^n \|_{1,p}, \quad \forall p < + \infty . \]
If we now choose a suitable \( m \), i.e. such that \( 1/h^2 \leq m \leq (1/h^2) + 1 \), then the proof is complete.  

**Step 4 : Inversion of Step 1.**

To complete the proof of theorem 3.2, let us use the same trick of Step 3, obtaining
\[ \| u - u_h \|_\infty \leq \| u - u^n \|_\infty + \| u^n - u_h^n \|_\infty + \| u_h^n - u_h \|_\infty; \]
according to (5.3), from (5.7) we get
\[ \| u^n - u_h^n \|_\infty \leq cn^{1-\alpha} h^{1-\varepsilon(p)}; \]
then, using lemmas 4.1 and 4.2, and (5.2),
\[ \| u - u_h \|_\infty \leq c(n^{-\alpha} + n^{1-\alpha} h^{1-\varepsilon(p)}); \]
if we now take \( n \) such that \( 1/h \leq n \leq (1/h) + 1 \), we finally have
\[ \| u - u_h \|_\infty \leq ch^{\alpha-\varepsilon(p)}, \quad \forall p < + \infty , \]
that is the thesis (3.5).  

**REFERENCES**


