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Iterative refinement of finite element approximations for elliptic problems


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ITERATIVE REFINEMENT OF FINITE ELEMENT APPROXIMATIONS FOR ELLIPTIC PROBLEMS (*)

by Lin QUN (1)

Communiqué par J A NITSCHEN

Résumé — On présente une extrapolation itérative d’approximations de problèmes elliptiques par des éléments fins de bas degré

Abstract — An iterative refinement of low-degree finite element approximations for elliptic problems is presented

1. We will consider the boundary value problem

\[ \Delta u + \sum a_i \frac{\partial u}{\partial x_i} + bu = -f \quad \text{in} \quad \Omega , \]

\[ u = 0 \quad \text{on} \quad \partial \Omega . \quad (1) \]

Here \( \Omega \subset \mathbb{R}^N \) is a bounded domain with boundary \( \partial \Omega \) sufficiently smooth. We will adopt the standard notations (cf. Gilbarg-Trudinger, 1977). Especially \( (., .) \) respective \( (., .)_1 \) denote the \( L_2(\Omega) \)-inner-product respective the Dirichlet integral and \( \| . \|_k \) the norm in \( H^k(\Omega) \).

The weak formulation of problem (1) is

\[ (u, v)_1 = (\sum a_i u \mid_i + bu + f, v) \quad \text{for} \quad v \in \tilde{H}_1 . \quad (2) \]

Our basic assumption is : problem (1) resp. (2) has a unique solution \( u \) to \( f \in H_0 \) with \( u \in \tilde{H}_1 \cap H_2 \) and \( \| u \|_2 \leq c \| f \| . \) Now let \( S_h \) be the space of linear finite

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éléments with isoparametric modifications in the boundary éléments such that $S_h \subset \bar{H}_1$ holds true. Due to an argument of Schatz (1974) for $h$ sufficiently small the Galerkin-approximation $u^0 = u_h \in S_h$ defined by

$$(u^0, \chi)_1 = \left( \sum a_i u^0 |_i + b u^0 + f, \chi \right) \quad \text{for} \quad \chi \in S_h$$

is uniquely defined. The error estimate

$$\| u - u^0 \| + h \| u - u^0 \|_1 \leq c h^2 \| u \|_2$$

is well known.

In Lin Qun (1978), (1980) we introduced a refinement of $u^0$ on the basis of the additional assumption : to $F \in H_0$ given the solution of

$$- \Delta U = F \quad \text{in} \quad \Omega,$$

$$U = 0 \quad \text{on} \quad \partial \Omega$$

resp. $U \in \bar{H}_1$ and

$$(U, v)_1 = (F, v) \quad \text{for} \quad v \in \bar{H}_1$$

is computable. Then given $u^0$ we can compute $\bar{u}^0$ defined by $\bar{u}^0 \in \bar{H}_1$ and

$$(\bar{u}^0, v)_1 = \left( \sum a_i u^0 |_i + b u^0 + f, v \right) \quad \text{for} \quad v \in \bar{H}_1.$$ (7)

This leads to a higher accuracy in the $H_1$-norm :

$$\| u - \bar{u}^0 \|_1 \leq c h^2 \| u \|_2.$$ (8)

Of course $\bar{u}^0$ is not an element of $S_h$.

Following a suggestion of Nitsche (private communication) we construct starting with the pair $(u^0, \bar{u}^0)$ iterates $(u^{\nu+1}, \bar{u}^{\nu+1})$ for $\nu \geq 0$ defined

$$u^{\nu+1} = \bar{u}^\nu + \varphi^\nu$$

with $\varphi^\nu \in S_h$ and

$$(\varphi^\nu, \chi)_1 = \left( \sum a_i \varphi^\nu |_i + b \varphi^\nu, \chi \right) =$$

$$\left( \sum a_i (\bar{u}^\nu - u^\nu) |_i + b (\bar{u}^\nu - u^\nu), \chi \right) \quad \text{for} \quad \chi \in S_h$$ (10)

and on the other hand by ($\nu \geq 0$)

$$(\bar{u}^\nu, v)_1 = \left( \sum a_i u^\nu |_i + b u^\nu + f, v \right) \quad \text{for} \quad v \in \bar{H}_1.$$ (11)
In Section 3 we give the proof of:

**Theorem 1:** Let \((u', \overline{u}')\) be defined as above. Then

\[
\| u - u' \| + \| u - \overline{u}' \|_1 \leq (ch)^{\nu+2} \| u \|_2
\]

is valid.

2. Our proof is based on the following operator framework (cf. Chatelin, 1981, Hackbusch, 1981). Let us consider the equation

\[
u = Ku + y
\]  

in a Banach-space \(X\) with \(K\) being a linear compact operator. Further let \(S\) be an approximating subspace and \(P : X \to S\) a bounded projection onto \(S\). The standard Galerkin solution is defined by

\[
u^0 = PKu^0 + Py
\]  

Now we construct iterates \(\overline{u}^\nu\) and \(u^{\nu+1}\) in the way

\[
\overline{u}^\nu = Ku^\nu + y, \quad u^{\nu+1} = \overline{u}^\nu + r^\nu
\]  

with \(r^\nu\) defined by

\[
r^\nu = PKr^\nu + PK(\overline{u}^\nu - u^\nu)
\]  

**Remark 1:** \(d^\nu = \overline{u}^\nu - u^\nu = Ku^\nu - u^\nu + y\) is the defect of the \(\nu\)-th iterate. Therefore \(r^\nu\) may be interpreted as the Galerkin-solution to the right hand side \(Kd^\nu\).

**Remark 2:** The approximations \(\overline{u}^0\) are also considered in Sloan (1976), but the higher iterates introduced there differ from ours.

**Lemma 1:** Suppose that \(K\) is compact, \(1\) is not an eigenvalue of \(K\) and \(\kappa = \| (I - P) K \|\) is sufficiently small.

Then \((I - PK)^{-1}\) exists as a bounded operator in \(X\) and the Galerkin solutions are well defined. Moreover

\[
u - u^\nu = (I - PK)^{-1} (I - P) K (u - u^{\nu-1}).
\]  

**Proof:** Since \((I - K)^{-1}\) is bounded for \(\kappa\) small enough also \((I - PK)^{-1}\) is bounded. As a consequence the Galerkin solution is uniquely defined. The identity

\[
(I - K)^{-1} = (I - PK)^{-1} + (I - PK)^{-1} (I - P) K (I - K)^{-1}
\]  

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will be useful. The solution $u$ of (12) may be written in the form

$$u = (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku.$$  \hfill (19)

Because of our construction we have

$$u^{r+1} = Ku^r + y + (I - PK)^{-1} PK(Ku^r + y - u^r)$$

$$= (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku^r.$$  \hfill (20)

Subtraction of (20) from (19) gives (17).

**Remark 3:** We mention that under our assumptions also $(I - KP)^{-1}$ exists and the recurrence relation

$$u - u^r = (I - KP)^{-1} K(I - P)(u - u^{r-1})$$  \hfill (21)

is valid. The proof is omitted.

By our assumptions $\| u^0 \|$ is bounded by a multiple of $\| y \|$. Because of

$$\| (I - PK)^{-1} \| \leq \frac{\gamma}{1 - \kappa \gamma}$$  \hfill (22)

with $\gamma$ being the norm of $\| (I - K)^{-1} \|$ we conclude from lemma 1:

**Corollary 1:** Let $\kappa = \| (I - P) K \|$ be less than the half of

$$\gamma^{-1} = \| (I - K)^{-1} \|^{-1}.$$  

Then error-estimates of the type

$$\| u - u^r \| \leq c \left\{ \frac{\kappa \gamma}{1 - \kappa \gamma} \right\}^\gamma \| y \|$$  \hfill (23)

hold true.

3. Now we come back to the situation discussed in section 1. We identify $X$ with the Hilbertspace $H_0 = L_2(\Omega)$. Since we want to work with the Ritz-method we have to impose the condition $S \subseteq \bar{H}_1$. For simplicity we focus our attention to the case $S = S_h$ is the space of linear finite elements with isoparametric modifications along the boundary. Further let $P = R_h$ be the standard Ritz-projection defined by $Pu \in S_h$ and

$$(Pu, \chi)_1 = (u, \chi)_1 \text{ for } \chi \in S_h.$$  \hfill (24)
The operator $K$ is defined by

$$w = Kv \Rightarrow w \in \hat{H}_1$$

and $(w, g)_1 = (v, -\sum (a_i g)_i + bg)$ for $g \in \hat{H}_1$. (25)

Under suitable conditions concerning the regularity of $a$, $b$ and since the original problem (1) resp. (2) is assumed to be uniquely solvable $K$ is a bounded operator from $H_0$ into $\hat{H}_1$ and hence compact as mapping of $H_0$ into itself.

By duality the error-estimate

$$\| u - Pu \| \leq ch \| u \|_1$$

is a consequence of (4). Because of

$$\| (I - P) K v \| \leq ch \| K v \|_1 \leq c' h \| v \|$$

we find

$$\kappa = \kappa_h = \| (I - P) K \| \leq ch$$

with some constant $c$.

The estimates derived in section 2 lead to

$$\| u - u^v \| \leq (ch)^v \| u - u^0 \|$$

and because of (4) to

$$\| u - u^v \| \leq (ch)^{v+2} \| u \|_2.$$ (30)

Finally the terms $\| u - \bar{u}^v \|_1$ are bounded in the same way since by definition

$$u - \bar{u}^v = K(u - u^v).$$ (31)

This completes the proof of theorem 1.

4. In this section we consider the model problem

$$- \Delta u = f(\cdot, u) \quad \text{in} \quad \Omega$$

$$u = 0 \quad \text{on} \quad \partial \Omega$$

in two or three space dimensions. The weak formulation of (32) is: Find $u \in \hat{H}_1$ such that

$$(u, v)_1 = (f(u), v) \quad \text{for} \quad v \in \hat{H}_1.$$ (33)
Our assumptions are:

(i) \( f(x, z) \) is twice continuously differentiable with respect to \( z \in \mathbb{R} \) and

\[
|f_{zz}(x, z)| \tag{34}
\]

is uniformly bounded.

(ii) For \( z = u(x) \in C^0(\Omega) \) the functions \( f(x, u(x)), f_z(x, u(x)) \) and \( f_{zz}(x, u(x)) \) are in \( C^0(\Omega) \).

(iii) \( u \) is an isolated solution of (32), i.e. the linear problem

\[
<w, g> = (f(u, w, g) \quad \text{for} \quad g \in \mathcal{H}_1 \tag{35}
\]

admits only \( w = 0 \) in \( \mathcal{H}_1 \).

Now let \( u^0 = u_h \in S_h \) be the solution of the corresponding Galerkin-problem

\[
(u^0, \chi)_1 = (f(u^0), \chi) \quad \text{for} \quad \chi \in S_h \tag{36}
\]

Corresponding to section 1 we define the iterates \( u^\nu \) for \( \nu \geq 0 \) by

\[
<w, g> = (f(u^\nu, g) \quad \text{for} \quad g \in \mathcal{H}_1 \tag{37}
\]

and

\[
u^{\nu+1} = u^\nu + \varphi^\nu \tag{38}
\]

with \( \varphi^\nu \in S_h \) and

\[
(\varphi^\nu, \chi)_1 = (f(u^0)(\varphi^\nu + u^\nu - u^\nu), \chi) \quad \text{for} \quad \chi \in S_h \tag{39}
\]

The counterpart of theorem 1 is:

**THEOREM 2**: Let \( (u^\nu, u^\nu) \) be defined as above. Then

\[
\| u - u^\nu \| + \| u - \mathcal{H}_1 \|_2 \leq c_1(c_2 h^2)^{\nu+1} \tag{40}
\]

is valid. The constants \( c_1, c_2 \) depend on \( u \) and bounds of \( f_z, f_{zz} \) but are independent of \( h \) and \( \nu \).

**Proof**: Let \( K : H_0 \rightarrow \mathcal{H}_1 \cap H_2 \) be the inverse of the Laplacian defined by

\[
w = Kv \iff (w, g)_1 = (v, g) \quad \text{for} \quad g \in \mathcal{H}_1 \tag{41}
\]

and let \( P = R_h \) be the Ritz operator defined by

\[
\Phi = Pv \iff \Phi \in S_h \quad \text{and} \quad (\Phi, \chi)_1 = (v, \chi)_1 \quad \text{for} \quad \chi \in S_h \tag{42}
\]
Problem (32) is equivalent to \( u = Kf(u) \). We may rewrite this in the form
\[
(I - PKf'(u^0))u = Kf(u) - PKf'(u^0)u.
\]

In terms of \( K \) and \( P \) the iterates \( \bar{u}^\nu \) and \( \varphi^\nu \) have the representation
\[
\bar{u}^\nu = Kf(u^\nu),
\]
\[
(I - PKf'(u^0))\varphi^\nu = PKf'(u^0)(\bar{u}^\nu - u^\nu).
\]

This leads to
\[
(I - PKf'(u^0))u^{\nu+1} = Kf(u^\nu) - PKf'(u^0)u^\nu.
\]

By comparison of (43) and (46) and by adding and subtracting appropriate terms we come to
\[
(I - PKf'(u^0))(u^{\nu+1} - u) = (I - P)Kf'(u^0)(u^\nu - u) +
\]
\[
+ K \{ f(u^\nu) - f(u) - f'(u)(u^\nu - u) + f'(u) - f(u^0) \} (u^\nu - u).
\]

The Ritz operator \( P \) is the orthogonal projection in \( H^1 \) onto \( S = S_h \). For \( v, w \in H_0 \) arbitrary we get
\[
((I - P)Kv, w) = ((I - P)Kv, Kw)_1
\]
\[
= ((I - P)Kv, (I - P)Kw)_1
\]
\[
\leq ch^2 \| Kv \|_2 \| Kw \|_2 \leq ch^2 \| v \| \| w \|.
\]

This implies that the norm of \( (I - P)K \) as mapping of \( H_0 \) into \( H_0 \) is bounded by \( ch^2 \). Next let \( a \) be a continuous function and \( v, w \in H_0 \). Then also \( K(aww) \) is in \( H_0 \) and
\[
\| K(aww) \| \leq c \| v \| \| w \|.
\]

This follows from
\[
\| K(aww) \|_0 = \sup \{ (Kaww, g) \| g \| = 1 \}
\]
and
\[
(K(aww), g) = (v, \{ aKg \} w)
\]
in combination with Sobolev's embedding lemma.

For \( h \) small enough the initial Galerkin solution \( u^0 \) is "near" to \( u \). Because of our assumption (iii) then the operator \( I - PKf'(u^0) \) will have a bounded inverse.

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By the aid of these arguments we derive from the recurrence relation (47) the corresponding error bound

\[ \| u^{\nu+1} - u \| \leq c_3 h^2 \| u^{\nu} - u \| + c_4 \| u^{\nu} - u \|^2 + c_5 \| u^0 - u \| \| u^{\nu} - u \|. \]  

(52)

For the sake of clarity we have numbered the constants. Since an estimate of the type

\[ \| u^0 - u \| \leq c h^2 \]  

(53)

holds true anyway we derive from (52)

\[ \| u^{\nu+1} - u \| \leq c_6 h^2 \| u^{\nu} - u \| + c_4 \| u^{\nu} - u \|^2 . \]  

(54)

Because of (53) by complete inductions there is a constant \( c_7 \) such that for \( h < h_0 \) with \( h_0 \) chosen appropriate the relation

\[ \| u^{\nu+1} - u \| \leq c_7 h^2 \| u^{\nu} - u \| \]  

(55)

holds true (55) together with (53) lead to the error bound stated in theorem 2 for \( u^{\nu} - u \).

Because of

\[ \tilde{u}^{\nu} - u = K (f(u^{\nu}) - f(u)) \]  

(56)

we come to

\[ \| \tilde{u}^{\nu} - u \|_2 \leq c \| f(u^{\nu}) - f(u) \| \leq c \| u^{\nu} - u \|. \]  

(57)

**Remark 3**: Whereas assumption (iii) is essential the two preceding ones can be reduced.

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