RAIRO
ANALYSE NUMÉRIQUE

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RAIRO – Analyse numérique, tome 16, n° 2 (1982), p. 113-128.
<http://www.numdam.org/item?id=M2AN_1982__16_2_113_0>

R.A.I.R.O. Analyse numérique/Numerical Analysis
(vol. 16, n° 2, 1982, p. 113 à 128)

THE ERROR ESTIMATES FOR THE INFINITE ELEMENT METHOD FOR EIGENVALUE PROBLEMS (*)

by Houde HAN [1]

Communicated by P. G. CIARLET

Résumé. — *Cet article analyse les estimations d'erreur pour la méthode des éléments « infinis ». Le domaine est partagé en un nombre infini de triangles. On utilise des éléments linéaires pour les deux espaces en cause.*

Abstract. — *This paper analyses the error estimates for infinite element method applied to eigenvalue problems. The domain is divided into infinitely many triangles and linear elements are used for the trial space and test space.*

1. INTRODUCTION

In the numerical solution of elliptic boundary value problems, it is well-known that the presence of corners in the domain can cause a loss of accuracy in the solution. Many methods have been developed to overcome the loss of accuracy, such as the use of singular functions [9], mesh refinements [10], and the infinite element method [11, 2]. The infinite element method may be considered as a kind of mesh refinement, but has the advantages that the refinement is easy to construct, the stiffness matrix is calculated more efficiently, and an approximate solution is obtained which itself has a singularity at the corner. Recently, we showed how the infinite element method may be applied to eigenvalue problems on domains with corners [1]. In this paper, we obtain the error estimates for the infinite element method, when applied to an eigenvalue problem.

(*) Received in June 1981.
(1) Department of Mathematics, University of Maryland, College Park, Maryland, 20742, U.S.A., and Department of Mathematics, Peking University, Beijin, The People's Republic of China.

R.A.I.R.O. Analyse numérique/Numerical Analysis, 0399-0516/1982/113/$ 5.00

We consider the following eigenvalue problem.

$$\Delta u + \lambda u = 0\,, \qquad \text{in } \Omega\,, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = 0\,, \qquad \text{on } \Gamma_0 \text{ and } \Gamma_M\,, \tag{1.2}$$

$$u = 0\,, \qquad \text{on } \Gamma^0\,. \tag{1.3}$$

Here Ω is an open polygonal domain in the x_1, x_2 plane and $\Gamma = \bigcup_{j=0}^{M} \Gamma_j$ is the boundary of Ω, with the Γ_j's denoting the side of Ω.

$\Gamma^0 = \bigcup_{j=1}^{M-1} \Gamma_j$ and $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u on Γ_0 and Γ_M. A_i ($i = 0, 1, \ldots, M$) denotes the vertice of Ω with φ_i being the interior angle of Ω as shown in figure 1.

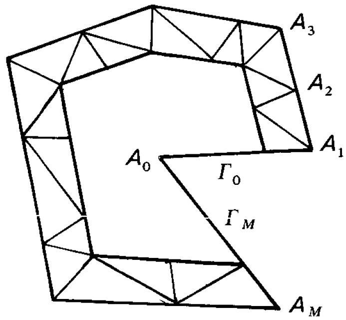

Figure 1.

For the sake of simplicity, we suppose

$$\pi < \varphi_0 \leqslant 2\pi; \quad 0 < \varphi_1\,, \quad \varphi_M \leqslant \frac{\pi}{2}; \quad 0 < \varphi_j \leqslant \pi \quad (j = 2, \ldots, M-1)\,. \tag{1.4}$$

Without losing generality, we assume A_0 is the origin of the rectangular coordinate system.

Let $W^{m,p}(\Omega)$ denote the Sobolev space on Ω with norm

$$\| u \|_{W^{m,p}(\Omega)} = \left\{ \sum_{i=0}^{m} \iint_{\Omega} \sum_{\alpha_1+\alpha_2=i} \left| \frac{\partial^i u}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}} \right|^p dx_1\, dx_2 \right\}^{1/p},$$

where m is a non-negative integer and p is a positive real number. As usual, when $p = 2$, $w^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$; when $m = 0$, $w^{0,p}(\Omega)$ is denoted by $L_p(\Omega)$. Moreover, we shall introduce the Sobolev space with weight, $H^{m,t}(\Omega)$, with the norm

$$\left.\begin{aligned} \| u \|^2_{H^{m,t}(\Omega)} &= \| u \|^2_{H^{m-1}(\Omega)} + \\ &\quad + \iint_\Omega \sum_{\alpha_1+\alpha_2=m} r^{2t} \left| \frac{\partial^m u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right|^2 dx_1\, dx_2,\ m \geqslant 1 \\ \| u \|^2_{H^{0,t}(\Omega)} &= \iint_\Omega r^{2t} | u |^2 dx_1\, dx_2 \end{aligned}\right\} \quad (1.5)$$

where $r^2 = x_1^2 + x_2^2$ and t is a real number.

Let

$$\overset{*}{H}{}^1(\Omega) = \{ u \mid u \in H^1(\Omega) \text{ and } u = 0 \text{ on } \Gamma^0 \quad \text{(in the trace sense)} \}.$$

$\overset{*}{H}{}^1(\Omega)$ is a subspace of $H^1(\Omega)$.

We know that the eigenvalue problem (1.1)-(1.3) has the following variational form : find a complex numbers λ and a nonzero $u \in \overset{*}{H}{}^1(\Omega)$ such that

$$B(u, v) = \lambda J(u, v), \quad \forall v \in \overset{*}{H}{}^1(\Omega), \quad (1.6)$$

where

$$B(u, v) = \iint_\Omega \left(\frac{\partial u}{\partial x_1} \frac{\partial \bar{v}}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial \bar{v}}{\partial x_2} \right) dx_1\, dx_2,$$

$$J(u, v) = \iint_\Omega u\bar{v}\, dx_1\, dx_2.$$

Let us now recall the procedure of obtaining the approximate solution of (1.6) using the infinite element method [1]. In the first step the domain Ω is divided into infinitely many similar element layers $D_1, D_2, \ldots, D_k, \ldots$, where D_k denotes the k-th layer. Every layer is divided into several triangles in the same manner as in [1]. $0 < \xi < 1$ is the constant of proportionality. Point A_0 is the center of similarity. Therefore

$$\Omega = \bigcup_{k=1}^{\infty} D_k.$$

Let

$$\Omega_N = \bigcup_{k=N+1}^{\infty} D_k \quad (N = 1, 2, \dots).$$

Let h denote the length of the side which is the longest among all triangles in Ω. Moreover, we suppose the angles of all triangles are greater than γ_0, where $0 < \gamma_0 < \pi/3$ is a constant. This criterion is called the smallest interior angle condition.

For this partition we introduce a closed subspace of $\overset{*}{H}{}^1(\Omega)$ denoted by $\overset{\circ}{S}(\Omega)$:

$$\overset{\circ}{S}(\Omega) = \{ u \mid u \in \overset{*}{H}{}^1(\Omega) \cap C(\Omega) \text{ and } u \text{ is a linear function on each triangle} \}.$$

Using the space $\overset{\circ}{S}(\Omega)$ instead of $\overset{*}{H}{}^1(\Omega)$ in the problem (1.6) we obtain the following eigenvalue problem : find a complex numbers λ and a nonzero $u \in \overset{\circ}{S}(\Omega)$, such that

$$B(u, v) = \lambda J(u, v), \quad \forall v \in \overset{\circ}{S}(\Omega). \tag{1.7}$$

The eigenvalue problem (1.7) is the discretized model of (1.6). (1.7) is equivalent to the following eigenvalue problem of the pencil of the infinite dimensional matrices

$$[Q_1 - \lambda Q_2] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \\ \vdots \end{bmatrix} = 0 \tag{1.8}$$

where

$$Q_1 = \begin{pmatrix} K & -A^T & & \bigcirc \\ -A & K & -A^T & \\ & \cdots\cdots & \cdots\cdots & \\ \bigcirc & & \cdots\cdots & \cdots \end{pmatrix}_{\infty \times \infty},$$

$$Q_2 = \begin{pmatrix} L & -\xi^2 D^T & & \bigcirc \\ -\xi^2 D & \xi^2 L & -\xi^4 D^T & \\ & \cdots\cdots & \cdots\cdots & \\ \bigcirc & & \cdots\cdots & \cdots \end{pmatrix}_{\infty \times \infty},$$

$K = K_0 + K_0'$ and $L = \xi^2 L_0 + L_0'$. The matrices K_0, K_0', A and L_0, L_0', D constitute the stiffness matrix of the k-th layer

$$\begin{pmatrix} K_0 & -A^T \\ -A & K_0' \end{pmatrix} - \lambda \xi^{2(k-1)} \begin{pmatrix} L_0 & -D^T \\ -D & L_0' \end{pmatrix}$$

corresponding to equation (1.1). We do not have a method to solve the eigenvalue problem (1.8) in the present form. By means of a technique in [1], the eigenvalue problem (1.8) was changed to a eigenvalue problem of the pencil of finite dimensional matrices :

$$[Q_1^N - \lambda Q_2^N] \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = 0 , \qquad (1.9)$$

where $y_1, ..., y_N$ are M-dimensional column vectors,

$$Q_1^N = \begin{pmatrix} K & -A^T & & & & \bigcirc \\ -A & K & -A^T & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -A & K & -A^T \\ \bigcirc & & & & -A & K - A^T X(0) \end{pmatrix}_{(M \times N) \times (M \times N)} ,$$

$$Q_2^N = \begin{pmatrix} L & -\xi^2 D^T & & & & \bigcirc \\ -\xi^2 D & \xi^2 L & -\xi^4 D^T & & & \\ & & \cdots & \cdots & \cdots & \\ & & & -\xi^{2(N-2)} D & \xi^{2(N-2)} L & -\xi^{2(N-1)} D^T \\ \bigcirc & & & & -\xi^{2(N-1)} D & \xi^{2(N-1)} L_0' \end{pmatrix}_{(M \times N) \times (M \times N)}$$

Here $X(0)$ is a solution of the matrix equation

$$- A + KX(0) - A^T(X(0))^2 = 0$$

of which the solution can be obtained using the direct method or the iterative method [2], [3].

The solutions of (1.9) are the approximate solutions of (1.6). In this paper we shall study error estimates.

2. THE VARIATIONAL FORM OF (1.9)

Prior to discussing the error estimate for the solution of (1.6) and (1.9) we introduce a variational form of the matrix eigenvalue problem (1.9). Consider the following eigenvalue problem. Find a complex numbers λ and a nonzero function $u \in \overset{\circ}{S}(\Omega)$ such that

$$B(u, v) = \lambda J_N(u, v), \quad \forall v \in \overset{\circ}{S}(\Omega), \tag{2.1}$$

where

$$J_N(u, v) = \iint_{\Omega/\Omega_N} u\bar{v}\, dx_1\, dx_2 . \tag{2.2}$$

For the problem (2.1), if $k \leqq N$ the stiffness matrix of the k-th layer is

$$\begin{pmatrix} K_0 & -A^T \\ -A & K_0' \end{pmatrix} - \lambda\xi^{2(k-1)} \begin{pmatrix} L_0 & -D^T \\ -D & L_0' \end{pmatrix}$$

and, when $k > N$, the stiffness matrix of k-th layer is

$$\begin{pmatrix} K_0 & -A^T \\ -A & K_0' \end{pmatrix}.$$

It is straightforward to show that the problem (2.1) is equivalent to the following eigenvalue problem :

$$\begin{aligned}
&(K - \lambda L)\, y_1 - (A^T - \lambda\xi^2 D^T)\, y_2 = 0\,, \\
&-(A - \lambda\xi^{2(k-1)} D)\, y_{k-1} + (K - \lambda\xi^{2(k-1)} L)\, y_k - \\
&\qquad - (A^T - \lambda\xi^{2k} D^T)\, y_{k+1} = 0\,, \quad k = 2, \dots, N-1\,, \qquad (2.3)\\
&-(A - \lambda\xi^{2(N-1)} D)\, y_{N-1} + (K - \lambda\xi^{2(N-1)} L_0')\, y_N - A^T y_{N+1} = 0\,, \\
&-A y_{k-1} + K y_k - A^T y_{k+1} = 0\,, \quad k = N+1, N+2, \dots
\end{aligned}$$

From the above equations it is seen that λ does not appear in the last part of (2.3). Therefore, we consider the system of infinitely many equations

$$\left.\begin{aligned} -A y_{k-1} + K y_k - A^T y_{k+1} = 0 \\ k = N+1, N+2, \dots \end{aligned}\right\}. \tag{2.4}$$

From Lemma 1.4 in [1] we know that, for any given y_N, problem (2.4) has a unique solution $\{ y_N, y_{N+1}, \dots \}$ which corresponds to the function $u_N \in H^1(\Omega_N)$ and

$$y_{k+1} = X(0)\, y_k\,, \quad k = N, N+1, \dots \tag{2.5}$$

We have :

LEMMA 2.1 : *Suppose* $\lambda^{h,N}$ *is an eigenvalue of* (2.3) *and* $u^{h,N}$ *is an eigenfunction corresponding to* $\lambda^{h,N}$. *Then* $y_0^{h,N}, y_1^{h,N}, \ldots, y_N^{h,N}$ *are not all zero-vectors* ($u^{h,N}$ *corresponds to the sequence* $y_0^{h,N}, y_1^{h,N}, \ldots, y_k^{h,N}, \ldots$).

Proof : Suppose the conclusion is false, then $y_0^{h,N} = y_1^{h,N} = \cdots = y_N^{h,N} = 0$. Since $u^{h,N}$ is an eigenfunction corresponding to $\lambda^{h,N}$, we know that

$$y_1^{h,N}, y_2^{h,N}, \ldots, y_k^{h,N}, \ldots$$

satisfy the system (2.3) in which $\lambda^{h,N}$ is used instead of λ. Moreover, from (3.5) we obtain

$$y_{N+l}^{h,N} = (X(0))^l \, y_N^{h,N} = 0 \,, \quad l = 1, 2, \ldots$$

Consequently, we have $y_0^{h,N} = y_1^{h,N} = \cdots = y_N^{h,N} = y_{N+1}^{h,N} = \cdots = 0$. Namely $u^{h,N} \equiv 0$. This contradicts the fact that $u^{h,N}$ is an eigenfunction. This contradiction shows that our conclusion is correct.

LEMMA 2.2 : *The eigenvalue problem* (2.3) *is equivalent to* (1.9).

Proof : Suppose $\lambda^{h,N}$ is an eigenvalue of (2.3) and $\{ y_1^{h,N}, \ldots, y_k^{h,N}, \ldots \}$ is an eigenvector corresponding to $\lambda^{h,N}$. Since $y_{N+1}^{h,N} = X(0) \, y_N^{h,N}$, we know that $\lambda^{h,N}$; $y_1^{h,N}, y_2^{h,N}, \ldots, y_N^{h,N}$ satisfies (1.9) and that, from Lemma 2.1,

$$y_1^{h,N}, \ldots, y_N^{h,N}$$

are not all zero vectors. Therefore $\lambda^{h,N}$; $y_1^{h,N}, \ldots, y_N^{h,N}$ is a solution of the eigenvalue problem (1.9).

On the other hand, if $\lambda^{h,N}, y_1^{h,N}, \ldots, y_N^{h,N}$ is a solution of (1.9) let

$$y_{N+l}^{h,N} = (X(0))^l \, y_N^{k,N} \,, \quad l = 1, 2, \ldots .$$

Obviously, $\lambda^{h,N}, y_1^{h,N}, y_2^{h,N}, \ldots, y_N^{h,N}, y_{N+1}^{h,N}, \ldots$ is a solution of the eigenvalue problem (2.3).

From the above lemma we have

LEMMA 2.3 : *The eigenvalue problem* (2.1) *is equivalent to* (1.9). *The variational form* (2.1), *instead of* (1.9), *will be used for the following discussion.*

3. ERROR ESTIMATE

Before the discussion of the error estimate we recall some results which are used in this paper.

LEMMA 3.1 : *There exists a constant* $\alpha > 0$ *such that*

$$B(u, u) \geqslant \alpha \| u \|^2_{H^1(\Omega)}, \quad \forall u \in \overset{*}{H}{}^1(\Omega). \tag{3.1}$$

LEMMA 3.2 : *For any* $f \in H^{0,t}(\Omega)$ (*where* $1 - \pi/\varphi_0 < t < 1$), *then*

$$B(u, v) = J(f, v), \quad \forall v \in \overset{*}{H}{}^1(\Omega) \tag{3.2}$$

has a unique solution $u \in H^{2,t}(\Omega) \cap \overset{*}{H}{}^1(\Omega)$ *and there exists a constant* $c > 0$ *independent of* f *such that*

$$\| u \|_{H^{2,t}(\Omega)} \leqq c \| f \|_{H^{0,t}(\Omega)}. \tag{3.3}$$

LEMMA 3.3 : *For any* $u \in H^{2,t}(\Omega) \cap \overset{*}{H}{}^1(\Omega)$ ($1 - \pi/\varphi_0 < t < 1$) *there exists a function* $u_I \in \overset{\circ}{S}(\Omega)$ *such that*

$$\| u - u_I \|_{H^1(\Omega)} \leqslant ch \| u \|_{H^{2,t}(\Omega)}, \tag{3.4}$$

where c is a constant which is independent of u and h. The proof of Lemma 3.1 *can be found in* [8]. *The Lemma* 3.2 *is quoted from* [4], *whereas Lemma* 3.3 *is from Theorem* 1 *in* [5].

LEMMA 3.4 : *For any* $f \in H^0(\Omega)$, *the problem*

$$B(u, v) = J(f, v), \quad \forall v \in \overset{\circ}{S}(\Omega) \tag{3.5}$$

has a unique solution $u^h \in \overset{\circ}{S}(\Omega)$ *and*

$$\| u^h \|_{H^1(\Omega)} \leqslant \frac{1}{\alpha} \| f \|_{H^0(\Omega)}. \tag{3.6}$$

Proof : Since $f \in H^0(\Omega)$, we know that f is a bounded linear functional on the space $\overset{\circ}{S}(\Omega)$. Moreover, from Lemma 3.1 we note that $B(u, v)$ is a positive definite bilinear form. Thus there exists a unique solution $u^h \in \overset{\circ}{S}(\Omega)$ such that

$$B(u^h, v) = J(f, v), \quad \forall v \in \overset{\circ}{S}(\Omega). \tag{3.5$'$}$$

Taking $v = u^h$ in (3.5)′ we obtain (3.6) from Lemma 3.1. Similarly, we can prove the following Lemma.

LEMMA 3.5 : *For any* $f \in H^0(\Omega)$ *the problem*

$$B(u, v) = J_N(f, v), \quad \forall v \in \overset{\circ}{S}(\Omega) \tag{3.7}$$

has a unique solution $u^{h,N} \in \overset{\circ}{S}(\Omega)$ *and*

$$\| u^{h,N} \|_{H^1(\Omega)} \leqq \frac{1}{\alpha} \| f \|_{H^0(\Omega)} . \tag{3.8}$$

From Lemma 3.2 we know that, for any $f \in H^0(\Omega)$, problem (3.2) always has a unique solution $u \in H^{2,t}(\Omega) \cap \overset{*}{H}{}^1(\Omega)$. Therefore, we denote by T the linear operator which maps f to u. Moreover, we know that T is a compact operator from $H^0(\Omega)$ to $H^0(\Omega)$. It satisfies

$$B(Tf, v) = J(f, v), \quad \forall v \in \overset{*}{H}{}^1(\Omega) . \tag{3.9}$$

An eigenvalue of T is a real number μ (because T is a self-adjoint operator) such that $Tu = \mu u$ for some non-zero function $u \in H^0(\Omega)$. Clearly, for any non-zero eigenvalue μ of T, we have that $\lambda = 1/\mu$ is an eigenvalue of (1.6). On the other hand, since λ is an eigenvalue of (1.6), $\mu = 1/\lambda$ is also an eigenvalue of T.

Similarly, from Lemma 3.4 we know that, for any $f \in H^0(\Omega)$, (3.5) has a unique solution $u^h \in \overset{\circ}{S}(\Omega) \subset H^1(\Omega)$. We denote by T_h the linear operator which maps f to u^h. T_h is compact from $H^0(\Omega)$ to $H^0(\Omega)$. T_h satisfies

$$B(T_h f, v) = J(f, v), \quad \forall v \in \overset{\circ}{S}(\Omega) . \tag{3.10}$$

Obviously, λ^h is an eigenvalue of (1.7) if and only if $\mu(h) = 1/\lambda^h$ is an eigenvalue of T_h.

Let T_h^N denote the linear operator which takes $f \in H^0(\Omega)$ to $u^{h,N}$; T_h^N satisfies

$$B(T_h^N f, v) = J_N(f, v), \quad \forall v \in \overset{\circ}{S}(\Omega) . \tag{3.11}$$

Obviously, for any non-zero eigenvalue $\mu^N(h)$ of T_h^N, $\lambda^{h,N} = 1/\mu^N(h)$ is an eigenvalue of problem (2.1). Namely, $\lambda^{h,N} = 1/\mu^N(h)$ is an eigenvalue of (1.9). Since $u^{h,N} \in \overset{\circ}{S}(\Omega) \subset H^1(\Omega)$, we know that T_h^N is a compact operator from $H^0(\Omega)$ to $H^0(\Omega)$.

Now the error estimates for the solutions of (1.6) and (1.9) are reduced to those for the eigenvalues and eigenfunctions of the compact operators T and T_h^N.

Let :

$$\| T \| = \sup_{\substack{f \neq 0 \\ f \in H^0(\Omega)}} \frac{\| Tf \|_{H^0(\Omega)}}{\| f \|_{H^0(\Omega)}}$$

denote the norm of the operator T. We need to estimate the error $\| T - T_h^N \|$.

Moreover, we have

$$\| T - T_h^N \| \leqslant \| T - T_h \| + \| T_h - T_h^N \| . \tag{3.12}$$

In order to get the error $\| T - T_h^N \|$, we estimate $\| T - T_h \|$ and $\| T_h - T_h^N \|$.
We have :

LEMMA 3.6 : *For any $f \in H^0(\Omega)$ there exists a constant C which is independent of f such that*

$$\inf_{v \in \overset{\circ}{S}(\Omega)} \| Tf - v \|_{H^1(\Omega)} \leqq Ch \| f \|_{H^0(\Omega)} . \tag{3.13}$$

Proof : By Lemma 3.3, we know that $Tf \in H^{2,t}(\Omega)$ and there exists a function $v_I \in \overset{\circ}{S}(\Omega)$ such that

$$\| Tf - v_I \|_{H^1(\Omega)} \leqq Ch \| Tf \|_{H^{2,t}(\Omega)} .$$

On the other hand, we know from Lemma 3.2

$$\| Tf \|_{H^{2,t}(\Omega)} \leqq C \| f \|_{H^{0,t}(\Omega)} \leqq C \| f \|_{H^0(\Omega)} ,$$

here C is a constant independent of h and f. Therefore we get

$$\| Tf - v_I \|_{H^1(\Omega)} \leqq Ch \| f \|_{H^0(\Omega)} .$$

Thus

$$\inf_{v \in \overset{\circ}{S}(\Omega)} \| Tf - v \|_{H^1(\Omega)} \leqq \| Tf - v_I \| \leqq Ch \| f \|_{H^0(\Omega)} .$$

LEMMA 3.7 : *There exists a constant C such that*

$$\| T - T_h \| \leqq Ch^2 . \tag{3.14}$$

Proof : Since $\overset{\circ}{S}(\Omega)$ is a subspace of $\overset{*}{H}{}^1(\Omega)$, for any $f \in H^0(\Omega)$ from (3.2) and (3.5)′, we obtain

$$B((T - T_h) f, v) = 0 , \quad \forall v \in \overset{\circ}{S}(\Omega) . \tag{3.15}$$

By Lemma 3.1 we have

$$\| (T - T_h) f \|_{H^1(\Omega)}^2 \leqq \frac{1}{\alpha} | B((T - T_h) f, (T - T_h) f) |$$

$$= \frac{1}{\alpha} | B((T - T_h) f, (T - T_h) f - v) | , \quad \forall v \in \overset{\circ}{S}(\Omega) .$$

Therefore, we get

$$\| (T - T_h) f \|_{H^1(\Omega)} \leqslant \frac{1}{\alpha} \inf_{v \in \mathring{S}(\Omega)} \| Tf - v \|_{H^1(\Omega)} .$$

From Lemma 3.6 it follows that

$$\| (T - T_h) f \|_{H^1(\Omega)} \leqslant Ch \| f \|_{H^0(\Omega)} . \tag{3.16}$$

On the other hand, for any $f, \psi \in H^0(\Omega)$ we have

$$\begin{aligned} J((T - T_h) f, \psi) &= B((T - T_h) f, T\psi) \\ &= B((T - T_h) f, T\psi - v) , \quad \forall v \in \mathring{S}(\Omega) . \end{aligned}$$

Consequently

$$\begin{aligned} | J((T - T_h) f, \psi) | &\leqq \| (T - T_h) f \|_{H^1(\Omega)} \inf_{v \in \mathring{S}(\Omega)} \| T\psi - v \|_{H^1(\Omega)} \\ &\leqslant Ch^2 \| f \|_{H^0(\Omega)} \| \psi \|_{H^0(\Omega)} . \end{aligned}$$

The last inequality is from (3.16) and Lemma 3.6. Thus we have

$$\| (T - T_h) f \|_{H^0(\Omega)} = \sup_{\| \psi \|_{H^0(\Omega)} = 1} | J((T - T_h) f, \psi) | \leqq Ch^2 \| f \|_{H^0(\Omega)} .$$

Finally, from the above inequality we get (3.14).

LEMMA 3.8 : *For any fixed* $0 < \varepsilon < 1$, *there exists a constant* $C(\varepsilon)$ *which is independent of* N *and* ξ *such that*

$$| J(u, v) - J_N(u, v) | \leqq C(\varepsilon) (\xi^N)^{1-\varepsilon} \| u \|_{H^0(\Omega)} \| v \|_{H^1(\Omega)} ,$$
$$\forall u \in H^0(\Omega) , \quad v \in H^1(\Omega) . \tag{3.17}$$

Proof : Based on the imbedding theorem of the Sobolev space [7] we know that for any real number $p \geqslant 1$ there exists a constant $C_1(p)$ such that

$$\| v \|_{L_p} \leqslant C_1(p) \| v \|_{H^1(\Omega)} , \quad \forall v \in H^1(\Omega) . \tag{3.18}$$

From the definition of $J_N(u, v)$ and upon repeated use of the Cauchy inequality, we obtain

$$\begin{aligned} | J(u, v) - J_N(u, v) | &= \left| \iint_{\Omega_N} u\bar{v} \, dx_1 \, dx_2 \right| \\ &\leqq \left\{ \iint_{\Omega_N} | u |^2 \, dx_1 \, dx_2 \right\}^{1/2} \left\{ \iint_{\Omega_N} | v |^2 \, dx_1 \, dx_2 \right\}^{1/2} \end{aligned}$$

$$\leqq \| u \|_{H^0(\Omega)} \left\{ \iint_{\Omega_N} | v |^2 \, dx_1 \, dx_2 \right\}^{1/2}$$

$$\leqq \| u \|_{H^0(\Omega)} (\text{meas } \Omega_N)^{\frac{1}{2} - \frac{1}{p}} \left(\iint_{\Omega_N} | v |^p \, dx_1 \, dx_2 \right)^{\frac{1}{p}}$$

$$\leqslant (\text{meas } \Omega)^{\frac{1}{2} - \frac{1}{p}} (\xi^N)^{1 - \frac{2}{p}} \| u \|_{H^0(\Omega)} \| v \|_{L_p(\Omega)} ,$$

$$\forall p > 2 , \quad \forall u \in H^0(\Omega) , \quad v \in H^1(\Omega) .$$

Taking $p = 2/\varepsilon$, $C(\varepsilon) = (\text{meas } \Omega)^{\frac{1-\varepsilon}{2}} C_1(2/\varepsilon)$ and using (3.18), we obtain (3.17).

LEMMA 3.9 : *For any fixed* $0 < \varepsilon < 1$, *there exists a constant* $C(\varepsilon)$, *such that*

$$\| T_h - T_h^N \| \leqq \frac{C(\varepsilon)}{\alpha} (\xi^N)^{1-\varepsilon} . \tag{3.19}$$

Proof : Equations (3.10) and (3.11) yield

$$B((T_h - T_h^N) f, v) = J(f, v) - J_N(f, v) , \quad \forall v \in \mathring{S}(\Omega) \tag{3.20}$$

and from Lemma 3.8 we derive

$$| B((T_h - T_h^N) f, v) | \leqslant \frac{C(\varepsilon)}{\alpha} (\xi^N)^{1-\varepsilon} \| f \|_{H^0(\Omega)} \| v \|_{H^1(\Omega)} , \quad \forall v \in \mathring{S}(\Omega) .$$

Taking $v = (T_h - T_h^N) f \in \mathring{S}(\Omega)$, and using the Lemma 3.1 we obtain

$$\| (T_h - T_h^N) f \|_{H^1(\Omega)} \leqq \frac{C(\varepsilon)}{\alpha} (\xi^N)^{1-\varepsilon} \| f \|_{H^0(\Omega)} .$$

Moreover,

$$\| (T_h - T_h^N) f \|_{H^0(\Omega)} \leqq \| (T_h - T_h^N) f \|_{H^1(\Omega)} .$$

Therefore, we get (3.19).

From (3.14) and (3.19) we obtain the following result.

THEOREM 3.1 : *For fixed* $0 < \varepsilon < 1$, *there exist two constants* C *and* $C(\varepsilon)$ *such that*

$$\| T - T_h^N \| \leqq Ch^2 + \frac{C(\varepsilon)}{\alpha} (\xi^N)^{1-\varepsilon} . \tag{3.21}$$

Suppose that μ is a non-zero eigenvalue of T with algebraic multiplicity m and $\delta > 0$ is a fixed constant such that the interval $[\mu - \varepsilon, \mu + \varepsilon]$ contains no other eigenvalues of T. Then there is an h_0 such that for $0 < h \leqq h_0$ there exists an integer $N_0(h)$ such that if $N \geqq N_0(h)$, then there are exactly m eigenvalues (counting algebraic multiplicities) of T_h^N lying on $[\mu - \delta, \mu + \delta]$. All other eigenvalues of T_h are located beyond $[\mu - \delta, \mu + \delta]$ (*cf.* [6]). In this case the operators T and T_h^N are self-adjoint. Let $\mu_1^N(h), \mu_2^N(h), \ldots, \mu_m^N(h)$ denote the eigenvalues of T_h^N lying on the interval $[\mu - \delta, \mu + \delta]$. From Theorem 3 in [6] and Theorem 3.1 we have the following theorem.

THEOREM 3.2 : *There are two constants C and $C(\varepsilon)$ such that, for* $0 < h < h_0$, $N \geqq N_0(h)$

$$\max_{1 \leqslant i \leqslant m} | \mu - \mu_i^N(h) | \leqq Ch^2 + \frac{C(\varepsilon)}{\alpha} (\xi^N)^{1-\varepsilon} . \tag{3.22}$$

With regard to the eigenfunctions, from Theorem 4 in [6], we have :

THEOREM 3.3 : *Let $\mu^N(h)$ be an eigenvalue of T_h^N such that $\mu^N(h) \to \mu$ as $h \to 0$, $N \to \infty$ and suppose that, for each pair h, N, w is a unit eigenfunction satisfying $(\mu^N(h) - T_h^N) w = 0$. Then there is a function u which is an eigenfunction of T corresponding to μ such that*

$$\| u - w \|_{H^0(\Omega)} \leqq Ch^2 + \frac{C(\varepsilon)}{\alpha} (\xi^N)^{1-\varepsilon} , \tag{3.23}$$

where C and $C(\varepsilon)$ are two constants independent of h and N.

4. CHOOSING THE INTEGER N

The formula (3.22) and (3.23) are the error estimates of the infinite element method for eigenvalue problems. Each of them contains two terms Ch^2 and $\frac{C(\varepsilon)}{\alpha} (\xi^N)^{1-\varepsilon}$. Therefore only when h is sufficiently small and N is sufficiently large we can get accurate approximate solutions. Now we shall discuss a question regarding how to choose N for a fixed partition so that

$$\left.\begin{array}{l} | \mu - \mu_i^N(h) | \leqslant Ch^2 , \quad i = 1, 2, \ldots, m , \\ \| w^{h,N} - u \|_{H^0(\Omega)} \leqslant Ch^2 \end{array}\right\} . \tag{4.1}$$

From (3.22), (3.16) and (3.23) we know that either $(\xi^N)^{1-\varepsilon} = h^2$ or $(\xi^N)^{1-\varepsilon} = Ch^2$ needs to be chosen (where c is constant) and that the constant ξ

is dependent on h, i.e., $\xi(h) \to 1$ when $h \to 0$. Furthermore, we have the following Lemma.

LEMMA 4.1 : *Suppose that the partitions satisfy the smallest interior angle condition. Then there exist two positive constants C_2 and C_3 which are independent of h such that*

$$C_2 h \leqslant 1 - \xi \leqslant C_3 h . \tag{4.1}$$

Proof : We consider the triangle of which the length of one side is equal to h. This triangle belongs to a quadrangle of the first layer. Let $\triangle A_i A_{i+1} AB$ denote this quadrangle.

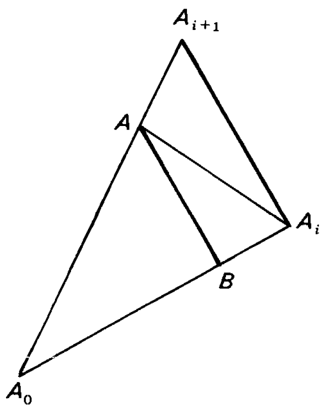

Figure 2.

Let

$$L_{\max} = \max \{ | A_0 A_1 |, | A_0 A_2 |, \ldots, | A_0 A_M | \} ,$$

and

$$L_{\min} = \min \{ | A_0 A_1 |, | A_0 A_2 |, \ldots, | A_0 A_m | \} .$$

i) If $| AA_{i+1} | = h$ or $| BA_i | = h$, then

$$L_{\min}(1 - \xi) \leqslant | AA_{i+1} | (\text{or } | BA_i |) \leqslant L_{\max}(1 - \xi) .$$

Taking $C_2 = \dfrac{1}{L_{\max}}$ and $C_3 = \dfrac{1}{L_{\min}}$, we obtain (4.1).

ii) $| BA | = h$ is impossible, because of $| A_i A_{i+1} | > | BA |$.

iii) If $| A_i A_{i+1} | = h$, we have

$$\frac{| A_i A_{i+1} |}{\sin \llcorner A_i A A_{i+1}} = \frac{| A_{i+1} A |}{\sin \llcorner A_{i+1} A_i A}. \tag{4.2}$$

Because of $| A_i A_{i+1} | = h \geqslant | A_{i+1} A |$, then $\llcorner A_i A A_{i+1} \geqslant \llcorner A_{i+1} A_i A$. Namely, $\pi - \gamma_0 \geqslant \llcorner A_i A A_{i+1} \geqslant \llcorner A_{i+1} A_i A \geqslant \gamma_0$. From (4.2) we get

$$\frac{(1 - \xi)}{h} = \frac{\sin \llcorner A_{i+1} A_i A}{\sin \llcorner A_i A A_{i+1}} | A_0 A_{i+1} |.$$

Consequently,

$$L_{\min} \sin \gamma_0 \leqslant \frac{(1 - \xi)}{h} \leqslant \frac{L_{\max}}{\sin \gamma_0}.$$

In this case $C_2 = L_{\min} \sin \gamma_0$ and $C_3 = \dfrac{L_{\max}}{\sin \gamma_0}$.

iv) If $| A A_i | = h$, similarly we get some results as in case (iii). This proof is completed.

Let $1 - \xi = \Delta$. From $(\xi^N)^{1-\varepsilon} = h^2$ we obtain $(1 - \varepsilon) N \ln (1 - \Delta) = 2 \ln h$. The conditions $h \ll 1$, and $\Delta \ll 1$, lead to $\ln (1 - \Delta) \approx - \Delta$. Furthermore, we have $(1 - \varepsilon) N\Delta \cong - 2 \ln h$. Namely,

$$N \approx \frac{- 2 \ln h}{(1 - \varepsilon) \Delta}.$$

By means of Lemma 4.1, we obtain

$$\frac{1}{(1 - \varepsilon) C_3} \left(- \frac{2 \ln h}{h} \right) \leqslant N \leqslant \frac{1}{(1 - \varepsilon) C_2} \left(- \frac{2 \ln h}{h} \right). \tag{4.3}$$

From (4.3) we know that, if $N \approx C\left(- \dfrac{\ln h}{h} \right)$, then we can get the error estimates (4.1). In this case the dimension of matrices Q_1^N and Q_2^N is $M \times N$ which can be approximated by $C\left(\dfrac{- \ln h}{h^2} \right)$. It is shown that, in order to get the approximate solutions of Equation (1.6) satisfying the error estimates (4.1), we only need to calculate an eigenvalue problem of a symmetric matrix with the dimension which is approximately $C\left(\dfrac{- \ln h}{h^2} \right)$.

ACKNOWLEDGMENT

The author thanks Professors Kellogg and Osborn for valuable dıscussıons

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