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EXTERNAL APPROXIMATION OF EIGENVALUE PROBLEMS IN BANACH SPACES (*)

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Abstract. — *We are concerned with approximate methods for solving the eigenvalue problem $Tu = \lambda u$, $u \neq 0$, for the linear bounded operator T in a Banach space X. The problem is approximated by an appropriate family of eigenvalue problems for operators $\{ T_h \}$. We present a theoretical framework which allows us to consider in the same way the methods for which T_h are defined on subspaces of X and those which are defined on spaces forming external approximation of X. Particularly, the paper contains theorems on sufficient conditions for stability and strong stability of $\{ T_h \}$.*

Résumé. — *On considère ici une classe de méthodes de résolution approchée du problème spectral de la forme $Tu = \lambda u$, où T est un opérateur linéaire, borné dans un espace Banach X. Les méthodes présentées remplacent le problème original par une famille de problèmes spectraux pour des opérateurs T_h. Les résultats sont présentés d'une manière qui permet de considérer à la fois les méthodes où les T_h sont définis sur des sous-espaces de X et celles où les espaces de définition de T_h forment une approximation externe de X. L'ouvrage contient certaines conditions suffisantes de stabilité et de stabilité forte de la famille $\{ T_h \}$.*

1. INTRODUCTION

Let X be a Banach space and $T \in \mathscr{L}(X)$ be a linear bounded operator on X. Let us consider the eigenvalue problem $Tu = \lambda u$, $u \neq 0$. Most methods used to solve this problem consist in approximation of the initial problem by a sequence of eigenvalue problems for $T_h \in \mathscr{L}(X_h)$, where X_h are finite dimensional subspaces of X and T_h are certain approximantes of T. This approach has been used in many papers, among others by J. Decloux, N. Nassif, J. Rappaz in [5] and by F. Chatelin in [2]. However, there are methods which cannot be presented within this unifying theoretical framework (e.g. the Aronszajn's method, *cf.* [1, 12]). Therefore we consider the more general case of approximation when the operators T_h are defined in spaces not contained in X. Strictly speaking we use an external approximation of X. We present some theorems

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concerning the approximation of eigenelements of T by eigenelements of T_h. Particularly we formulate new theorems about sufficient conditions for stability and strong stability of $\{ T_h \}$.

Let us introduce a family of Banach spaces $\{ X_h \}_{h \in \mathscr{H}}$ with the norms $\| \cdot \|_h$, where $\mathscr{H} \subset \mathbb{R}^+$ has an accumulation point at 0. We assume that there exist uniformly bounded linear maps $r_h : X \xrightarrow{\text{on}} X_h$. Let F be a normed space such that there exist an isomorphism $\omega : X \to F$ and uniformly bounded linear maps $p_h : X_h \to F$. We adopt the following definition :

DEFINITION 1 : *An approximation $\{ X_h, r_h, p_h \}$ of X is said to be an external approximation convergent in F if for any $u \in X$*

$$\lim_{h \to 0} \| \omega u - p_h r_h u \|_F = 0 .$$

The above definition is weaker than that used customarily (*cf.* [11, 6]).

Next, let us introduce a family $\{ T_h \}_{h \in \mathscr{H}}$ of linear operators where $T_h \in \mathscr{L}(X_h)$. We will assume that :

$A1$: The approximation $\{ X_h, r_h, p_h \}$ of X is convergent in F;

$A2$: For any $u \in X$ $\lim_{h \to 0} \| r_h Tu - T_h r_h u \|_h = 0$.

2. STABILITY OF $\{ T_h \}$

Let $\rho(T)$ and $\rho(T_h)$ denote, as usually, the resolvent sets of operators T and T_h respectively. We additionally assume that either the operators T_h have no residual spectrum or that the residual spectrum of T_h does not contain the points of $\rho(T)$ (since not only finite dimensional approximation is considered). We will use the following definition of stability *cf.* [4, 2] :

DEFINITION 2 : *The approximation $\{ T_h \}$ is stable at $z \in \rho(T)$ iff $\exists h(z)$,*

$$\forall h \leqslant h(z) : z \in \rho(T_h) \quad \textit{and} \quad \| (z - T_h)^{-1} \| \leqslant M(z) < \infty .$$

Now we are going to formulate some sufficient conditions for stability of $\{ T_h \}$ in terms of external approximation of T.

Let $N(r_h)$ denote the null space of r_h. Let us introduce the set of families of complementary subspaces of $N(r_h)$ in X

$$\mathscr{F} = \{ \{ V_h \}_{h \in \mathscr{H}}, V_h \subset X, V_h \oplus N(r_h) = X \} .$$

LEMMA 1 : *If there exists $\{ V_h \}_{h \in \mathscr{H}} \in \mathscr{F}$ such that*

$$\delta_h = \delta(V_h) := \sup_{\substack{v \in V_h \\ \|v\| = 1}} \| \omega T v - p_h T_h r_h v \|_F \to 0 , \tag{2.1}$$

$$\varepsilon_h = \varepsilon(V_h) := \sup_{\substack{v \in V_h \\ \|v\| = 1}} \| \omega v - p_h r_h v \|_F \to 0 , \tag{2.2}$$

then $\{ T_h \}$ *is stable at any* $\lambda \in \rho(T)$.

Proof : Let $\lambda \in \rho(T)$. Hence, there exists $c > 0$ such that

$$\| (\lambda - T) u \| \geqslant c \| u \| \quad \forall u \in X ,$$

and for $\tilde{c} = c/\| \omega^{-1} \|$, $\| \omega(\lambda - T) u \|_F \geqslant \tilde{c} \| u \| \, \forall u \in X$. Let us take an arbitrary $u_h \in X_h$. Then there exists $v_h \in V_h$ such that $r_h v_h = u_h$. We have $\| v_h \| \geqslant (1/d) \| u_h \|_h$ and $\forall x_h \in X_h \, \| x_h \|_h \geqslant 1/d \, \| p_h x_h \|_F$, where

$$d \geqslant \max (\| p_h \|, \| r_h \|)$$

for any h. Hence

$$\| (\lambda - T_h) u_h \|_h = \| (\lambda - T_h) r_h v_h \|_h \geqslant \frac{1}{d} \| p_h(\lambda - T_h) r_h v_h \|_F =$$

$$= \frac{1}{d} \| \omega(\lambda - T) v_h + \lambda(p_h r_h - \omega) v_h + (\omega T - p_h T_h r_h) v_h \|_F \geqslant$$

$$\geqslant \frac{1}{d^2} \| u_h \|_h (\tilde{c} - | \lambda | \varepsilon_h - \delta_h) .$$

Thus, for given $\lambda \in \rho(T)$ there exists h_0 such that for $h < h_0$

$$\| (\lambda - T_h) u_h \|_h \geqslant \frac{\tilde{c}}{2 d^2} \| u_h \|_h ,$$

what means, according to Definition 2, that $\{ T_h \}$ is stable at λ.

Remark 1 : In the case of an internal approximation of X, when $F = X$, $X_h = V_h \subset X$ and ω and p_h are identity maps, and r_h are projections of X on X_h, the condition (2.2) is automatically satisfied with $\varepsilon_h = 0$. In turn, the condition (2.1) takes the form $\| (T - T_h) \mid X_h \| \to 0$ i.e. the assumption of Lemma 1 in [5].

In the general case of an external approximation we have $\varepsilon_h \neq 0$. Thus, we must analyse how $\varepsilon(V_h)$ depends on $\{ V_h \} \in \mathscr{F}$. To do this we introduce the following numbers characterizing the subspaces V_h :

$$\gamma(V_h) := \sup_{\substack{v \in V_h \\ \|v\| = 1}} \| Q_h v \| , \tag{2.3}$$

where Q_h $(h \in \mathscr{H})$ are some given linear and bounded projections of X onto $N(r_h)$.

Let $\hat{V}_h = (1 - Q_h) X$. In this case $\gamma(\hat{V}_h) = 0$.

We can state the following result :

LEMMA 2 : *Let us assume that* $\varepsilon(\hat{V}_h) \to 0$ *as* $h \to 0$. *Then* $\varepsilon(V_h) \to 0$ *for* $\{ V_h \} \in \mathscr{F}$ *if and only if* $\gamma(V_h) \to 0$.

Proof :

$$\varepsilon(V_h) = \sup_{\substack{v \in V_h \\ \|v\| = 1}} \| \omega Q_h v + \omega(1 - Q_h) v - p_h r_h(1 - Q_h)v \|_F \geqslant$$

$$\geqslant \sup_{\substack{v \in V_h \\ \|v\| = 1}} \left\{ \frac{1}{\| \omega^{-1} \|} \| Q_h v \| - \| (1 - Q_h) v \| \varepsilon(\hat{V}_h) \right\}$$

$$\geqslant \frac{1}{\| \omega^{-1} \|} \gamma(V_h) - (1 + \gamma(V_h)) \varepsilon(\hat{V}_h) .$$

The implication " $\Rightarrow$ " follows from the above inequality.

It is easy to see that

$$\varepsilon(V_h) \leqslant \sup_{\substack{v \in V_h \\ \|v\| = 1}} \{ \| \omega \| . \| Q_h v \| + \| (1 - Q_h) v \| \varepsilon(\hat{V}_h) \} \leqslant \gamma(V_h) \| \omega \| + \varepsilon(\hat{V}_h)$$

which ends the proof of Lemma 2.

In the case when the X_h are infinite dimensional spaces the condition (2.2) becomes very strong, so another version of Lemma 1 will be more useful in this special case. Let us introduce the following

DEFINITION 3 : *The family* $\{ V_h \}$, $V_h \subset X$ *is asymptotically equivalent to* $\{ X_h \}$ *with respect to* $\{ r_h \}$ $(r_h \in \mathscr{L}(X, X_h), r_h X = r_h V_h = X_h)$ *if the* r_h *are uniformly bounded and* $\inf_{\substack{x \in V_h \\ \|x\| = 1}} \| r_h x \|_h \geqslant c > 0, \forall h \in \mathscr{H}$.

LEMMA 3 : *If there exist* $\{ \hat{r}_h \}$ *and* $\{ \hat{V}_h \}$ *asymptotically equivalent to* $\{ X_h \}$ *with respect to* $\{ \hat{r}_h \}$ *such that*

$$\hat{\delta}(\hat{V}_h) := \sup_{\substack{v \in \hat{V}_h \\ \|v\| = 1}} \| (T - (r_h |_{V_h})^{-1} T_h r_h) v \| \to 0 ,$$

then $\{ T_h \}$ *is stable at any* $\lambda \in \rho(T)$.

Proof : Let us take $u_h \in X_h$. Let $v_h \in V_h$ be such that $\hat{r}_h v_h = u_h$:

$$\| (\lambda - T_h)u_h \|_h = \| (\lambda - T_h) \hat{r}_h v_h \|_h = \| \hat{r}_h(\hat{r}_h |_{V_h})^{-1} (\lambda - T_h) \hat{r}_h v_h \|_h \geqslant$$

$$\geqslant c \| \lambda v_h - T v_h + (T - (\hat{r}_h |_{V_h})^{-1} T_h \hat{r}_h) v_h \|$$

$$\geqslant c \| (\lambda - T) v_h \| - \hat{\delta}(\hat{V}_h) \| v_h \| .$$

Since $\lambda \in \rho(T)$, there exists a constant $c_1 > 0$ such that $\| (\lambda - T) v_h \| \geqslant c_1 \| v_h \|$. Moreover, $\| v_h \| \geqslant \frac{1}{\| \hat{r}_h \|} \| u_h \|_h$. If $c_2 := \sup_h \| \hat{r}_h \|$, then

$$\| (\lambda - T_h) u_h \| \geqslant \left\{ \frac{c . c_1}{c_2} - \frac{\hat{\delta}(\hat{V}_h)}{c_2} \right\} \| u_h \|_h ,$$

what proves Lemma 3.

Now, we are going to give a short analysis of the assumptions of the above lemma. To do this we restrict our considerations to the case of separable Hilbert spaces.

LEMMA 4 : *For an arbitrary separable Hilbert space X and a family of separable Hilbert spaces X_h there exist uniformly bounded maps $r_h : X \to X_h$ such that the orthogonal complements of the null spaces of r_h form a family asymptotically equivalent to $\{ X_h \}$ with respect to $\{ r_h \}$.*

Proof: Let $\{ u_n \}_{n=1}^{\infty}$ and $\{ u_n^h \}_{n=1}^{\infty}$ be orthonormal bases in X and X_h respectively. If X_h is k-dimensional, we put $u_j^h = 0$ for $j > k$. Transformation $\varphi : X \to l^2$ and $\varphi_h : X_h \to l^2$ are defined as follows :

$$\varphi u = \{ (u, u_1), (u, u_2), \dots \} \quad \text{for} \quad u \in X ,$$
$$\varphi_h v = \{ (v, u_1^h)_h, (v, u_2^h)_h, \dots \} \quad \text{for} \quad v \in X_h .$$

Thus $\forall u \in X \; \| \varphi u \|_{l^2} = \| u \|$ and $\forall \{ x_n \} \in l^2$

$$\| \varphi^{-1} \{ x_n \} \|^2 = \left\| \sum_{n=1}^{\infty} x_n u_n \right\|^2 = \sum_{n=1}^{\infty} x_n^2 = \| \{ x_n \} \|_{l^2}^2 .$$

Similarly $\| \varphi_h \| = 1$ and $\varphi_h^{-1} : \varphi_h X_h \to X_h$, $\| \varphi_h^{-1} \| = 1$. Let P_h be the orthogonal projection from l^2 onto $\varphi_h X_h$. Let

$$r_h := \varphi_h^{-1} P_h \varphi : X \to X_h , \tag{2.5}$$
$$V_h := \varphi^{-1} \varphi_h X_h . \tag{2.6}$$

For any $v \in X \; \| r_h v \|_h \leqslant \| v \|$ and since $\varphi V_h = \varphi_h X_h$, $r_h |_{V_h} = \varphi_h^{-1} \varphi |_{V_h}$ and $(r_h |_{V_h})^{-1} = \varphi^{-1} \varphi_h$. Thus $\| (r_h |_{V_h})^{-1} \| = 1$. Hence $\{ V_h \}$ is asymptotically equivalent to $\{ X_h \}$ with respect to $\{ r_h \}$.

Now, let us take arbitrary elements $v \in V_h$ and $x \in N(r_h)$. For v there exists $u_v \in X_h$ such that $(v, u_i) = (u_v, u_i^h)$, $i = 1, 2, \dots$ Hence $(v, x) = \sum_{i=1}^{\infty} (u_v, u_i^h)(x, u_i)$.

Since $\varphi x \perp \varphi_h X_h$, $\sum_{i=1}^{\infty} (x, u_i)(u, u_i^h) = 0$ for any $u \in X_h$, so it also holds for $u = u_v$. Thus $(v, x) = 0$ for any $v \in V_h$ and $x \in N(r_h)$, what means that V_h is orthogonal to $N(r_h)$.

Let Q_h be orthogonal projection onto $N(r_h)$, and V_h be complementary subspace of $N(r_h)$ in X. Thus

$$\inf_{\substack{v \in V_h \\ \|v\| = 1}} \| r_h v \|_h = \inf_{\substack{v \in V_h \\ \|v\| = 1}} \| r_h Q_h v + r_h(1 - Q_h) v \| =$$

$$= \inf_{\substack{v \in V_h \\ \|v\| = 1}} \|(1-Q_h) v\| \cdot \left\| r_h \frac{(1-Q_h) v}{\|(1-Q_h) v\|} \right\| \geqslant \inf_{\substack{v \in V_h \\ \|v\| = 1}} \|(1-Q_h) v\| \cdot \inf_{\substack{x \perp N(r_h) \\ \|x\| = 1}} \| r_h x \|_h .$$

Using the notation (2.3) we obtain

$$\inf_{\substack{v \in V_h \\ \|v\| = 1}} \| r_h v \|_h \geqslant (1 - \gamma(V_h)) \cdot \inf_{\substack{x \perp N(r_h) \\ \|x\| = 1}} \| r_h x \|_h .$$

This leads us to the following remark :

Remark 2 : Let $\{ N(r_h)^{\perp} \}$ be asymptotically equivalent to $\{ X_h \}$ with respect to $\{ r_h \}$. If $\exists c_0 > 0$ such that $\forall h < h_0$ $1 - \gamma(V_h) \geqslant c_0$ then the family $\{ V_h \}$ is also asymptotically equivalent to $\{ X_h \}$ with respect to $\{ r_h \}$.

Remark 3 : If $\{ V_h \}$ satisfies the condition (2.2), then $\{ V_h \}$ is asymptotically equivalent to $\{ X_h \}$ with respect to $\{ r_h \}$.

This follows from the inequalities : $\forall v \in V_h$, $\| v \| = 1$:

$$\| r_h v \|_h \geqslant \frac{1}{\| p_h \|} [\| \omega v \|_F - \varepsilon(V_h)] .$$

Since $\| p_h \| \leqslant \alpha$ and $\| \omega v \|_F \geqslant \frac{1}{\| \omega^{-1} \|} \| v \|$, we have

$$\| r_h v \| \geqslant \frac{1}{\alpha} \left[\frac{1}{\| \omega^{-1} \|} - \varepsilon(V_h) \right]$$

for any $v \in V_h$ and $\| v \| = 1$.

3. APPROXIMATION OF EIGENELEMENTS OF T

In this section the proofs of the theorems are based on the ideas contained in [5] and [2].

Let Γ be a Jordan curve in the resolvent set $\rho(T)$. If $\{T_h\}$ is stable for all $\lambda \in \Gamma$, then $\Gamma \subset \rho(T_h)$ for sufficiently small $h < h_0$. Hence, we can define the spectral projectors $E : X \to X$ and $E_h : X_h \to X_h$ by

$$E = \frac{1}{2\pi i} \int_\Gamma (\lambda - T)^{-1} \, d\lambda \,, \quad E_h = \frac{1}{2\pi i} \int_\Gamma (\lambda - T_h)^{-1} \, d\lambda \,.$$

LEMMA 5 : *If the assumption A2 is satisfied and $\{T_h\}$ is stable on Γ, then* $\forall v \in X \lim_{h \to 0} \| r_h Ev - E_h r_h v \|_h = 0$.

Proof : From the definition of E and E_h and from the identity

$$r_h(\lambda - T)^{-1} - (\lambda - T_h)^{-1} r_h = (\lambda - T_h)^{-1} (T_h r_h - r_h T)(\lambda - T)^{-1}$$

it follows that for given $v \in X$

$$\| r_h Ev - E_h r_h v \| \leqslant \frac{|\Gamma|}{2\pi} \sup_\Gamma \| (\lambda - T_h)^{-1}(T_h r_h - r_h T)(\lambda - T)^{-1} v \| =$$

$$= \frac{|\Gamma|}{2\pi} \sup_{u \in U} \| (\lambda - T_h)^{-1}(T_h r_h - r_h T) u \| \,,$$

where $U = \{ u \in X : u = (\lambda - T)^{-1} v, \lambda \in \Gamma \}$.

The operators $(\lambda - T_h)^{-1}$ are uniformly bounded for $\lambda \in \Gamma$ and $h < h_0$ since the stability of $\{T_h\}$ on Γ. Thus, by the assumption $A2$,

$$\forall u \in X \; \| (\lambda - T_h)^{-1}(T_h r_h - r_h T) u \| \to 0 \,.$$

Moreover,

$$\| (\lambda - T_h)^{-1}(T_h r_h - r_h T) \| \leqslant \| (\lambda - T_h)^{-1} r_h T \| + \| \lambda(\lambda - T_h)^{-1} r_h \| + \| r_h \| \,,$$

so the operators $(\lambda - T_h)^{-1}(T_h r_h - r_h T)$ are uniformly bounded for $\lambda \in \Gamma$ and $h < h_0$. Thus, since the set U is compact,

$$\sup_{u \in U} \| (\lambda - T_h)^{-1}(T_h r_h - r_h T) u \| \to 0 \,.$$

LEMMA 6 : *If $A1$ and $A2$ are satisfied and $\{ T_h \}$ is stable on Γ, then*

$$\forall v \in EX \quad \inf_{y_h \in E_h X_h} \| \omega v - p_h y_h \|_F \to 0 .$$

Proof : Since

$$\inf_{y_h \in E_h X_h} \| \omega v - p_h y_h \|_F \leqslant \| \omega v - p_h r_h v \|_F + \| p_h \| \; \| r_h Ev - E_h r_h v \|_h ,$$

the proof follows immediately from Lemma 5.

As usually, $\sigma(T)$ denotes the spectrum of T. Let $\Omega \subset \mathbb{C}$ be an open domain with the boundary $\Gamma \subset \rho(T)$ which is a Jordan curve. Finally, let

$$K(\lambda, \delta) := \{ z \in \mathbb{C} : | z - \lambda | \leqslant \delta \} .$$

THEOREM 1 : *If the assumptions $A1$ and $A2$ are satisfied and $\{ T_h \}$ is stable in $\rho(T)$ then* :

1° *if $\Omega \cap \sigma(T) \neq 0$ then $\sigma(T_h) \cap \Omega \neq 0$ for sufficiently small h,*

2° *if $\lambda_0 \in \sigma(T)$ and $\exists \delta_0 > 0 : K(\lambda_0, \delta_0) \cap \sigma(T) = \{ \lambda_0 \}$ then $\forall 0 < \delta < \delta_0$, $0 \neq \sigma(T_h) \cap K(\lambda_0, \delta_0) \subset K(\lambda_0, \delta)$ for sufficiently small h,*

3° *if $\lambda_h \in \sigma(T_h)$ and $\lambda_h \to \lambda_0$ then $\lambda_0 \in \sigma(T)$.*

Proof : It follows from Lemma 5 that $\forall v \in EX \inf_{y_h \in E_h X_h} \| r_h v - y_h \|_h \to 0$. If $v \neq 0$ then, since $A1$, $r_h v \neq 0$ for sufficiently small h. Thus 1° is proved. For the proof of 2° it is enough to remark, that for

$$0 < \delta < \delta_0 \; K(\lambda, \delta_0) \setminus \operatorname{int} K(\lambda, \delta) \subset \rho(T)$$

and thus, by the stability of $\{ T_h \}$, $K(\lambda, \delta_0) \setminus \operatorname{int} K(\lambda, \delta)$ is contained in $\rho(T_h)$ for $h < h_0$. Assume now that $\lambda_h \in \sigma(T_h)$ and $\lambda_h \to \lambda_0 \notin \sigma(T)$. Thus there exists $\delta > 0$ such that $K(\lambda_0, \delta) \subset \rho(T)$ and from the stability $K(\lambda_0, \delta) \subset \rho(T_h)$ for $h < h_0$, what means that for $h < h_1$, $\lambda_h \in \rho(T_h)$.

The above theorem gives convergence of eigenvalues, but without preservation of the algebraic multiplicities. Namely, we have only

THEOREM 2 : *If $A1$ and $A2$ are satisfied and $\{ T_h \}$ is stable on Γ then*

1° $\dim EX = \infty \Rightarrow \dim E_h X_h \to \infty$

2° $\dim EX = n \Rightarrow \dim p_h E_h X_h \geqslant n.$

Proof : Let $\{ u_i \}_{i=1}^{\infty}$ be a linearly independent set of elements of EX. From Lemma 6 it follows that for every finite number

$$N \; \forall \varepsilon \, \exists h_\varepsilon \, \forall h < h_\varepsilon \, \forall i = 1, \dots, N \, \exists x_i^h \in E_h X_h : \| \omega u_i - p_h x_i^h \|_F \leqslant \varepsilon .$$

Thus $\forall N < \infty \ \exists h_N \ \forall h < h_N \dim p_h E_h X_h \geqslant N$, hence 1°.

Let now $\dim EX = n$. By Lemma 6 we have

$$\sup_{\substack{v \in EX \\ \|v\| = 1}} \inf_{y_h \in E_h X} \| \omega v - p_h y_h \|_F \to 0 .$$

Using the known notation (*cf.* [7] chap. IV) : for closed subspaces Y, Z of X

$$\delta(Y, Z) = \sup_{\substack{y \in Y \\ \|y\| = 1}} \inf_{z \in Z} \| y - z \| , \tag{3.1}$$

we have $\delta(\omega EX, p_h E_h X_h) \to 0$. It is known that if $\delta(Y, Z) < 1$ then $\dim Y \leqslant \dim Z$ (*cf.* [7] chap. IV, Corollary 2.6). Thus

$$n = \dim \omega EX \leqslant \dim p_h E_h X_h .$$

Under additional assumptions we can state the following result :

THEOREM 3 : *One supposes $A1$, $A2$ and stability of $\{ T_h \}$ on Γ. Moreover let $\| p_h u_h - f \|_F \to 0$, where $u_h \in X_h$, imply that f belongs to ωX, and let the norms in F and X_h be asymptotically equivalent (i.e. if $u_h \in X_h$ and $\| p_h u_h \|_F \to 0$ then $\| u_h \|_h \to 0$). Then if $x_h \in E_h X_h$ and $\| p_h x_h - f \|_F \to 0$ then $f \in \omega EX$.*

Proof : If $\| p_h x_h - f \| \to 0$ then there exists $x_0 \in X$ such that $f = \omega x_0$. It remains to show that $Ex_0 = x_0$. From the inequality

$$\| \omega x - p_h x_h \|_F \geqslant \| \omega(Ex_0 - x_0) \| - \| \omega Ex_0 - p_h E_h r_h x_0 \|_F - \| p_h E_h(r_h x_0 - x_h) \|_F$$

we get

$$\| Ex_0 - x_0 \| \leqslant \| \omega^{-1} \| [\| \omega x_0 - p_h x_h \|_F + \| \omega Ex_0 - p_h r_h Ex_0 \|_F + \\ + \| p_h \| \ \| r_h Ex_0 - E_h r_h x_0 \|_h + \| p_h E_h \| \ \| r_h x_0 - x_h \|_h] .$$

The convergence $\| p_h x_h - \omega x_0 \| \to 0$ implies $\| p_h r_h x_0 - p_h x_h \|_F \to 0$ and thus, by the additional assumption on p_h, $\| r_h x_0 - x_h \|_h \to 0$. By Lemma 5 and $A1$ we have : $\forall \varepsilon \ \exists h_0 \ \forall h < h_0 \ \| Ex_0 - x_0 \| \leqslant \varepsilon$, thus $Ex_0 = x_0$.

4. STRONG STABILITY OF $\{ T_h \}$

Let $\Omega \subset \mathbb{C}$ be a domain limited by the Jordan curve $\Gamma \subset \rho(T)$. Let E and E_h be the spectral projections associated with the spectrum of T and T_h inside Γ. We will assume that $\dim EX < \infty$. With respect to the convergence of eigenvectors it is very important to have the same dimensions of $E_h X_h$ (or $p_h E_h X_h$)

and EX. We will use the notion of strongly stable approximation $\{ T_h \}$ similar to that introduced by F. Chatelin in [4].

DEFINITION 4 : *An approximation* $\{ T_h \}$, *stable on* Γ, *is strongly stable on* Γ *if* $\dim EX = \dim p_h E_h X_h$ *for h small enough.*

The convergence of external approximation (i.e. $A1$), the consistency of $\{ T_h \}$ to T (i.e. $A2$) and the stability of $\{ T_h \}$ are not sufficient for strong stability of $\{ T_h \}$, so we need a stronger assumption.

LEMMA 7 : *If* $\{ T_h \}$ *is stable on* Γ *and*

$$\| (T_h r_h - r_h T)(\lambda - T)^{-1} \|_h \to 0 \quad \textit{for} \quad \lambda \in \Gamma \tag{3.2}$$

then $\| r_h E - E_h r_h \|_{\mathscr{L}(X, X_h)} \to 0$.

Proof : Repeating argumentation of the proof of Lemma 5 we get $\| r_h E - E_h r_h \| \leqslant c_0 \| (T_h r_h - r_h T)(\lambda - T)^{-1} \|$ for a some constant c_0.

LEMMA 8 : *If there exists* $\{ V_h \} \in \mathscr{F}$ *such that* $\forall h < h_0$

$$\eta_h := \inf_{\substack{x \in V_h \\ \|x\| = 1}} \| p_h r_h x \|_F \geqslant \varepsilon_0 > 0$$

then

$$\delta(p_h E_h X_h, \omega EX) \leqslant \frac{1}{\varepsilon_0} \| p_h E_h r_h - \omega E \| .$$

Proof : Let $\tilde{V}_h$ be a subspace of V_h such that $r_h \tilde{V}_h = E_h X_h$. Then

$$\| p_h E_h r_h - \omega E \| \geqslant \sup_{\substack{x \in X \\ \|x\| = 1}} \inf_{y \in EX} \| p_h E_h r_h x - \omega y \| \geqslant$$

$$\geqslant \sup_{\substack{x \in \tilde{V}_h \\ \|x\| = 1}} \inf_{y \in EX} \| p_h r_h x - \omega y \| \geqslant \inf_{\substack{x \in \tilde{V}_h \\ \|x\| = 1}} \| p_h r_h x \| \sup_{\substack{x_h \in E_h X_h \\ \|p_h x_h\| = 1}} \inf_{y \in EX} \| p_h x_h - \omega y \| .$$

According to (3.1) the last factor is equal to $\delta(p_h E_h X_h, \omega EX)$.

THEOREM 4 : *If the assumptions* $A1$, (2.1), (2.2), (3.2) *are satisfied, then* $\{ T_h \}$ *is strongly stable on* Γ.

Proof : It follows from (2.2) that

$$\eta_h \geqslant \inf_{\substack{x \in V_h \\ \|x\| = 1}} \| \omega x \|_F - \sup_{\substack{x \in V_h \\ \|x\| = 1}} \| p_h r_h x - \omega x \|_F \geqslant \frac{1}{\| \omega^{-1} \|} - \varepsilon_h ,$$

thus $\eta_h \geqslant \varepsilon_0 > 0$ for sufficiently small h. Moreover, since $\dim EX < \infty$, by Lemma 7

$$\| p_h E_h r_h - \omega E \| \leqslant \| p_h \| \; \| E_h r_h - r_h E \| + \| (p_h r_h - \omega) E \| \to 0 .$$

Hence, from Lemma 8 we get $\delta(p_h E_h X_h, \omega EX) < 1$ for h small enough and thus $\dim p_h E_h X_h \leqslant \dim \omega EX$. The oposit inequality have been obtained in Theorem 2, thus $\dim p_h E_h X_h = \dim EX$.

The assumption (2.2), which is very strong in the case of infinite dimensional spaces X_h, can be ommited as it is shown in the following.

THEOREM 5 : *Let* $A1$ *be satisfied. Moreover, let* $\{ V_h \}$ *be asymptotically equivalent to* $\{ X_h \}$ *with respect to* $\{ r_h \}$ *and* $\{ X_h \}$ *be asymptotically equivalent to* $\{ p_h X_h \}$ *with respect to* $\{ p_h \}$. *If*

$$\| [T - (r_h |_{V_h})^{-1} T_h r_h] (\lambda - T)^{-1} \| \to 0 \quad \textit{for} \quad \lambda \in \Gamma \tag{3.3}$$

then $\{ T_h \}$ *is strongly stable on* Γ.

Proof : It follows from (3.3) that

$$\exists c > 0 \; \forall h < h_0 \; \forall \lambda \in \Gamma \; \| (r_h |_{V_h})^{-1} (\lambda - T_h) r_h (\lambda - T)^{-1} \| \geqslant c .$$

On the other hand

$$\| (r_h |_{V_h})^{-1} (\lambda - T_h) r_h (\lambda - T)^{-1} \| \leqslant \| \lambda - T_h \| \; \| (r_h |_{V_h})^{-1} \| \; \| r_h \| \; \| (\lambda - T)^{-1} \| .$$

Thus, by the uniform boundness of $\| (r_h |_{V_h})^{-1} \|$ and $\| r_h \|$ we obtain that $\| \lambda - T_h \| \geqslant c_1 > 0$ for $h < h_0$ and $\lambda \in \Gamma$, what gives the stability of $\{ T_h \}$ on Γ.

Moreover, (3.3) implies (3.2). Thus, by Lemma 7, $\| r_h E - E_h r_h \| \to 0$, what implies $\| p_h E_h r_h - \omega E \| \to 0$, since $\dim EX < \infty$. The assumption on asymptotic equivalence of $\{ V_h \}$, $\{ X_h \}$ and $\{ p_h X_h \}$ guaranties the existence of positive lower bound for η_h. Hence, by Lemma 8, $\delta(p_h E_h X_h, \omega EX) \to 0$. Thus $\dim p_h E_h X_h \leqslant \dim \omega EX$ what together with Theorem 2 gives : $\dim p_h E_h X_h = \dim E_h X_h = \dim EX$ for sufficiently small h.

The condition (3.3) imposed on the approximation is some modification of radial convergence introduced in [2, 3] for the case of internal approximation.

5. APPLICATION

Let X be a Hilbert space with the scalar product $a(,)$. Let b be a bounded sesquilinear form defined on $X \times X$. The eigenvalue problem for two forms

$$b(u, v) = \lambda a(u, v) \quad \forall v \in X \tag{5.1}$$

is considered. This problem is equivalent to the eigenproblem for an operator T defined by : $b(u, v) = a(Tu, v)\ \forall u, v \in X$. Let V be a dense subspace of X. We will consider approximate methods of solving the problem (5.1) which are generated by sequences of sesquilinear forms a_n and b_n defined on $V \times V$. It is assumed that a_n $n = 0, 1, \ldots$ are symmetric and positive definite and b_n are bounded with respect to a_n, i.e. $\forall u, v \in V \mid b_n(u, v) \mid \leqslant c_n\, a_n^{1/2}(u, u)\, a_n^{1/2}(v, v)$. Let X_n be the closure of V in the norm $a_n^{1/2}$, $n = 0, 1, \ldots$ The n-th approximate eigenvalue problem has the form

$$\begin{aligned} &\text{find } \lambda \in \mathbb{C} \quad \text{and} \quad 0 \neq u \in X_n \text{ such that} \\ &b_n(u, v) = \lambda a_n(u, v) \quad \forall v \in V , \end{aligned} \tag{5.2}$$

which is equivalent to the eigenproblem for an operator T_n defined by a_n and $b_n : b_n(u, v) = a_n(T_n u, v)\ \ \forall v \in V, u \in X_n$. Under the assumptions

$$a_0 \leqslant a_n \leqslant a , \tag{5.3}$$

a is quasi-bounded with respect to a_0, i.e. there exists a symmetric operator $\hat{L}$ in X_0, with dense domain V, such that $a(u, v) = a_0(\hat{L}u, v)\ \forall u, v \in V$ (*cf.* [1]),

$$\tag{5.4}$$

the approximation (5.2) can be described in terms of external approximation (for details see [8]).

From (5.3) and (5.4) it follows that a is quasi-bounded with respect to a_n, $n = 1, 2, \ldots$ Let $\hat{L}_n$ be the symmetric operator defined by $a(u, v) = a_n(\hat{L}_n u, v)$ $\forall u, v \in V$, and let L_n denote its selfadjoint extension in X_n. L_n is positive definite. Thus, there is a unique positive definite and self-adjoint square root $L_n^{1/2}$ of L_n and the domain $D(L_n)$ of L_n is dense in $D(L_n^{1/2})$. It can be proved (see [8]) that $D(L_n^{1/2}) = X$ and $\forall u, v \in X$ $a(u, v) = a_n(L_n^{1/2} u, L_n^{1/2} v)$. Let us put $r_n := L_n^{1/2}$. It is easy to show (see [8]) that $\| r_n \|_{\mathscr{L}(X, X_n)} = \| r_n^{-1} \|_{\mathscr{L}(X_n X)} = 1$. We define $p_n := r_n^{-1}$. The approximation $\{ X_n, r_n, p_n \}$ is convergent in X due to Definition 1. The following property can be proved (see [8]) :

LEMMA 9 : *Let* (5.3) *and* (5.4) *be satisfied and moreover*

$$\forall u \in V \sup_{\substack{v \in V \\ \|v\| = 1}} | a_n(u, v) - a(u, v) | \to 0 , \tag{5.5}$$

$$\sup_{\substack{u,v \in V \\ \|u\| = \|v\| = 1}} | b_n(u, v) - b(u, v) | \to 0. \tag{5.6}$$

Let $\| u_n \|_n \leqslant M$ and $\| v_n \|_n \leqslant M$ $n = 0, 1, \ldots$ for some M.
If $a_n(u_n, w) \to a(u, w)$ $\forall w \in V$, and $a_n(v_n, w) \to a(v, w)$ $\forall w \in V$ imply

$$b_n(u_n, v_n) \to b(u, v) , \tag{5.7}$$

then $\{ T_n \}$ is stable at any $\lambda \in \rho(T)$.

Let us remark, that in the considered case the condition (2.1) of Lemma 1 implies $A2$ and (3.2). Thus we have

COROLLARY 1 : *If the assumptions* (5.3)-(5.7) *are satisfied then the method is convergent in the sense of Theorems* 1 *to* 4.

The class of methods described above has been investigated by R. D. Brown in [1] by using the another theory. He adopts the theory of discrete convergence of Banach spaces in the form developed by Stummel [10]. His results are similar to those obtained above.

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