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A TECHNIQUE OF UPSTREAM TYPE APPLIED TO A LINEAR NONCONFORMING FINITE ELEMENT APPROXIMATION OF CONVECTIVE DIFFUSION EQUATIONS (*)

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Summary — We present a technique of upstream type in a linear nonconforming finite element approximation of convective diffusion equations. It is then shown that this scheme satisfies the discrete maximum principle and leads to an $O(h)$ error estimate in $H^1$-norm. Some numerical examples are given for the model problem.

INTRODUCTION

In this note a technique of upstream type is introduced in a linear nonconforming finite element approximation of convective diffusion equations. The nonconforming element under consideration here is so-called a piecewise linear element using Loof connections, which were thoroughly investigated by Crouzeix and Raviart [7] and Temam [16] from the theoretical interest.

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occurred in studying the approximations of incompressible flow problems. For practical computations, see also the recent book by Thomasset [17].

On the other hand, several techniques of upstream type to the usual piecewise linear element were developed in recent years in Japan (Baba and Tabata [1], Ikeda [10], Kanayama [11], Kikuchi and Ushijima [13]). Our present method is an extension of one of such techniques to the considered nonconforming element, which is obtained along the way of the modification of the bilinear form corresponding to the convective term, mentioned in Kikuchi and Ushijima [13]. Then we introduce barycentric domains corresponding to mid-points of sides of all triangles belonging to the triangulation $T_h$ in order to define the lumped regions. Recently, Dervieux and Thomasset [8] also proposed the barycentric domain associated with the linear nonconforming finite element in order to derive an upwind scheme to the convective term. However, their scheme is different from our scheme.

An outline of the paper is as follows. In Section 1, notation and the model problem are presented. Section 2 is devoted to the construction of a lumping method based on the considered nonconforming element. In Section 3, our technique of upstream type is proposed. Then we show the discrete maximum principle for our scheme in Section 4, and an $O(h)$ $H^1$ error estimate in Section 5. In Section 6, we give some numerical examples.

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1. NOTATION AND PRELIMINARIES

Let $\Omega$ be a polygonal bounded connected domain of $\mathbb{R}^2$ with the boundary $\Gamma$. For a non-negative integer $m$, let $H^m(\Omega)$ be the usual $m$th order Sobolev space equipped with the norm and the semi-norm

$$
\| v \|_{m,\Omega} = \left( \sum_{|\alpha| \leq m} \| D^\alpha v \|_{0,\Omega}^2 \right)^{1/2},
$$

$$
| v |_{m,\Omega} = \left( \sum_{|\alpha| = m} \| D^\alpha v \|_{0,\Omega}^2 \right)^{1/2},
$$

where $\| \cdot \|_{0,\Omega}$ is the norm of $L^2(\Omega)$. The scalar product in $L^2(\Omega)$ is given by $(\cdot, \cdot)$. We set as usual

$$
H^1_0(\Omega) = \{ v \in H^1(\Omega); v |_{\Gamma} = 0 \}.
$$

R.A.I.R.O. Analyse numérique/Numerical Analysis
We consider the following stationary convective diffusion equation

\begin{equation}
\begin{cases}
-\nu \Delta u + (\mathbf{b} \cdot \nabla) u = f & \text{in } \Omega, \\
u u = u_0 & \text{on } \Gamma,
\end{cases}
\end{equation}

where \(\nu\) is a positive constant, \(\mathbf{b} = b(x) \in C^1(\Omega)^2\), \(f \in L^2(\Omega)\) and \(u_0 \in H^1(\Omega)\).

Let \(a(u, v)\) and \(b(u, v)\) be two bilinear forms on \(H^1(\Omega) \times H^1(\Omega)\) defined by

\begin{align}
a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \, dx, \\
b(u, v) &= \int_\Omega \mathbf{b} \cdot \nabla u \cdot v \, dx.
\end{align}

Furthermore, we set

\[ t(u, v) = va(u, v) + b(u, v). \tag{1.8} \]

We consider the variational formulation (II) for (E):

\begin{equation}
\begin{cases}
\text{Find } u \in H^1(\Omega) \text{ such that } \\
t(u, v) = (f, v) \quad \text{for all } v \in H^1_0(\Omega), \\
u - u_0 \in H^1_0(\Omega).
\end{cases}
\end{equation}

This problem has a unique solution under the condition that there exists a positive constant \(\alpha_1\) such that

\[ va_0 - \frac{1}{2} \text{div } \mathbf{b} \geq \alpha_1 > 0 \quad \text{in } \Omega, \tag{1.11} \]

where \(\alpha_0 > 0\) is less than or equal to the minimum eigenvalue of \(-\Delta\) with Dirichlet boundary condition.

It is well known that the maximum principle holds for the solution of (II) in the following form (cf. Courant and Hilbert [6]):

Assume that the solution \(u\) of (II) is continuous on \(\overline{\Omega}\) and twice continuously differentiable in \(\Omega\). Then it holds that

\[ \max_{x \in \Omega} u(x) \leq \max_{x \in \Gamma} u_0(x) \quad \text{when } f \leq 0 \quad \text{in } \Omega. \tag{1.12} \]

### 2. Nonconforming Finite Element and Lumping Operator

In this section we shall consider an approximation of upstream type for the convective term using the linear nonconforming finite element.
Let \( \{ T_h \} \) be a family of triangulation of \( \bar{\Omega} \) made of open triangles \( K \), that is
\[
\bar{\Omega} = \bigcup_{K \in T_h} K, \tag{2.1}
\]
where any two triangles are either disjoint or share at most one side or one vertex. Let \( h_K \) be the maximum side length of \( K \in T_h \) and \( \rho_K \) be the diameter of the inscribed circle in \( K \). Moreover, we set \( h = \max_{K \in T_h} h_K \).

In what follows, we assume that \( \{ T_h \} \) is regular. That is, when \( h \) tends to 0, there exists a constant \( \sigma > 0 \), independent of \( h \) and \( K \), such that
\[
\sigma_K = h_K/\rho_K \leq \sigma \quad \text{for all } K \in T_h. \tag{2.2}
\]

Let us recall the linear nonconforming finite element studied by Crouzeix and Raviart [7]. Let \( B_i \), \( 1 \leq i \leq N \), be the mid-points of sides lying in the interior of \( \Omega \) and \( B_i^\tau \), \( N + 1 \leq i \leq N + M \), be the mid-points of sides lying on \( \Gamma \). Let \( V_h \) be the linear nonconforming finite element approximate space of \( H^1(\Omega) \) defined by
\[
V_h = \{ v_h \in L^2(\Omega) : v_h \text{ is linear on } K \in T_h \text{ and is continuous at } B_i, \quad 1 \leq i \leq N + M \}. \tag{2.3}
\]
Furthermore, we define
\[
V_{0h} = \{ v_h \in V_h ; v_h = 0 \text{ at } B_i, N + 1 \leq i \leq N + M \}. \tag{2.4}
\]
Observe that \( V_h \not\subset H^1(\Omega) \) and \( V_{0h} \not\subset H^1_0(\Omega) \).

Let \( w_{ih} \), \( 1 \leq i \leq N + M \), be the elements of \( V_h \) such that
\[
w_{ih}(B_j) = \delta_{ij} \quad \text{for } 1 \leq i, j \leq N + M, \tag{2.5}
\]
where \( \delta_{ij} \) is the Kronecker delta. Then the sets of functions \( \{ w_{ih} ; 1 \leq i \leq N + M \} \), and \( \{ w_{ih} ; 1 \leq i \leq N \} \), form bases of \( V_h \), and \( V_{0h} \), respectively. This element, however, satisfies the following compatibility conditions:

\( (N1) \) For any \( K_1, K_2 \in T_h \), it holds that
\[
\int_{\partial K_1 \cap \partial K_2} (v_h |_{K_1} - v_h |_{K_2}) \, d\gamma = 0 \quad \text{for all } v_h \in V_h, \tag{2.6}
\]
where \( \partial K_1 \cap \partial K_2 \).

\( (N2) \) For any \( K \in T_h \), it holds that
\[
\int_{\partial K \cap \Gamma} v_h \, d\gamma = 0 \quad \text{for all } v_h \in V_{0h}. \tag{2.7}
\]
We provide the space $V_h$ with the following norm and semi-norm:

$$
\| v_h \|_{1,h} = \left( \sum_{K \in T_h} \| v_h \|^2_{1,K} \right)^{1/2},
$$

(2.8)

$$
\| v_h \|_h = \left( \sum_{K \in T_h} |v_h|^2_{1,K} \right)^{1/2}.
$$

(2.9)

The above conditions (NI) and (N2) imply that $\| . \|_h$ is a norm over the space $V_{oh}$.

Next, we define the barycentric domain associated with the linear nonconforming finite element. For any $K \in T_h$ with vertices $P_{i,K}$, $1 \leq i \leq 3$, let $B_{i,K}$ be the mid-point of the side $K'$ opposite to $P_{i,K}$, $1 \leq i \leq 3$, and $G_K$ be the barycenter of $K$. Consider the triangle $S_{ij}$, $1 \leq i, j \leq 3$, $i \neq j$, with vertices $G_K$, $B_{i,K}$ and $P_{k,K}$, where $k \neq i, j$. We say that $S_{ij}$ is a barycentric fragment of $K$. Then, with each $B_{i,K}$, $1 \leq i \leq 3$, we associate a barycentric subdomain $S^K_i$ as follows:

$$
S^K_i = \bigcup_{j \neq i} S_{ij}.
$$

(2.10)

If $K_1$ and $K_2$ are adjacent elements having $B_i$ as its common mid-point, we say that $\Omega_i = S^{K_1}_i \cup S^{K_2}_i$ is the barycentric domain with respect to $B_i$. If $B_i \in \Gamma$, we set $\Omega_i = S^K_i$. Furthermore, with each $B_i$, $1 \leq i \leq N + M$, we associate the index set

$$
\Lambda_i = \{ j \neq i ; B_j \text{ is the mid-point of the side of a triangle having } B_i \text{ as another one} \}.
$$

(2.11)

For any $j \in \Lambda_i$, $1 \leq i \leq N$, we set as follows:

$$
\Gamma^S_{ij} = \partial S^K_i \cap \partial S^K_j.
$$

(2.12)

If $B_i$ is the mid-point of the side lying in the interior of $\Omega$, we have

$$
\partial \Omega_i = \bigcup_{j \in \Lambda_i} \Gamma^S_{ij}.
$$

(2.13)

In our linear nonconforming finite element approximation, this barycentric domain plays the role of the lumping region in the usual conforming finite element approximation (see Kikuchi and Ushijima [13]).

Let $\overline{w}_{jh}$ be the characteristic function of $\Omega_i$, and $\overline{V}_h$ be the linear space spanned by the functions $\overline{w}_{jh}$, $1 \leq j \leq N + M$. Let $L_h$ be the lumping operator from vol 18, n°3, 1984.
It is easily seen that the lumping operator \( L_h \) satisfies the following properties:

\[
\| v_h \|_{0, \Omega} = \| L_h v_h \|_{0, \Omega} \quad \text{for all} \quad v_h \in V_h, \quad (2.15)
\]

\[
\| v_h - L_h v_h \|_{0, \Omega} \leq h \| v_h \|_h \quad \text{for all} \quad v_h \in V_h. \quad (2.16)
\]
3. UPSTREAM-LIKE SCHEME OF THE NONCONFORMING TYPE

We define the following approximate bilinear forms on $V_h \times V_h$:

$$a_h(u_h, v_h) = \sum_{K \in T_h} \int_K \nabla u_h \cdot \nabla v_h \, dx,$$  \hspace{1cm} (3.1)

$$b_h(u_h, v_h) = \sum_{K \in T_h} \int_K (b \cdot \nabla u_h) \, v_h \, dx.$$  \hspace{1cm} (3.2)

Then we set for any $u_h, v_h \in V_h$

$$t_h(u_h, v_h) = va_h(u_h, v_h) + b_h(u_h, v_h).$$  \hspace{1cm} (3.3)

In [14] we have considered the following approximate problem of Galerkin type:

$$(\Pi_h) \begin{cases} 
\text{Find } u_h \in V_h \text{ such that} \\
\quad t_h(u_h, v_h) = (f, v_h) \text{ for all } v_h \in V_{oh}, \\
\quad u_h - u_{oh} \in V_{oh},
\end{cases}$$  \hspace{1cm} (3.4, 3.5)

where $u_{oh} \in V_h$ is chosen so that $u_{oh}(B_i) = u_0(B_i), N + 1 \leq i \leq N + M$.

Now, we shall consider the modification of $b_h(u_h, v_h)$ by using the lumping process with the barycentric domain, following the procedure stated in Kikuchi and Ushijima [13] for the case of conforming piecewise linear approximation. In the first time, we rewrite $b_h(u_h, v_h)$ as follows:

$$b_h(u_h, v_h) = b^1_h(u_h, v_h) + b^2_h(u_h, v_h),$$  \hspace{1cm} (3.6)

where

$$b^1_h(u_h, v_h) = \sum_{K \in T_h} \int_K (\text{div } u_h \ b) \, v_h \, dx,$$  \hspace{1cm} (3.7)

and

$$b^2_h(u_h, v_h) = - \sum_{K \in T_h} \int_K (\text{div } b) \, u_h \, v_h \, dx.$$  \hspace{1cm} (3.8)

Then we modify $b^1_h(u_h, v_h)$ by $b^1_h(u_h, L_h v_h)$. According to the patch-wise application of the Gauss divergence formula, it can be easily verified that

$$b^1_h(u_h, L_h v_h) = \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}} (b \cdot n_j) \ u_h \, \nu_j v_h(B_j)$$
$$+ \sum_{j=N+1}^{N+M} \int_{\partial \Omega_j \cap \Gamma} (b \cdot n_j) \ u_h \, \nu_j v_h(B_j),$$  \hspace{1cm} (3.9)
where \( n_j \) is the unit outer normal vector along \( \partial \Omega_j \). Taking account of (3.9) we define the modified form \( \tilde{b}_h^1(u_h, v_h) \) as follows:

\[
\tilde{b}_h^1(u_h, v_h) = \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma^S_{jk}} (b.n_j) u_h^{jk} \, d\gamma \, v_h(B_j) + \sum_{j=N+1}^{N+M} \int_{\partial \Omega_j \cap \Gamma} (b.n_j) \, d\gamma \, u_h(B_j) \, v_h(B_j),
\]

(3.10)

where

\[
\begin{align*}
\lambda_{jk} &= \lambda_{jk} u_h(B_j) + (1 - \lambda_{jk}) u_h(B_k), \\
\lambda_{jk} &= 1 - \lambda_{kj},  \\
|\lambda_{jk}| &\leq \Lambda \quad (\Lambda \text{ is a constant independent of } j, k \text{ and } h). 
\end{align*}
\]

(3.11)

Furthermore, \( b_h^2(u_h, v_h) \) is modified by \( b_h^2(L_h u_h, L_h v_h) \) which is denoted by \( \tilde{b}_h^2(u_h, v_h) \). Then we have

\[
\tilde{b}_h^2(u_h, v_h) = -\sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma^S_{jk}} (b.n_j) \, d\gamma \, u_h(B_j) \, v_h(B_j) - \sum_{j=N+1}^{N+M} \int_{\partial \Omega_j \cap \Gamma} (b.n_j) \, d\gamma \, u_h(B_j) \, v_h(B_j).
\]

(3.12)

Thus we can define the modified form \( \tilde{b}_h(u_h, v_h) \) as follows:

\[
\tilde{b}_h(u_h, v_h) = \tilde{b}_h^1(u_h, v_h) + \tilde{b}_h^2(u_h, v_h) = \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma^S_{jk}} (b.n_j) (u_h^{jk} - u_h(B_j)) \, d\gamma \, v_h(B_j).
\]

(3.13)

**Remark 1**: If we take \( \lambda_{jk} \) as follows, then (3.13) yields the upstream scheme for the convective term:

\[
\lambda_{jk} = \begin{cases} 
1 & \text{if } \int_{\Gamma^S_{jk}} (b.n_j) \, d\gamma \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

(3.14)

Finally, we define our modified form of \( i_h(u_h, v_h) \):

\[
i_h(u_h, v_h) = \nu a_h(u_h, v_h) + \tilde{b}_h(u_h, v_h).
\]

(3.15)
Hence, our approximate problem is written as follows:

\[
\begin{align*}
\text{(P}_h\text{)}: & \quad \text{Find } \tilde{u}_h \in V_h \text{ such that } \\
& \quad \tilde{t}_h(\tilde{u}_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_{0h}, \\
& \quad \tilde{u}_h - u_{0h} \in V_{0h}.
\end{align*}
\] (3.16) (3.17)

4. DISCRETE MAXIMUM PRINCIPLE

This section is devoted to the study of the discrete maximum principle for the upstream-like scheme (P_h). From now on, we assume that the triangulation T_h is of acute type, that is, it holds that

\[
\tau_K \leq 0 \quad \text{for all } K \in T_h,
\] (4.1)

where

\[
\tau_K = \max_{i \neq j} \{ \cos (\nabla \mu_i, \nabla \mu_j)_{\mathbb{R}^2} \} 
\] (4.2)

and \( \mu_i, 1 \leq i \leq 3, \) are barycentric coordinates of \( x \in K \) with respect to the mid-points \( B_{iK}, 1 \leq i \leq 3, \) of the sides of \( K. \)

Remark 2: We note that the above definition of the acuteness is equivalent to the usual one (cf. Fujii [9]). Hence (4.1) implies that all the angles of the triangles of \( T_h \) are less than or equal to \( \pi/2. \)

Let

\[
\tilde{b}_{jk} = \tilde{b}_h(w_{kh}, w_{jh}) \quad \text{for } 1 \leq j, k \leq N + M ,
\] (4.3)

then we have

**Lemma 1**: It holds that

\[
\tilde{b}_{jj} = - \sum_{k \in \Lambda_j} \int_{\Gamma^S_{jk}} (1 - \lambda_{jk}) (b, n_j) d\gamma ,
\] (4.4)

\[
\tilde{b}_{jk} = \begin{cases} 
\int_{\Gamma^S_{jk}} (1 - \lambda_{jk}) (b, n_j) d\gamma & \text{(if } k \in \Lambda_j), \\
0 & \text{(if } k \notin \Lambda_j),
\end{cases}
\] (4.5)

and

\[
\sum_{k=1}^{N+M} \tilde{b}_{jk} = 0 \quad \text{for } 1 \leq j \leq N + M .
\] (4.6)

**Proof**: By the definition of \( \tilde{b}_h(\cdot, \cdot) \) we have

\[
\tilde{b}_{jk} = \sum_{l=1}^{N+M} \delta_{jl} \sum_{m \in \Lambda_l} \int_{\Gamma^S_{lm}} (b, n_l) (1 - \lambda_{lm}) (\delta_{km} - \delta_{kl}) d\gamma ,
\]
where $\delta_{ij}$ is the Kronecker delta. Hence, (4.4) and (4.5) hold. On the other hand, it holds that
\[
\sum_{j=1}^{N+M} \tilde{b}_{jk} = \tilde{b}_{jj} + \sum_{k \in \Lambda_j} \tilde{b}_{jk}.
\]
Then (4.6) follows from (4.4) and (4.5).

If we take $\lambda_{jk}$ as in (3.14), then we have from Lemma 1,
\[
\tilde{b}_{jk} \leq 0 \quad \text{for} \quad 1 \leq j, k \leq N + M, \quad j \neq k.
\tag{4.7}
\]
We set
\[
a_{jk} = a_n(w_{kh}, w_{jh}) \quad \text{for} \quad 1 \leq j \leq N, \quad 1 \leq k \leq N + M.
\tag{4.8}
\]
By an analogous discussion to Kikuchi [12] we have
\[
\sum_{j=1}^{N+M} a_{jk} = 0 \quad \text{for} \quad 1 \leq j \leq N, \\
a_{jk} \leq 0 \quad \text{for} \quad 1 \leq j \leq N, \quad 1 \leq k \leq N + M, \quad j \neq k.
\tag{4.9}
\]
The proof of (4.9) can be found in [14].

Now let us return to the scheme (4.11). Observe that (4.11) is equivalent to the following linear system:
\[
\begin{aligned}
TU + T_1 V &= F, \\
V &= G,
\end{aligned}
\tag{4.10}
\tag{4.11}
\]
where
\[
T = (\tilde{t}_{ij}) = (va_{ij} + \tilde{b}_{ij}) \quad \text{for} \quad 1 \leq i, j \leq N, \\
T_1 = (\tilde{t}_{ij}) \quad \text{for} \quad 1 \leq i \leq N, \quad N + 1 \leq j \leq N + M, \\
U = (U_j) = (\tilde{u}_h(B_j)) \quad \text{for} \quad 1 \leq j \leq N, \\
V = (U_j) \quad \text{for} \quad N + 1 \leq j \leq N + M, \\
F = (F_j) = (f(B_j) \cdot \text{mes (supp } w_{jh})/3) \quad \text{for} \quad 1 \leq j \leq N, \\
G = (G_j) = (u_0(B_j)) \quad \text{for} \quad N + 1 \leq j \leq N + M.
\tag{4.12}
\]
We define the matrix
\[
T_0 = (\tilde{t}_{ij}) \quad \text{for} \quad 1 \leq i \leq N, \quad 1 \leq j \leq N + M.
\tag{4.13}
\]
Then we have the following Lemma from (4.6), (4.7) and (4.9).
**Lemma 2:** If we take $\lambda_{jk}$ as in (3.14), then the matrix $T_0$ is of non-negative type, that is

\[
\sum_{j=1}^{N+M} \tilde{t}_{ij} \geq 0 \quad \text{for} \quad 1 \leq i \leq N.
\]

**Theorem 1:** Assume that the triangulation $T_h$ is of acute type and that the matrix $T$ is invertible. If we take $\lambda_{jk}$ as in (3.14), then we have

\[
\max_{1 \leq j \leq N} U_j \leq \max \left( 0, \max_{1 \leq j \leq M} G_{N+j} \right) \quad \text{if} \quad \max_{1 \leq j \leq N} F_j \leq 0. \quad (4.15)
\]

**Proof:** This fact comes from general considerations due to Ciarlet [3]. For the sake of completeness we, however, give a direct proof as follows.

First we prove (4.15) in the case of

\[
\max_{1 \leq j \leq N} F_j < 0. \quad (4.16)
\]

Let $U_i = \max_{1 \leq j \leq N} U_j$. When $U_i \leq 0$, then the assertion is trivial. Then we let $U_i > 0$. Assume that $U_i > \max_{1 \leq j \leq M} U_{N+j}$. Since

\[
\sum_{j=1}^{N} \tilde{t}_{ij} U_j + \sum_{j=1}^{M} \tilde{t}_{i,N+j} U_{N+j} = F_i \quad \text{for} \quad 1 \leq i \leq N,
\]

we have

\[
\tilde{t}_{ii} U_i = \sum_{j=1}^{N} (-\tilde{t}_{ij}) U_j + \sum_{j=1}^{M} (-\tilde{t}_{i,N+j}) U_{N+j} + F_i
\]

\[
\leq - U_i \sum_{j=1}^{N+M} \tilde{t}_{ij} + F_i, \quad (4.17)
\]

where we use in the last inequality the fact that the matrix $T_0$ is non-negative type by Lemma 2. Therefore we have

\[
0 > F_i \geq U_i \sum_{j=1}^{N+M} \tilde{t}_{ij} \geq 0, \quad (4.18)
\]

which is a contradiction. Thus we obtain (4.15).

Next, we prove (4.15) for the case

\[
\max_{1 \leq j \leq N} F_j \leq 0. \quad (4.19)
\]
We set a column vector $F_\varepsilon$ for any $\varepsilon > 0$ as follows:

$$F_\varepsilon = (F_{\varepsilon j}) = (F_j - \varepsilon) \quad \text{for} \quad 1 \leq j \leq N.$$  \hspace{1cm} (4.20)

Then we have $\max_{1 \leq j \leq N} F_{\varepsilon j} < 0$. Let $U_\varepsilon$ be the solution of the following equation:

$$TU + T_1 V = F_\varepsilon, \quad V = G.$$  \hspace{1cm} (4.21)\hspace{1cm} (4.22)

For the solution $U_\varepsilon$ of (4.21)-(4.22), (4.15) holds by the first half of this proof. Furthermore, since the matrix $T$ is invertible, we have

$$U_\varepsilon = T^{-1}(F_\varepsilon - T_1 V)$$  \hspace{1cm} (4.23)

and

$$U_\varepsilon \to U \quad \text{as} \quad \varepsilon \to 0.$$  \hspace{1cm} (4.24)

Thus our assertion is completely proved.

**Remark 3**: In Theorem 1, if the triangulation $T_h$ is of strictly acute type, namely if all the angles of triangles of $T_h$ are less than $\pi/2$, then the matrix $T$ is invertible. Another condition to assert the invertibility of $T$ is that $\text{div} \, \mathbf{b}$ is non-positive in $\Omega$, which will be shown in Theorem 3.

### 5. ERROR ESTIMATE FOR UPSTREAM-LIKE APPROXIMATION

In the first time, we show that the modified form $\tilde{b}_h(\cdot, \cdot)$ is admissible in the sense defined in Kikuchi and Ushijima [13].

**Theorem 2**: Assume that there exists a constant $D > 0$ independent of $j$, $k$ and $h$ such that

$$h \cdot \text{mes} (\Gamma_{jk}^S) \leq D \cdot \text{mes} (S_{jk}),$$  \hspace{1cm} (5.1)

then there exists a constant $C > 0$ independent of $h$ ($0 < h \leq \bar{h}$) such that

$$| b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h) | \leq Ch \| u_h \|_{1,h} \| v_h \|_{1,h}$$  \hspace{1cm} (5.2)

for all $u_h, v_h \in V_h$.

**Proof**: We follow the proof of Proposition 2 of [13] with suitable modifications. We may write for all $u_h, v_h \in V_h$

$$b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h) = b_h^1(u_h, v_h) - b_h^2(u_h, v_h) + b_h^1(u_h, L_h v_h) - b_h^2(u_h, L_h v_h)$$

$$- \tilde{b}_h^1(u_h, v_h) + \tilde{b}_h^2(u_h, v_h) - \tilde{b}_h^2(u_h, v_h).$$
Using (2.13) and (2.14), one can easily check that
\[ | b_h^I(u_h, v_h) - b_h^I(u_h, L_h v_h) | \leq C_1 h \| u_h \|_{1,h} \| v_h \|_{1,h} \] (5.3)
and
\[ | b_h^2(u_h, v_h) - b_h^2(u_h, v_h) | \leq C_2 h \| u_h \|_{1,h} \| v_h \|_{1,h} \] (5.4)
with the appropriate constants \( C_i, 1 \leq i \leq 2 \).

Thus it suffices to estimate \( b_h^1(u_h, L_h v_h) - b_h^1(u_h, v_h) \). From (3.7) and (3.8) we find that
\[
\begin{align*}
b_h^1(u_h, L_h v_h) - b_h^1(u_h, v_h) &= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b} \cdot \mathbf{n}) (u_h - u_h^k) \, d\gamma v_h(B_j) \\
&\quad + \sum_{j=N+1}^{N+M} \int_{\Omega_j \cap \Gamma} (\mathbf{b} \cdot \mathbf{n}) (u_h - u_h(B_j)) \, d\gamma v_h(B_j) = I_1 + I_2. (5.5)
\end{align*}
\]

Taking into account that \( \Gamma_{jk}^S = \Gamma_{kj}^S \) and \( \mathbf{n}_j = - \mathbf{n}_k \), we have
\[
I_1 = 1/2 \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b} \cdot \mathbf{n}) \{ \lambda_{jk} (u_h - u_h(B_j)) \\
+ (1 - \lambda_{jk}) (u_h - u_h(B_k)) \} \, d\gamma (v_h(B_j) - v_h(B_k)). (5.6)
\]

On the other hand, since \( u_h \) is linear on \( \overline{S}_{jk} \) it is easy to check that for any \( x \in \Gamma_{jk}^S \)
\[ | u_h(x) - u_h(B_j) | \leq h \text{ mes} (S_{jk})^{-1/2} \| \nabla u_h \|_{L^2(S_{jk})} \] (5.7)
and
\[ | v_h(B_j) - v_h(B_k) | \leq h \text{ mes} (S_{jk})^{-1/2} \| \nabla v_h \|_{L^2(S_{jk})}. (5.8) \]

Then from the properties of \( \lambda_{jk} \), (5.7), (5.8) and the assumption (5.1), we find that
\[ | I_1 | \leq C_3 h \| u_h \|_{1,h} \| v_h \|_{1,h}. \] (5.9)

Next, let us estimate for \( I_2 \). Here we assume that the mid-points of sides lying on \( \Gamma, B_i, N + 1 \leq i \leq N + M \), are located consecutively on the boundary in anti-clockwise orientation such that
\[ B_j = \text{mid-point of } \overline{P_j P_{j+1}} \quad \text{for } N + 1 \leq j \leq N + M, \] (5.10)
where \( P_j, N + 1 \leq j \leq N + M \), are boundary vertices with \( P_{N+M+1} = P_{N+1} \).
Then we may write

\[ I_2 = \sum_{j=N+1}^{N+M} \int_{P_j} (b(B_j) \cdot n_j) (u_h - u_h(B_j)) \, d\gamma \, v_h(B_j) \]

\[ + \sum_{j=N+1}^{N+M} \int_{P_j} (b - b(B_j)) \cdot n_j (u_h - u_h(B_j)) \, d\gamma \, v_h(B_j). \quad (5.11) \]

Since \( u_h \) is linear on \( P_j P_{j+1}, N + 1 \leq j \leq N + M \), the first term of \( I_2 \) vanishes. Since the second term of \( I_2 \) is equal to

\[ \sum_{j=N+1}^{N+M} \left\{ \int_{P_j} (b - b(B_j)) \cdot n_j (u_h - u_h(B_j)) \, d\gamma \right\} v_h(B_j), \]

using (5.1), (5.7) and the following fact

\[ | (b - b(B_j)) \cdot n_j | \leq C_4 h \quad \text{for} \quad x \in \overline{P_j B_j} \quad (\text{resp.} \overline{B_j P_{j+1}}), \]

we conclude that there exists a constant \( C_4 \) such that

\[ | I_2 | \leq C_4 h \| u_h \|_{1,h} \| v_h \|_{1,h}. \quad (5.12) \]

Combining (5.3), (5.4), (5.9) and (5.12), we obtain (5.2).

**Remark 4**: Since from Proposition 4.13 of Temam [16] the following discrete Poincaré inequality holds

\[ \| v_h \|_{0,\Omega} \leq C(\Omega) \| v_h \|_h \quad \text{for all} \quad v_h \in V_{oh} \]

with a constant \( C(\Omega) > 0 \) independent of \( h \in (0, \bar{h}] \), \( \| . \|_{1,h} \) and \( \| . \|_h \) are equivalent norms on \( V_{oh} \).

Hence we find a constant \( C > 0 \) independent of \( h \in (0, \bar{h}] \) such that

\[ | b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h) | \leq C h \| u_h \|_h \| v_h \|_h \quad \text{for all} \quad u_h, v_h \in V_{oh}. \]

In what follows, we shall restrict our attention to the homogeneous Dirichlet problem for simplicity, which is denoted by \((E^0)\). Thus we consider the following problem:

\[ (\Omega^0_h) \left\{ \begin{array}{l}
\text{Find} \ \tilde{u}_h \in V_{oh} \text{ such that} \\
\tilde{t}_h(\tilde{u}_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in V_{oh}.
\end{array} \right. \]
**Lemma 3**: It holds that for any \( v_h \in V_{oh} \)

\[
\mathcal{E}_h^1(v_h, v_h) + \frac{1}{2} \mathcal{E}_h^2(v_h, v_h) = \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2} \left( b \cdot n_j \right) \left( v_h(B_j) - v_h(B_k) \right)^2 \left( \lambda_{jk} - \frac{1}{2} \right) \, d\gamma. \tag{5.15}
\]

**Proof**: Let \( v_h \) be an arbitrary element in \( V_{oh} \). From (3.10) and (3.12) we have

\[
\mathcal{E}_h^1(v_h, v_h) + \frac{1}{2} \mathcal{E}_h^2(v_h, v_h) = \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \left( b \cdot n_j \right) \left( v_h^k - \frac{1}{2} v_h(B_j) \right) v_h(B_j) \, d\gamma
\]

Since \( \Gamma_{jk} = \Gamma_{kj} \) and \( n_j = -n_k \), we obtain by using (3.11)

\[
\mathcal{E}_h^1(v_h, v_h) + \frac{1}{2} \mathcal{E}_h^2(v_h, v_h)
\]

\[
= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2} \left( b \cdot n_j \right) \left\{ \left( v_h^k - \frac{1}{2} v_h(B_j) \right) v_h(B_j) - \left( v_h^j - \frac{1}{2} v_h(B_k) \right) v_h(B_k) \right\} \, d\gamma
\]

\[
= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2} \left( b \cdot n_j \right) \left( \lambda_{jk} - \frac{1}{2} \right) \left\{ v_h(B_j)^2 - 2 v_h(B_j) v_h(B_k) + v_h(B_k)^2 \right\} \, d\gamma
\]

\[
= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2} \left( b \cdot n_j \right) \left( v_h(B_j) - v_h(B_k) \right)^2 \left( \lambda_{jk} - \frac{1}{2} \right) \, d\gamma.
\]

**Theorem 3**: Assume that there exists a constant \( \alpha' > 0 \) such that

\[
v C(\Omega)^{-2} - \frac{1}{2} \cdot \text{div} \, b \geq \alpha' > 0 \quad \text{in} \quad \Omega, \tag{5.17}
\]

where \( C(\Omega) \) is the constant in the discrete Poincaré inequality (5.13). If we take \( \lambda_{jk} \) as in (3.14), then the problem \((\tilde{\mathbf{u}}^0_h)\) has a unique solution \( \tilde{u}_h \in V_{oh} \).

**Proof**: It is sufficient to show the \( V_{oh} \)-coercivity of \( \tilde{\mathcal{I}}_h(\cdot, \cdot) \) and Lemma 3, we have for all \( v_h \in V_{oh} \)

\[
\mathcal{I}_h(v_h, v_h) = \nabla_h(v_h, v_h) + \mathcal{E}_h^1(v_h, v_h) + \frac{1}{2} \mathcal{E}_h^2(v_h, v_h) + \frac{1}{2} \mathcal{E}_h^2(v_h, v_h)
\]

\[
= v \left\| v_h \right\|_h^2 + \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2} \left( b \cdot n_j \right) \left( v_h(B_j) - v_h(B_k) \right)^2 \left( \lambda_{jk} - \frac{1}{2} \right) \, d\gamma
\]

\[
- \frac{1}{2} \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \left( b \cdot n_j \right) \, d\gamma \, v_h(B_j)^2. \tag{5.18}
\]
By virtue of the choice of $\lambda_{jk}$, we find that

$$
\sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}} \frac{1}{2} (\mathbf{b} \cdot \mathbf{n}_j) (v_h(B_j) - v_h(B_k))^2 \left( \lambda_{jk} - \frac{1}{2} \right) d\gamma \geq 0 . \quad (5.19)
$$

Then we have

$$
\tilde{t}_h(v_h, v_h) \geq \nu \| v_h \|_h^2 - \frac{1}{2} \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}} (\mathbf{b} \cdot \mathbf{n}_j) \frac{1}{2} \left( \lambda_{jk} - \frac{1}{2} \right) d\gamma (v_h(B_j))^2
$$

$$
= \nu \| v_h \|_h^2 - \frac{1}{2} \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}} (\mathbf{b} \cdot \mathbf{n}_j) \frac{1}{2} \left( \lambda_{jk} - \frac{1}{2} \right) d\gamma (v_h(B_j))^2
$$

$$
+ \frac{N+M}{2} \int_{\partial \Omega_j \cap \Gamma} (\mathbf{b} \cdot \mathbf{n}_j) \frac{1}{2} \left( \lambda_{jk} - \frac{1}{2} \right) d\gamma (v_h(B_j))^2
$$

where in the last equality we used the fact that $v_h(B_j) = 0$ for $N+1 \leq j \leq N+M$.

According to the patch-wise application of the Gauss divergence formula, it holds that

$$
\tilde{t}_h(v_h, v_h) \geq \nu \| v_h \|_h^2 - \frac{1}{2} \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}} (\text{div} \mathbf{b}) | L_h v_h |^2 dx . \quad (5.20)
$$

Then for any constant $\varepsilon$ satisfying $0 < \varepsilon < \min \{ 1, \alpha' C(\Omega)^2 / \nu \}$ we have

$$
\tilde{t}_h(v_h, v_h) \geq \nu \varepsilon \| v_h \|_h^2 \quad \text{for all } v_h \in V_{\text{oh}} . \quad (5.21)
$$

**THEOREM 4:** Assume the hypotheses of Theorem 3. Then $t_h(\cdot, \cdot)$ is coercive on $V_{\text{oh}}$ for any sufficiently small $h$.

**Proof.** For all $v_h \in V_{\text{oh}}$ we have

$$
t_h(v_h, v_h) = \tilde{t}_h(v_h, v_h) + b_h(v_h, v_h) - \tilde{b}_h(v_h, v_h) . \quad (5.22)
$$

Therefore, from Theorem 3 and Remark 4 it holds that

$$
t_h(v_h, v_h) \geq (\nu \varepsilon - C h) \| v_h \|_h^2 \quad \text{for all } v_h \in V_{\text{oh}} , \quad (5.23)
$$

where $C$ is the positive constant independent of $h$ in (5.14).

Thus, we find that there is a constant $h_0 > 0$ such that it holds for some constant $\alpha^* > 0$

$$
t_h(v_h, v_h) \geq \alpha^* \| v_h \|_h^2 \quad \text{for all } v_h \in V_{\text{oh}} , \quad (5.24)
$$

provided that $h \leq h_0$. 

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Now, let us derive a bound for the error $\| \tilde{u}_h - u_h \|_h$, where $\tilde{u}_h$ is the solution of $(\Pi_h^\lambda)$ with $\lambda_{jk}$ as in (3.14) and $u_h$ is the solution of $(\Pi_h^0)$ which is the Galerkin approximation of $(E^0)$.

**Theorem 5**: Assume the hypotheses of Theorem 3. Then we have

$$\| \tilde{u}_h - u_h \|_h \leq Ch \| f \|_{0,\Omega}$$  \hspace{1cm} (5.25)

for some constant $C > 0$.

**Proof**: Since

$$\tilde{t}_h(\tilde{u}_h, v_h) = (f, v_h) = t_h(u_h, v_h) \quad \text{for all} \quad v_h \in V_{0h},$$  \hspace{1cm} (5.26)

we may write

$$t_h(u_h, v_h) - \tilde{t}_h(u_h, v_h) + \tilde{t}_h(u_h, v_h) = \tilde{t}_h(\tilde{u}_h, v_h) \quad \text{for all} \quad v_h \in V_{0h}. \hspace{1cm} (5.27)$$

Hence we have

$$\tilde{t}_h(\tilde{u}_h - u_h, v_h) = t_h(u_h, v_h) - \tilde{t}_h(u_h, v_h)$$

$$= b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h).$$  \hspace{1cm} (5.28)

According to Remark 4, it holds that

$$| \tilde{t}_h(\tilde{u}_h - u_h, v_h) | \leq Ch \| u_h \|_h \| v_h \|_h.$$  \hspace{1cm} (5.29)

Taking $v_h = \tilde{u}_h - u_h$ in (5.29) we obtain from the coercivity of $\tilde{t}_h(\cdot, \cdot)$

$$\| \tilde{u}_h - u_h \|_h^2 \leq Ch \| u_h \|_h \| \tilde{u}_h - u_h \|_h.$$  \hspace{1cm} (5.30)

Hence we have

$$\| \tilde{u}_h - u_h \|_h \leq C' h \| u_h \|_h.$$  \hspace{1cm} (5.30)

On the other hand, from the coercivity of $t_h(\cdot, \cdot)$ we can show that

$$\| u_h \|_h \leq 1/\alpha^* \| f \|_{0,\Omega}.$$  \hspace{1cm} (5.31)

Then, from (5.30) and (5.31) the assertion follows.

**Remark 5**: In order to obtain the error estimate for $u - u_h$ in the norm $\| . \|_h$, we can apply the primal hybrid finite element method introduced by Raviart-Thomas [15], since the linear nonconforming finite element is one of the hybrid elements. Then we have the following result:

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**Theorem 6**: In addition to the hypotheses of Theorem 3, suppose that \( \{ T_h \} \) is regular. If \( u \in H^2(\Omega) \cap H_0^1(\Omega) \), then we have
\[
\| u - u_h \|_h \leq C h \| u \|_{2,\Omega}.
\] (5.32)

Using Theorems 5 and 6, we can derive an error estimate for the upstream-like approximation with the linear nonconforming finite element.

**Theorem 7**: Assume the hypotheses of Theorem 6. Then we have
\[
\| u - \tilde{u}_h \|_h \leq C h (\| u \|_{2,\Omega} + \| f \|_{0,\Omega}).
\] (5.33)

6. NUMERICAL EXAMPLES

As an illustration, here we adopt one of the problems treated in Kikuchi and Ushijima [13]. Namely, our model problem is:

\[
\begin{aligned}
- \nu \Delta u + (b \cdot \nabla) u &= 1 & \text{in } \Omega &= (0, 1) \times (0, 1), \\
\quad u &= 0 & \text{on } \Gamma,
\end{aligned}
\] (6.1)

where \( b = (1, 0) \). In [13], the following initial boundary value problem (6.2) is taken as an approximation of (6.1) for sufficiently small \( \nu \) in the region far from \( x_1 = 1 \).

\[
\begin{aligned}
\frac{\partial u}{\partial x_1} - \nu \frac{\partial^2 u}{\partial x_2^2} &= 1 & \text{in } \Omega, \\
\quad u(0, x_2) &= 0 & \text{for } 0 < x_2 < 1, \\
\quad u(x_1, 0) &= u(x_1, 1) = 0 & \text{for } 0 < x_1 < 1.
\end{aligned}
\] (6.2)

Examples of employed meshes are pictured in figure 2, where \( N \) denotes a number of elements along the side \( x_2 = 0 \) (or \( x_1 = 0 \)) of the domain \( \Omega \). Figures 3, 4, 5 and 6 show the distributions of the numerical solutions \( u_h \) and \( \tilde{u}_h \) along the line \( x_1 = 1/2 \) of the square domain \( \Omega \), where \( u_h \) is the linear nonconforming finite element approximation and \( \tilde{u}_h \) is the linear nonconforming finite element approximation of upstream type. In these figures, continuous curves are the profiles of numerical solutions of the problem (6.2), which are denoted by PEA. Among these results, the Galerkin method gives a strongly oscillating solution for the coarse meshes and the small values of \( \nu \), but gives an improved one for sufficiently fine meshes. On the other hand, our method gives a non-oscillating and reasonable solution.

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Fig. 2. — Finite element meshes for $N = 5$. 
Fig. 3. — Distributions of $u_h(0.5, x_2)$ and $\tilde{u}_h(0.5, x_2)$ with $\nu = 0.01$ and $N = 5$. 
Fig. 4. — Distributions of $u_h(0.5, x_2)$ and $\tilde{u}_h(0.5, x_2)$ with $\nu = 0.01$ and $N = 10$. 

$v = 0.01$

$N = 10$

$\triangle = u_h$

$\square = \tilde{u}_h$

--- = PEA
Fig. 5. — Distributions of $u_h(0.5, x_2)$ and $\hat{u}_h(0.5, x_2)$ with $\nu = 0.001$ and $N = 10$. 
Fig. 6. — Distributions of $u_h(0.5, x_2)$ and $\bar{u}_h(0.5, x_2)$ with $v = 0.001$ and $N = 20$. 

$v = 0.001$

$N = 20$

$\triangle = u_h$

$\square = \bar{u}_h$

$\triangle = \text{PEA}$
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