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FINITE ELEMENT APPROXIMATION OF STEADY NAVIER-STOKES EQUATIONS WITH MIXED BOUNDARY CONDITIONS (*)

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Abstract. — *We consider the steady Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary Γ. As boundary conditions we require that the normal velocity component and the tangential stress components vanish on Γ. Problems of this type arise as subproblems when dealing with fluid flows subject to surface tension. The continuous problem is discretized using a non-conforming mixed finite element method with quadratic elements for the velocities and linear elements for the pressure. For sufficiently small data both the continuous and the discrete problem have unique solutions. We obtain $O(h^{1/2})$ error estimates for the H^1-norm of the velocities and the L^2-norm of the pressure and an $O(h)$ error estimate for the L^2-norm of the velocities. The suboptimality of the error estimates is due to the non-conformity of the method. However, this cannot be avoided as is shown by a Babuška-type paradox.*

Résumé. — *Nous considérons les équations de Navier-Stokes dans un domaine $\Omega \subset \mathbb{R}^3$ au bord Γ régulier. Comme condition de limite nous supposons que la direction normale du champ de vitesse et les directions tangentielles de la tension superficielle sont nulles sur Γ. De tels problèmes apparaissent quand on considère l'écoulement d'un fluide soumis à une surtension superficielle. On utilise une méthode non conforme d'éléments finis mixtes avec des éléments quadratiques pour le champ de vitesse et des éléments linéaires pour la pression. Le problème continu et le problème discret possèdent des solutions uniques si la force extérieure est assez petite. On établit une majoration d'erreur d'ordre $O(h^{1/2})$ pour la norme H^1 du champ de vitesse et la norme L^2 de la pression et d'ordre $O(h)$ pour la norme L^2 du champ de vitesse. La majoration d'erreur n'est pas optimale en raison de la non-conformité de la méthode. Un exemple analogue au paradoxe de Babuška montre que cette non-conformité est indispensable.*

1. INTRODUCTION

The flow inside a volume $\Omega \subset \mathbb{R}^3$ of fluid governed by an exterior force $\underline{f}$ and surface tension is described by the Navier-Stokes equations

$$\left.\begin{aligned} -\nu\,\Delta\underline{u} + \nabla p + (\underline{u}.\nabla)\,\underline{u} &= \underline{f} \\ \operatorname{div}\underline{u} &= 0 \end{aligned}\right\} \text{ in } \Omega \tag{1.1}$$

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with the mixed boundary conditions

$$\underline{u}.\underline{n} = \underline{n}.\underline{\underline{T}}(\underline{u}, p).\underline{\tau}_k = 0 \quad \text{on} \quad \Gamma := \partial\Omega\,, \quad k = 1, 2\,, \tag{1.2}$$

(*cf.* [2, 3, 9]). Here $\underline{n}$ denotes the outward normal to Ω, $\underline{\tau}_k$, $k = 1, 2$, orthonormal vectors spanning the tangent plane and

$$\underline{\underline{T}}(\underline{u}, p) := -p\delta_{ij} + \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad 1 \leqslant i, j \leqslant 3\,,$$

the stress tensor. The boundary Γ of the fluid is not known a priori. It is determined by the condition that its mean curvature is proportional to the normal stress component :

$$2\,\kappa H = \underline{n}.\underline{\underline{T}}(\underline{u}, p).\underline{n} \quad \text{on}\ \Gamma\,.$$

The existence, uniqueness and regularity of solutions to this problem are investigated in [2, 3].

As a first step towards error estimates for finite element approximations of this problem, we consider problem (1.1) with boundary conditions (1.2) in a *fixed* bounded domain $\Omega \subset \mathbb{R}^3$ with three times continuously differentiable boundary Γ. In order to simplify the notation we assume in addition that Ω is *convex.*

We consider a non-conforming mixed finite element method. For sufficiently small data $\nu^{-2}\underline{f}$ both the continuous and the discrete problem have unique solutions. We obtain $0(h^{1/2})$ error estimates for the H^1-norm of the velocities and the L^2-norm of the pressure and an $0(h)$ error estimate for the L^2-norm of the velocities. The suboptimality of the error estimates is due to the non-conformity. However, this cannot be avoided as is shown in the last section by a Babuška-type paradox.

2. FINITE ELEMENT DISCRETIZATION

Denote by $H^k(\Omega)$, $k \geqslant 0$, and $L^2(\Omega) := H^0(\Omega)$ the usual Sobolev and Lebesgue spaces equipped with the seminorm

$$|\,v\,|_k := \left\{ \int_\Omega \sum_{|\alpha| = k} |\,D^\alpha v(x)\,|^2\, dx \right\}^{1/2} \tag{2.1}$$

and norm

$$\|\,v\,\|_k := \left\{ \sum_{l=0}^{k} |\,v\,|_l^2 \right\}^{1/2}. \tag{2.2}$$

Since no confusion can arise, we use the same notation for the corresponding norm and seminorm on $H^k(\Omega)^3$. The inner product of $L^2(\Omega)$ and $L^2(\Omega)^3$ will be denoted by $(u, v)_0$.

Let

$$\mathscr{S} := \text{span}\,\{\,\underline{u}(\underline{x}) = \underline{\beta} \wedge \underline{x} : \beta \text{ is an axis of symmetry of } \Omega\,\} \qquad (2.3)$$

where $\wedge$ denotes the vector product. Put

$$\begin{aligned} X &:= \{\,\underline{u} \in H^1(\Omega)^3 : \underline{u}.\underline{n} = 0 \text{ on } \Gamma\,\}/\mathscr{S}\,, \\ M &:= \left\{ p \in L^2(\Omega) : \int_\Omega p\,dx = 0 \right\} \end{aligned} \qquad (2.4)$$

and denote by

$$\underline{\underline{D}}(\underline{u})_{ij} := \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad 1 \leqslant i, \quad j \leqslant 3\,, \qquad (2.5)$$

the deformation tensor. We introduce the following three bilinear resp. trilinear forms for $\underline{u}, \underline{v}, \underline{w} \in H^1(\Omega)^3, p \in L^2(\Omega)$:

$$a(\underline{u}, \underline{v}) := \frac{\nu}{2} \int_\Omega \underline{\underline{D}}(\underline{u})\,\underline{\underline{D}}(\underline{v})\,dx\,, \qquad (2.6a)$$

$$b(\underline{u}, p) := -\int_\Omega p \operatorname{div} \underline{u}\,dx\,, \qquad (2.6b)$$

$$N(\underline{u}, \underline{v}, \underline{w}) := \int_\Omega [(\underline{u}.\nabla)\,\underline{v}]\,\underline{w}\,dx\,. \qquad (2.6c)$$

The weak formulation of problem (1.1), (1.2) to which we will refer as *Problem* $(\mathscr{N})$ then is :

$$\begin{aligned} &\text{Find } (\underline{u}, p) \in X \times M \text{ such that} \\ &a(\underline{u}, \underline{v}) + b(\underline{v}, p) + N(\underline{u}, \underline{u}, \underline{v}) = (\underline{f}, \underline{v})\,, \quad \forall \underline{v} \in X\,, \\ &b(\underline{u}, q) = 0 \quad \forall q \in M\,. \end{aligned}$$

The corresponding linear problem without the non-linear term $(N(\underline{u}, \underline{u}, \underline{v})$ will be refered to as *Problem* $(\mathscr{S})$. Problem (S) always has a unique solution and the regularity estimate

$$\|\,\underline{u}\,\|_2 + \|\,p\,\|_1 \leqq c\,\|\,\underline{f}\,\|_0 \qquad (2.7)$$

holds (*cf.* [2, 3, 9]). If the data $\nu^{-2} \| \underline{f} \|_0$ are sufficiently small, Problem (N) also has a unique solution and the regularity estimate (2.7) holds (*cf.* [2, 3, 9]).

Let $\Omega_h \subset \overline{\Omega}$ be a family of polyhedrons satisfying the assumptions :

$(A1)$ each vertex of Ω_h lies on Γ.

$(A2)$ the length of all edges of Ω_h can be bounded from below and from above by $\underline{c}h$ and $\overline{c}h$ resp. with constants $\underline{c}$, $\overline{c}$ independent of h.

We divide each Ω_h into tetrahedrons with edges of length $0(h)$ such that the resulting family $\mathcal{T}_h$ satisfies the usual regularity assumptions for finite elements (*cf.* [6]). For simplicity we assume :

$(A3)$ each face of Ω_h is the face one $T \in \mathcal{T}_h$.

Denote by $\mathscr{P}_h$ the set of vertices of Ω_h. Let S_h^r be the space of continuous finite elements corresponding to $\mathcal{T}_h$ which are piecewise polynomials of degree $\leqq r$. Put

$$
\begin{aligned}
X_h &:= \{ u \in (S_h^2)^3 : \underline{u}.\underline{n} = 0 \quad \forall Q \in \mathscr{P}_h \}/\mathscr{S} \\
M_h &:= \left\{ p \in S_h^1 : \int_{\Omega_h} p \, dx = 0 \right\}.
\end{aligned}
\tag{2.8}
$$

Note that $X_h \subset H^1(\Omega_h)^3$, but $X_h \not\subset X$ and $\underline{u}.\underline{n}_h \neq 0$ on $\Gamma_h := \partial\Omega_h$ where $\underline{n}_h$ is the outer normal to Ω_h.

We denote the seminorm and norm of $H^k(\Omega_h)$ by $|\,.\,|_{k,h}$ and $\|\,.\,\|_{k,h}$ resp. The inner product of $L^2(\Omega)$ is denoted by $(.,.)_{0,h}$. Finally, we introduce discrete analogues of the forms a, b and N :

$$
a_h(u, v) := \frac{\nu}{2} \int_{\Omega_h} \underline{\underline{D}}(\underline{u}) \, \underline{\underline{D}}(\underline{v}) \, dx \,, \tag{2.9a}
$$

$$
b_h(\underline{u}, p) := - \int_{\Omega_h} p \operatorname{div} \underline{u} \, dx \,, \tag{2.9b}
$$

$$
N_h(\underline{u}, \underline{v}, \underline{w}) := \frac{1}{2} \int_{\Omega_h} ([(\underline{u}.\nabla)\,\underline{v}]\,\underline{w} - [(\underline{u}.\nabla)\,\underline{w}]\,\underline{v}) \, dx \,, \tag{2.9c}
$$

$$
\forall \underline{u}, \underline{v}, \underline{w} \in H^1(\Omega_h)^3 \,, \quad p \in L^2(\Omega_h) \,.
$$

The discrete approximation of Problem (N) to which we will refer as *Problem* $(\mathcal{N}_h)$ then is :

$$
\begin{aligned}
&\text{Find } (\underline{u}_h, p_h) \in X_h \times M_h \text{ such that} \\
&a_h(\underline{u}_h, \underline{v}_h) + b_h(\underline{v}_h, p_h) + N_h(\underline{u}_h, \underline{u}_h, \underline{v}_h) = (\underline{f}, \underline{v}_h)_{0,h} \,, \quad \forall \underline{v}_h \in X_h \,, \\
&b_h(\underline{u}_h, q_h) = 0 \,, \quad \forall q_h \in M_h \,.
\end{aligned}
$$

The corresponding linear problem without the non-linear term $N_h(\underline{u}_h, \underline{u}_h, \underline{v}_h)$ will be refered to as *Problem* $(\mathcal{S}_h)$.

In the sequel $c, c_0, c_1, \ldots$ denote various constants which are independent of h but have different values depending on the context. Moreover, we will often use the Green's formula

$$\int_\Omega \{ -\nu \nabla \underline{u} + \nabla p \} \underline{v} = \int_\Omega \left\{ \frac{\nu}{2} \underline{\underline{D}}(u) \underline{\underline{D}}(v) - p \operatorname{div} \underline{v} \right\} - \int_\Gamma n \underline{\underline{T}}(u, p) \underline{v} \quad (2.10)$$

(*cf.* (2.7) in [2]) which holds for $\underline{v} \in H^1(\Omega)^3$, $p \in H^1(\Omega)$ and $\underline{u} \in H^2(\Omega)^3$ with $\operatorname{div} \underline{u} = 0$.

3. ERROR ESTIMATES FOR THE LINEAR PROBLEM

In this section we want to establish error estimates for the linear Problems (S) and (S_h). Recall that $\Gamma := \partial\Omega$, $\Gamma_h := \partial\Omega_h$ and that $\underline{n}$ and $\underline{n}_h$ denote the normal to Ω and Ω_h, resp.

LEMMA 3.1 : *There is an $h_0 > 0$ such that the boundary estimates*

$$\| \underline{u} \cdot \underline{n}_h \|_{L^2(\Gamma_h)} \leqq c \cdot h^{1/2} \| \underline{u} \|_{1,h}, \qquad \forall \underline{u} \in X_h, \quad (3.1)$$

and

$$\| \underline{u} \cdot \underline{n}_h \|_{L^2(\Gamma_h)} \leqq ch \| \underline{u} \|_1, \qquad \forall \underline{u} \in X, \quad (3.2)$$

hold for all $0 < h \leqq h_0$.

Proof : Let S be a face of Ω_h and Q be a vertex of S. Denote by $T \in \mathcal{C}_h$ the tetrahedron which has S as face and by $\underline{n}_Q$ the normal to Ω in Q. Since $\Gamma \in C^2$ and all vertices of Ω_h lie on Γ, we have $| \underline{n}_h - \underline{n}_Q | \leqq ch$ and thus

$$\int_S | \underline{u} \cdot \underline{n}_h |^2 \leqq ch^2 \int_S | u |^2 + \int_S | \underline{u} \cdot \underline{n}_Q |^2 . \quad (3.3)$$

Let $\varphi : T \to \hat{T}$, be the linear transformation which maps T into the tetrahedron $\hat{T}$ with vertices (0, 0, 0), (0, 1, 0), (1,.0, 0), (0, 0, 1) such that $\varphi(Q) = 0$. Put

$$v(x) := \underline{u} \cdot \underline{n}_Q(\varphi^{-1}(x)), \qquad \forall x \in \hat{T} .$$

If $\underline{u} \in X_h$, then v belongs to the space of quadratic polynomials on $\hat{T}$ vanishing in the origin. On this space, $| \cdot |_{H^1(\hat{T})}$ and $\| \cdot \|_{H^1(\hat{T})}$ are equivalent norms.

The trace theorem and a homogeneity argument therefore imply

$$\int_S |\underline{u}.\underline{n}_Q|^2 \leqq ch^2 \int_{\varphi(S)} |v|^2 \leqq ch^2 |v|^2_{H^1(\hat{T})} \leqq ch |\underline{u}|^2_{H^1(T)} .$$

Recalling (3.3) and summing over all faces of Ω_h yields (3.1). To prove (3.2), denote by Δ the part of $\Omega \setminus \Omega_h$ which has S as face. Associate with $Q \in S$ the point $\mu(Q) \in \Gamma$ which lies on the line through Q with direction $\underline{n}_h$. Since $\underline{u}(\mu(Q)).\underline{n}(\mu(Q)) = 0$ we conclude

$$\begin{aligned}
\int_S |\underline{u}.\underline{n}_h|^2 \, dQ &\leqq \max_{Q \in S} |\underline{n}_h - \underline{n}(\mu(Q))|^2 \int_S |\underline{u}|^2 \, dQ \\
&\quad + \int_S |\underline{u}.\underline{n}(\mu(Q))|^2 \, dQ \\
&\leqq ch^2 \int_S |\underline{u}|^2 \, dQ \\
&\quad + \int_S \left| \int_0^1 \frac{d}{ds} [\underline{u}(Q + s(\mu(Q) - Q)).\underline{n}(\mu(Q))] \, ds \right|^2 dQ \\
&\leqq ch^2 \int_S |\underline{u}|^2 \, dQ + ch^2 \|\underline{u}\|^2_{H^1(\Delta)^3}
\end{aligned}$$

where we have used $|Q - \mu(Q)| \leqq 0(h^2)$. Summing over all faces of Γ_h and recalling the continuous imbedding $H^1(\Omega_h)^3 \to L^2(\Gamma_h)^3$ this implies (3.2). Note that the norm of the above injection depends on meas (Ω_h) and diam (Ω_h) and can therefore be bounded independently of h for sufficiently small h. □

Since X_h contains all continuous, piecewise quadratic finite elements which vanish on Γ_h, we conclude from [4] with the same arguments as in [10] that

$$\inf_{p_h \in M_h \setminus \{0\}} \sup_{\underline{u}_h \in X_h \setminus \{0\}} \frac{b_h(\underline{u}_h, p_h)}{\|\underline{u}_h\|_{1,h} \|p_h\|_{0,h}} \geqq \beta > 0 \tag{3.4}$$

holds with a constant β independent of h. The second Korn inequality, the generalized Poincaré-Friedrichs inequality (*cf.* [9]) and Lemma (3.1) imply

$$\begin{aligned}
c_0 \|\underline{u}\|^2_{1,h} &\leqq \int_{\Omega_h} |\underline{\underline{D}}(\underline{u})|^2 + (c_0 + c_1) \|\underline{u}\|^2_{0,h} \\
&\leqq (1 + c_0 c_2 + c_1 c_2) \int_{\Omega_h} |\underline{\underline{D}}(\underline{u})|^2 + (c_0 + c_1) c_2 \left| \int_{\Gamma_h} \underline{u}.\underline{n}_h \right|^2 \\
&\leqq c_3 \int_{\Omega_h} |\underline{\underline{D}}(\underline{u})|^2 + c_4 h \|\underline{u}\|^2_{1,h}, \qquad \forall \underline{u} \in X_h .
\end{aligned}$$

Hence there is an $h_0 > 0$ and a constant $\alpha > 0$ independent of h such that

$$a_h(\underline{u}, \underline{u}) = \frac{\nu}{2} \int_{\Omega_h} | \underline{\underline{D}}(\underline{u}) |^2 \geqq \alpha\nu \| \underline{u} \|_{1,h}^2 , \qquad \forall \underline{u} \in X_h , \quad 0 < h \leqq h_0 . \quad (3.5)$$

The continuity of a_h, b_h, equations (3.4), (3.5) and standard results on mixed problems (*cf.* [5]) imply the unique solvability of Problem (S_h) for all $0 < h \leqq h_0$.

THEOREM 3.2 : *Let* $(\underline{u}, p) \in X \times M$ *and* $(\underline{u}_h, p_h) \in X_h \times M_h$, $0 < h \leqq h_0$, *be the unique solution of Problem* (S) *and* (S_h) *resp. Then the error estimate*

$$\| \underline{u} - \underline{u}_h \|_{1,h} + \| p - p_h \|_{0,h} \leqq c h^{1/2} \| \underline{f} \|_0 \qquad (3.6)$$

holds.

Proof : Let p_h^* be the best approximation of p in the $\| \cdot \|_{0,h}$-norm by elements of M_h. Standard approximation results (*cf.* [6]) and equation (2.7) imply

$$\| p - p_h^* \|_{0,h} \leqq ch \| \underline{f} \|_0 . \qquad (3.7)$$

Denote by $\underline{u}_h^*$ the interpolating function of $\underline{u}$ by elements of $(S_h^2)^3$. Since $\underline{u} \in H^2(\Omega)^3 \subset X$, $\underline{u}_h^*$ is well defined, lies in X_h and satisfies

$$\| \underline{u} - \underline{u}_h^* \|_{1,h} \leqq ch \| \underline{u} \|_2 \leqq ch \| \underline{f} \|_0 . \qquad (3.8)$$

The functions $\underline{u} \in H^2(\Omega)^3$, $p \in H^1(\Omega)$ are solutions of

$$\left.\begin{aligned} - \nu \, \Delta \underline{u} + \nabla p &= \underline{f} \\ \operatorname{div} \underline{u} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\underline{u} . \underline{n} = \underline{n} . \underline{\underline{D}}(\underline{u}) . \underline{\tau}_k = 0 , \quad k = 1, 2, \text{ on } \Gamma . \qquad (3.9)$$

Multiplying (3.9) with $\underline{v}_h \in X_h$, integrating over Ω_h and using (2.10) yields

$$(\underline{f}, \underline{v}_h)_{0,h} = a_h(\underline{u}, \underline{v}_h) + b_h(\underline{v}_h, p) - \int_{\Gamma_h} \underline{n}_h \, \underline{\underline{T}}(\underline{u}, p) \, \underline{v}_h .$$

Hence, we have

$$a_h(\underline{u} - \underline{u}_h, \underline{v}_h) + b_h(\underline{v}_h, p - p_h) = \int_{\Gamma_h} \underline{n}_h \, \underline{\underline{T}}(\underline{u}, p) \, \underline{v}_h , \qquad \forall \underline{v}_h \in X_h , \quad (3.10)$$

where the right hand side is due to the non-conformity. To simplify the notation, put

$$R := \sup_{\underline{v}_h \in X_h \setminus \{0\}} \frac{1}{\| \underline{v}_h \|_{1,h}} \int_{\Gamma_h} \underline{n}_h \, \underline{\underline{T}}(\underline{u}, p) \, \underline{v}_h . \qquad (3.11)$$

First, we estimate $\| p - p_h \|_{0,h}$. From (3.4), (3.7), (3.10) and (3.11) we conclude

$$\begin{aligned}\beta \| p_h - p_h^* \|_{0,h} &\leq \sup_{\underline{v}_h \in X_h \setminus \{0\}} \frac{1}{\| \underline{v}_h \|_{1,h}} b_h(\underline{v}_h, p_h - p_h^*) \\ &\leq ch \| \underline{f} \|_0 + \sup_{\underline{v}_h \in X_h \setminus \{0\}} \frac{1}{\| \underline{v}_h \|_{1,h}} b_h(\underline{v}_h, p_h - p) \\ &\leq ch \| \underline{f} \|_0 + c \| \underline{u} - \underline{u}_h \|_{1,h} + R\end{aligned}$$

and thus

$$\| p - p_h \|_{0,h} \leq ch \| f \|_0 + c \| \underline{u} - \underline{u}_h \|_{1,h} + R\,. \tag{3.12}$$

Put $\underline{w}_h := \underline{u}_h - \underline{u}_h^*$. From (3.5), (3.8) we get

$$\begin{aligned}\alpha\nu \| \underline{w}_h \|_{1,h}^2 &\leq a_h(\underline{w}_h, \underline{w}_h) \leq \\ &\leq ch \| \underline{f} \|_0 \| \underline{w}_h \|_{1,h} + a_h(\underline{u}_h - \underline{u}, \underline{w}_h) \\ &\leq \{ ch \| \underline{f} \|_0 + R \} \| \underline{w}_h \|_{1,h} + b_h(\underline{w}_h, p - p_h) \\ &\leq \{ ch \| \underline{f} \|_0 + R \} \| \underline{w}_h \|_{1,h} + ch \| \underline{f} \|_0 \| p - p_h \|_{0,h} \\ &\quad + b_h(\underline{u}_h - \underline{u}, p - p_h) \\ &\leq c' \{ ch \| \underline{f} \|_0 + R \} \{ \| \underline{w}_h \|_{1,h} + ch \| \underline{f} \|_0 \}\end{aligned} \tag{3.13}$$

where we have used (3.12) and

$$b_h(\underline{u} - \underline{u}_h, q_h) = 0\,, \qquad \forall q_h \in M_h\,.$$

Next, we estimate R. Let $\underline{v} \in X_h$. The trace theorem, eq. (2.7) and Lemma (3.1) imply

$$\begin{aligned}&\int_{\Gamma_h} \underline{n}_h \, \underline{\underline{T}}(\underline{u}, p) \, \underline{v}_h \leq \\ &\leq \| \underline{n}_h \, \underline{\underline{T}}(\underline{u}, p) \, \underline{n}_h \|_{L^2(\Gamma_h)} \| \underline{v}_h \cdot \underline{n}_h \|_{L^2(\Gamma_h)} \\ &\quad + \sum_{k=1}^{2} \| \underline{n}_h \, \underline{\underline{T}}(\underline{u}, p) \, \underline{\tau}_{hk} \|_{L^2(\Gamma_h)} \| \underline{v}_h \cdot \underline{\tau}_{hk} \|_{L^2(\Gamma_h)} \\ &\leq ch^{1/2} \| \underline{f} \|_0 \| \underline{v}_h \|_{1,h} + c \| \underline{v}_h \|_{1,h} \sum_{k=1}^{2} \| \underline{n}_h \, \underline{\underline{D}}(\underline{u}) \, \underline{\tau}_{hk} \|_{L^2(\Gamma_h)}\,.\end{aligned} \tag{3.14}$$

Here, $\underline{\tau}_{hk}$, $k = 1, 2$ are orthonormal vectors spanning the tangent space at Γ_h. Let S, Δ and $\mu(Q)$ be as in the proof of Lemma 3.1. Since $\underline{n}\underline{\underline{D}}(\underline{u}) \, \underline{\tau}_k = 0$ on Γ and $| \underline{n}_h - \underline{n}(\mu(Q)) | = 0(h)$, $| \underline{\tau}_{hk} - \underline{\tau}_k(\mu(Q)) | = 0(h)$ and $| \overline{Q} - \mu(Q) | = 0(h^2)$, we conclude with the same arguments as in the proof of Lemma 3.1

$$\int_S | \underline{n}_h \underline{\underline{D}}(\underline{u}) \underline{\tau}_{hk} |^2 \leqq ch^2 \int_S | \underline{\underline{D}}(\underline{u}) |^2 + \int_S | \underline{n}(\mu(Q)) . \underline{\underline{T}}(\underline{u}, p)) . \underline{\tau}_k(\mu(Q)) |^2 \, dQ \leqq$$
$$\leqq ch^2 \int_S | \underline{\underline{D}}(\underline{u}) |^2 + ch^2 \| \underline{u} \|^2_{H^2(\Delta)^3} .$$

Summing over all faces of Γ_h and using the trace theorem, this implies

$$\| \underline{n}_h \underline{\underline{D}}(\underline{u}) \underline{\tau}_{hk} \|_{L^2(\Gamma_h)} \leqq ch \| \underline{f} \|_0 . \tag{3.15}$$

Combining equations (3.12)-(3.15), we finally obtain the desired error estimate. □

Note, that — regardless of the polynomial degree of the finite elements — the estimate

$$R \leqq ch^{1/2} \| \underline{f} \|_0 \tag{3.16}$$

can only be improved by requiring $\underline{u}_h . \underline{n}_h = 0$ on Γ_h for all $\underline{u}_h \in X_h$. The example of § 5 shows that this assumption is not appropriate for the problem under consideration.

By a standard duality argument one can prove the error estimate

$$\| \underline{u} - \underline{u}_h \|_{0,h} \leqq ch \| \underline{f} \|_0 . \tag{3.17}$$

We omit the proof here, since we give a detailed proof of the corresponding error estimate for the non-linear problem in the next section.

4. ERROR ESTIMATES FOR THE NON-LINEAR PROBLEM

The aim of this section is to prove :

THEOREM 4.1 : *There is an $h_0 > 0$ and a constant $K > 0$ which does not depend on h such that Problems (N) and (N_h), $0 < h \leqslant h_0$, have unique solutions $(\underline{u}, p) \in X \times M$ and $(\underline{u}_h, p_h) \in X_h \times M_h$ resp., provided $\nu^{-2} \| \underline{f} \|_0 < K$. Moreover, the error estimate*

$$\| \underline{u} - \underline{u}_h \|_{0,h} + h^{1/2} \| \underline{u} - \underline{u}_h \|_{1,h} + h^{1/2} \| p - p_h \|_{0,h} \leqslant ch \| \underline{f} \|_0 \tag{4.1}$$

holds.

Proof : Using Sobolev's imbedding theorem one easily proves that the trilinear forms N and N_h are continuous on X^3 and X_h^3 resp. (*cf.* Lemma 2.1 in Chap. IV of [7]). The norm of N_h can be bounded independently of h for sufficiently small values of h. A standard fixed point argument (*cf.* [8] and Chap. IV of [7]) then yields the unique solvability of Problems (N) and (N_h) for sufficiently small data $\nu^{-2} \| \underline{f} \|_0$. Together with Theorem 3.2 it also implies the error estimate for the H^1-norm of the velocities. Since $\underline{u} \in H^2(\Omega)^3$, $p \in H^1(\Omega)$ is a weak solution of (1.1), we may multiply (1.1) with $\underline{v}_h \in X_h$, integrate over Ω_h and use (2.10) to obtain

$$(\underline{f}, \underline{v}_h)_{0,h} = a_h(\underline{u}, \underline{v}_h) + b_h(\underline{v}_h, p) + N_h(\underline{u}, \underline{u}, \underline{v}_h) - \\ - \int_{\Gamma_h} \left\{ \underline{n}_h \underline{\underline{T}}(\underline{u}, p) \underline{v}_h - \frac{1}{2} (\underline{u}\underline{n}_h)(\underline{u}\underline{v}_h) \right\}.$$

Hence, we have

$$a_h(\underline{u}-\underline{u}_h, \underline{v}_h) + b_h(\underline{v}_h, p-p_h) = \int_{\Gamma_h} \left\{ \underline{n}_h \underline{\underline{T}}(\underline{u}, p) \underline{v}_h - \frac{1}{2} (\underline{u}\underline{n}_h)(\underline{u}\underline{v}_h) \right\} + \\ + N_h(\underline{u}_h - \underline{u}, \underline{u}, \underline{v}_h) + N_h(\underline{u}, \underline{u}_h - \underline{u}, \underline{v}_h) + N_h(\underline{u}_h - \underline{u}, \underline{u}_h - \underline{u}, \underline{v}_h), \quad \forall \underline{v}_h \in X_h \quad (4.2)$$

and

$$b_h(\underline{u} - \underline{u}_h, q_h) = 0, \quad \forall q_h \in M_h. \quad (4.3)$$

The boundary integrals are due to the non-conformity.

To simplify the notation, put

$$\underline{e} := \underline{u} - \underline{u}_h, \quad \varepsilon := p - p_h,$$
$$R := \sup_{\underline{v}_h \in X_h \setminus \{0\}} \frac{1}{\| \underline{v}_h \|_{1,h}} \int_{\Gamma_h} \underline{n}_h \underline{\underline{T}}(\underline{u}, p) \underline{v}_h,$$
$$R_1 := \sup_{\underline{v}_h \in X_h \setminus \{0\}} \frac{1}{2 \| \underline{v}_h \|_{1,h}} \int_{\Gamma_h} (\underline{u}\underline{n}_h)(\underline{u}\underline{v}_h)$$

and denote by p_h^* the best L^2-approximation of p in M_h and by $\underline{u}_h^*$ the interpolant of $\underline{u}$ in $(S_h^2)^3$. Recall that $\underline{u}_h^* \in X_h$ and $R \leqslant ch^{1/2} \| \underline{f} \|_0$ (*cf.* (3.16)). The trace theorem implies

$$\int_{\Gamma_h} |(\underline{u}\underline{n}_h)(\underline{u}\underline{v}_h)| \leqslant \| \underline{u}\underline{n}_h \|_{L^4(\Gamma_h)} \| \underline{u} \|_{L^4(\Gamma_h)^3} \| \underline{v}_h \|_{L^2(\Gamma_h)^3} \leqslant \\ \leqslant c \| \underline{u}\underline{n}_h \|_{L^4(\Gamma_h)} \| \underline{u} \|_2 \| \underline{v}_h \|_{1,h}. \quad (4.4)$$

With the same arguments as in the proof of Lemma 3.1 we conclude

$$\| \underline{u}\underline{n}_h \|_{L^4(\Gamma_h)} \leqslant ch \| \underline{u} \|_{W^{1,4}(\Omega)^3} \leqslant c' h \| \underline{u} \|_2. \quad (4.5)$$

Equations (4.4) and (4.5) imply

$$R_1 \leqslant ch \| \underline{f} \|_0^2. \quad (4.6)$$

From equations (3.4), (3.7), (3.16), (4.2), (4.6) and the H^1-error estimates of the velocities we get

$$\beta \| p_h - p_h^* \|_{0,h} \leqslant \sup_{\underline{v}_h \in X_h \setminus \{0\}} \frac{1}{\| \underline{v}_h \|_{1,h}} b_h(\underline{v}_h, p_h - p_h^*) \leqslant$$
$$\leqslant c \| p - p_h^* \|_{0,h} + c \| \underline{u} - \underline{u}_h \|_{1,h} + c \| \underline{u} - \underline{u}_h \|_{1,h}^2 + R + R_1 \leqslant c h^{1/2} \| \underline{f} \|_0 .$$

Together with (3.7) this implies the error estimate (4.1) for the pressure.

The L^2-error estimate for the velocities follows from a standard duality argument. Let $(\underline{\mu}, \rho) \in X \times M$ be the weak solution of

$$- \nu \Delta \underline{\mu} + \nabla \rho - \underline{\underline{D}}(\underline{\mu}) \underline{u} + \frac{1}{2} \{ (\nabla \underline{u})^t \underline{\mu} + (\nabla \underline{\mu})^t \underline{u} \} = \underline{e} \chi_{\Omega_h} \quad \text{in } \Omega \tag{4.7}$$
$$\operatorname{div} \underline{\mu} = 0$$
$$\underline{\mu n} = \underline{n} \underline{\underline{D}} (\underline{\mu}) \underline{\tau}_k = 0 , \quad k = 1, 2, \text{ on } \Gamma .$$

Here χ_{Ω_h} is the characteristic function of the set Ω_h. From [9] it follows that (4.7) has a unique solution and that the regularity estimate

$$\| \underline{\mu} \|_2 + \| \rho \|_1 \leqslant c \| \underline{e} \|_{0,h} \tag{4.8}$$

holds. Multiplying (4.7) with $\underline{e} \in H^1(\Omega_h)^3$, integrating over Ω_h and using (2.10), we obtain

$$\| \underline{e} \|_{0,h}^2 = a_h(\underline{\mu}, \underline{e}) + b_h(\underline{e}, \rho) + N_h(\underline{e}, \underline{u}, \underline{\mu}) + N_h(\underline{u}, \underline{e}, \underline{\mu}) -$$
$$- \int_{\Gamma_h} \left\{ \underline{n}_h \underline{\underline{T}}(\underline{\mu}, \rho) \underline{e} + \frac{1}{2} (\underline{u n}_h) (\underline{\mu e}) \right\} . \tag{4.9}$$

Let ρ_h^* be the best L^2-approximation of ρ in M_h and $\underline{\mu}_h^*$ be the interpolate of $\underline{\mu}$ in $(S_h^2)^3$. Equations (4.2), (4.3), (4.9) and standard approximation results for finite elements [6] imply

$$\| \underline{e} \|_{0,h}^2 = a_h(\underline{\mu} - \underline{\mu}_h^*, \underline{e}) + b_h(\underline{e}, \rho - \rho_h^*) + b_h(\underline{\mu} - \underline{\mu}_h^*, \varepsilon) + N_h(\underline{e}, \underline{u}, \underline{\mu} - \underline{\mu}_h^*) +$$
$$+ N_h(\underline{u}, \underline{e}, \underline{\mu} - \underline{\mu}_h^*) + N_h(\underline{e}, \underline{e}, \underline{\mu}_h^*)$$
$$+ \int_{\Gamma_h} \left\{ \underline{n}_h \underline{\underline{T}}(\underline{u}, \rho) \underline{e} - \frac{1}{2} (\underline{u n}_h) (\underline{\mu e}) - \underline{n}_h \underline{\underline{T}}(\underline{\mu}, \rho) \underline{\mu}_h^* - \frac{1}{2} (\underline{u n}_h) (\underline{u \mu}_h^*) \right\}$$
$$\leqslant \{ ch \| \underline{e} \|_{1,h} + ch \| \varepsilon \|_{0,h} + c \| \underline{e} \|_{1,h}^2 \} \| \underline{e} \|_{0,h}$$
$$+ \int_{\Gamma_h} \left\{ | \underline{n}_h \underline{\underline{T}}(\underline{\mu}, \rho) \underline{e} | + \frac{1}{2} | \underline{u n}_h | | \underline{\mu e} | + | \underline{n}_h \underline{\underline{T}}(\underline{u}, p) \underline{\mu}_h^* | + \frac{1}{2} | \underline{u n}_h | | \underline{u \mu}_h^* | \right\} . \tag{4.10}$$

Next, we estimate the four boundary integrals in (4.10) separately. Lemma 3.1 and equation (3.15) imply

$$\int_{\Gamma_h} | \underline{n}_h \underline{\underline{T}}(\underline{u}, p) \underline{\mu}_h^* | \leqslant$$

$$\leqslant \| \underline{n}_h \underline{\underline{T}}(\underline{u}, p) \underline{n}_h \|_{L^2(\Gamma_h)} \{ \| (\underline{\mu} - \underline{\mu}_h^*) \underline{n}_h \|_{L^2(\Gamma_h)} + \| \underline{\mu} \underline{n}_h \|_{L^2(\Gamma_h)} \}$$

$$+ \sum_{k=1,2} \| \underline{\mu}_h^* \|_{L^2(\Gamma_h)^3} \| \underline{n}_h \underline{\underline{T}}(\underline{u}, p) \underline{\tau}_{hk} \|_{L^2(\Gamma_h)}$$

$$\leqslant c \{ \| \underline{u} \|_2 + \| p \|_1 \} \{ \| \underline{\mu} - \underline{\mu}_h^* \|_{1,h} + h \| \underline{\mu} \|_1 \} + ch \| \underline{\mu}_h^* \|_{1,h} \| \underline{f} \|_0$$

$$\leqslant ch \| \underline{f} \|_0 \| \underline{e} \|_{0,h} \,. \tag{4.11}$$

Similarly, we get

$$\int_{\Gamma_h} | \underline{n}_h \underline{\underline{T}}(\underline{\mu}, \rho) \underline{e} | \leqslant$$

$$\leqslant \| \underline{n}_h \underline{\underline{T}}(\underline{\mu}, \rho) \underline{n}_h \|_{L^2(\Gamma_h)} [\| (\underline{u} - \underline{u}_h^*) \underline{n}_h \|_{L^2(\Gamma_h)} + \| (\underline{u}_h^* - \underline{u}_h) \underline{n}_h \|_{L^2(\Gamma_h)}]$$

$$+ \sum_{k=1,2} \| \underline{e} \|_{L^2(\Gamma_h)^3} \| \underline{n}_h \underline{\underline{T}}(\underline{\mu}, \rho) \underline{\tau}_{hk} \|_{L^2(\Gamma_h)}$$

$$\leqslant c \| \underline{e} \|_{0,h} [\| \underline{u} - \underline{u}_h^* \|_{1,h} + h^{1/2} \| \underline{e} \|_{1,h}] + ch \| \underline{e} \|_{0,h} \| \underline{e} \|_{1,h}$$

$$\leqslant ch \| \underline{f} \|_0 \| \underline{e} \|_{0,h} + ch^{1/2} \| \underline{e} \|_{1,h} \| \underline{e} \|_{0,h} \,. \tag{4.12}$$

Lemma 3.1 and Sobolev's imbedding theorem imply

$$\frac{1}{2} \int_{\Gamma_h} | \underline{u} \underline{n}_h | \, | \underline{u} \underline{\mu}_h^* | \leqslant ch \| \underline{u} \|_1 \| \underline{u} \|_{W^{1,4}(\Omega_h)} \| \underline{\mu}_h^* \|_{W^{1,4}(\Omega_h)}$$

$$\leqslant ch \| \underline{f} \|_0^2 \| \underline{e} \|_{0,h} \,. \tag{4.13}$$

Finally, equation (4.5) yields

$$\frac{1}{2} \int_{\Gamma_h} | \underline{u} \underline{n}_h | \, | \underline{\mu} \underline{e} | \leqslant ch \| \underline{f} \|_0 \| \underline{\mu} \|_{W^{1,4}(\Omega_h)} \| \underline{e} \|_{1,h} \leqslant ch \| \underline{f} \|_0 \| \underline{e} \|_{1,h} \| \underline{e} \|_{0,h} \,. \tag{4.14}$$

Equations (4.10)-(4.14) together with the error estimates for $\| \underline{e} \|_{1,h}$ and $\| \varepsilon \|_{0,h}$ now complete the proof. □

5. A BABUŠKA-TYPE PARADOX

The proof of (3.16) shows that the error estimates of Theorems 3.2 and 4.1 could only be improved by requiring $\underline{u}.\underline{n}_h = 0$ on Γ_h for all $\underline{u} \in X_h$. However, the solutions of Problem (S_h) with X_h replaced by

$$\tilde{X}_h := \{ \underline{u} \in (S_h^2)^3 : \underline{u}.\underline{n}_h = 0 \quad \text{on } \Gamma_h \} \tag{5.1}$$

do not converge in general to the solution of Problem (S).

Example 5.1 : Let $\Omega \subset \mathbb{R}^2$ be the interior of the unit circle with centre in the origin.

Denote by $w \in C^\infty([0, 1])$ a function with

$$w(r) = \begin{cases} 0, & 0 \leqq r \leqq \frac{1}{3} \\ \frac{3}{2} r, & \frac{2}{3} \leqq r \leqq 1 \end{cases} \tag{5.2}$$

and put

$$\underline{u}(x, y) := w(r).(-\sin \varphi, \cos \varphi)^T \tag{5.3}$$

where (r, φ) are polar coordinates in Ω. Obviously, $\underline{u} \in C^\infty(\bar{\Omega})^2$ and $\underline{u}.\underline{n} = 0$ on $\Gamma := \partial\Omega$. An easy calculation yields

$$\operatorname{div} \underline{u} = 0 \quad \text{in } \Omega, \qquad \underline{n}\underline{\underline{D}}(\underline{u}) = 0 \quad \text{on } \Gamma.$$

Let Ω_h be as in § 2 and assume that h is small enough such that $\Gamma_h := \partial\Omega_h$ lies inside the annulus around the origin with radii $\frac{2}{3}$ and 1. Denote by $(\underline{u}_h, p_h) \in \tilde{X}_h \times M_h$ the unique solution of

$$\begin{aligned} a_h(\underline{u}_h, \underline{v}_h) + b_h(\underline{v}_h, p_h) &= (-\Delta \underline{u}, \underline{v}_h)_{0,h}, \quad \forall \underline{v}_h \in \tilde{X}_h, \\ b_h(\underline{v}_h, q_h) &= 0, \quad \forall q_h \in M_h. \end{aligned} \tag{5.4}$$

(The arguments in the proof of (3.4) and Korn's first inequality [9] imply that (5.4) fits into the abstract framework of [5].) Inserting $\underline{u}_h$ as test function in (5.4) and using Korn's first inequality we get

$$\| \underline{u}_h \|_{1,h} \leqq c \| \underline{u} \|_1$$

with a constant c independent of h. Let S_1, S_2 be two adjacent faces of Ω_h with common vertex Q and denote by $\underline{n}_1$, $\underline{n}_2$ the normal to S_1 and S_2 resp. The continuity of $\underline{u}_h$ and $\underline{u}_h.\underline{n}_i = 0$ on S_i, $i = 1, 2$, imply $\underline{u}_h(Q) = 0$.

Using the same arguments as in the proof of Lemma (3.1) this yields

$$\| \underline{u}_h \|_{L^2(\Gamma_h)^2} \leqq c_0 \, h^{1/2} \, \| \underline{u}_h \|_{1,h} \tag{5.5}$$

and thus

$$\| \underline{u} - \underline{u}_h \|_{1,h} \geqq c_1 \, \| \underline{u} - \underline{u}_h \|_{L^2(\Gamma_h)^2} \geqq$$
$$\geqq c_1 \, \| \underline{u} \|_{L^2(\Gamma_h)^2} - c_1 \, c_0 \, h^{1/2} \, \| \underline{u}_h \|_{1,h} \geqq c_2 - c_3 \, h^{1/2} \, \| \underline{u} \|_1 \, .$$

Since the constants $c_0, \dots, c_3$ can be bounded independently of h, there is a constant $c > 0$ independent of h such that

$$\| \underline{u} - \underline{u}_h \|_{1,h} \geqq c > 0$$

for sufficiently small $h > 0$. Indeed, the $\underline{u}_h$, p_h approximate the solution of a Stokes equation with *homogeneous Dirichlet* boundary condition. To see this, let $\hat{\underline{u}} \in H_0^1(\Omega)^2$, $\hat{p} \in M$ be the unique solution of

$$a(\hat{\underline{u}}, \underline{v}) + b(\underline{v}, \hat{p}) = (-\,\Delta \underline{u}, \underline{v})_0 \qquad \forall \underline{v} \in H_0^1(\Omega)^2 \, ,$$
$$b(\hat{\underline{u}}, q) = 0 \qquad \forall q \in M \, .$$

By standard regularity results we have

$$\| \hat{\underline{u}} \|_2 + \| \hat{p} \|_1 \leqq c \, \| \Delta \underline{u} \|_0 \leqq c' \, \| \underline{u} \|_2 \, .$$

Replacing $\underline{u}_h^*$ in the proof of Theorem 3.2 by the interpolant $\hat{\underline{u}}_h$ of $\hat{\underline{u}}$ by *linear* finite elements corresponding to $\mathcal{T}_h$ and noting $\hat{\underline{u}}_h \in \tilde{X}_h$, we conclude

$$\| \hat{\underline{u}} - \underline{u}_h \|_{1,h} + \| \hat{p} - p_h \|_{0,h} \leqq ch \, \| \underline{u} \|_2 + R_2 \, ,$$

where

$$R_2 := \sup_{\underline{v}_h \in \tilde{X}_h \setminus \{0\}} \frac{1}{\| \underline{v}_h \|_{1,h}} \int_{\Gamma_h} \underline{n}_h \, \underline{\underline{T}}(\underline{u}, p) \, \underline{v}_h \, .$$

Since (5.5) holds for all $\underline{v}_h \in \tilde{X}_h$, we have

$$R_2 \leqq ch^{1/2} \{ \| \hat{\underline{u}} \|_2 + \| \hat{p} \|_1 \} \leqq c' \, h^{1/2} \, \| \underline{u} \|_2 \, . \quad \square$$

Finally, let us remark that a similar result is well known as Babuška-paradox in plane elasticity ([1]). The above example, however, seems to be new in fluid dynamics. It shows that problems with mixed boundary conditions behave essentially different from those with Dirichlet boundary conditions.

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