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CONVERSE RESULTS IN THE WALSH THEORY OF OVERCONVERGENCE (*)

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Résumé. — Récemment, J. Szabados a obtenu un nouveau théorème réciproque dans la théorie de la sur-convergence de Walsh, fondé sur l'Interpolation de Lagrange. Ici, nous développons un théorème réciproque similaire, fondé sur l'Interpolation d'Hermite, qui généralise le résultat de Szabados.

Abstract. — Recently, J. Szabados has obtained a new converse theorem in the Walsh overconvergence theory, based on Lagrange interpolation. Here, we similarly develop a related converse theorem, based on Hermite interpolation, which generalizes Szabados' result.

1. INTRODUCTION

Let $A_p$ denote the collection of functions analytic in $|z| < p$, and, as usual, let $\pi_m$ denote the collection of all complex polynomials of degree at most $m$. For any $f(z) \in A_p$ with $p > 1$, and for any positive integer $n$, let $L_{n-1}(z; f)$ denote the Lagrange polynomial interpolant in $\pi_{n-1}$ of $f(z)$ in the $n$-th roots of unity, i.e.,

$$L_{n-1}(\omega; f) = f(\omega),$$

where $\omega$ is any $n$th root of unity. With $f(z) := \sum_{k=0}^{\infty} a_k z^k$ in $|z| < p$, and for each positive integer $l$, set

$$Q_{n-1,l}(z; f) := \sum_{j=0}^{l-1} \sum_{k=0}^{n-1} a_{k+jn} z^k,$$

so that $Q_{n-1,l}(z; f)$ is also an element of $\pi_{n-1}$. Then, the original and oft-cited

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beautiful result of J. L. Walsh [6, p. 153] on overconvergence is the case $l = 1$ of

**Theorem A ([1])** : For any $f(z) \in A_p$ with $p > 1$, and for any positive integer $l$,

\[
\lim_{n \to \infty} \{ L_{n-1}(z ; f) - Q_{n-1,l}(z ; f) \} = 0, \quad \text{for all } |z| < p^{l+1}, \tag{1.3}
\]

the convergence being uniform and geometric on any closed subset of $|z| < p^{l+1}$. Moreover, the result is best possible (in the sense that (1.3) is not valid at each point of $|z| = p^{l+1}$ for all $f(z)$ in $A_p$).

Now Theorem A, in the terminology of approximation theory, is a direct theorem in the Walsh overconvergence theory, in that the assumption $f(z) \in A_p$ leads to the overconvergence result of (1.3). Recently, Szabados [4] obtained the following interesting converse theorem to Theorem A. For notation, let $A_1 C$ denote the collection of all $f(z)$ in $A_1$ which are continuous on $|z| = 1$.

**Theorem B ([4])** : Assume that $f(z) \in A_1 C$. If $p > 1$, if $l$ is a positive integer, and if the sequence

\[
\{ L_{n-1}(z ; f) - Q_{n-1,l}(z ; f) \}_{n=1}^\infty \tag{1.4}
\]

is uniformly bounded on every closed subset of $|z| < p^{l+1}$, then $f(z) \in A_p$.

It may be asked if the conclusion of Theorem B (namely, that $f(z) \in A_p$) is best possible, i.e., with the hypothesis of Theorem B, could $f(z) \in A_{p'}$ where $p' > p$, in general? On considering the particular function $\tilde{f}(z) := (p - z)^{-1}$ which, with (1.3) satisfies the hypothesis of Theorem B, one sees that $\tilde{f}(z)$ is an element of $A_p$, but is clearly not an element of $A_{p'}$ for any $p' > p$. In this sense, Theorem B is best possible, as was remarked by Szabados [4].

There are now many known direct theorems in the Walsh overconvergence theory on the difference of interpolating polynomials (cf. [1], Rivlin [2], [5, chap. 4]). It is natural to ask if there are similar converse theorems which complement Szabados' Theorem B. Here, we show that such a converse theorem can be similarly derived for Hermite polynomial interpolation.

### 2. STATEMENT OF A NEW RESULT.

We first state a direct theorem for Hermite interpolation in the Walsh overconvergence theory. To fix notations, for any $f(z) \in A_p$ with $p > 1$, for a fixed positive integer $r$, and for every positive integer $n$, let $h_{rn}(z ; f)$ denote the Hermite polynomial interpolant in $\pi_{r-1}$ of $f$, $f'$, ..., $f^{(r-1)}$ in the $n$th roots of unity, i.e.,

\[
h_{rn-1}^{(j)}(\omega ; f) = f^{(j)}(\omega), \quad j = 0, 1, ..., r - 1, \tag{2.1}
\]
where \( \omega \) is any \( n \)th root of unity. Again, with \( f(z) := \sum_{k=0}^{\infty} a_k z^k \) in \( |z| < \rho \), and for any positive integer \( l \), set

\[
\tilde{\tilde{Q}}_{n-1,l}(z; f) := \sum_{k=0}^{r-1} a_k z^k + \sum_{j=1}^{l-1} \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+(r+j-1)n} z^k,
\]

(2.2)

where (cf. [1])

\[
\beta_{j,r}(z) := \sum_{k=0}^{r-1} \left( \begin{array}{c} r + j - 1 \\ k \end{array} \right) (z - 1)^k, \quad j = 1, 2, \ldots,
\]

(2.3)

and where the last sum in (2.2) is defined here, and subsequently, to be zero when \( l = 1 \). Note that \( \tilde{\tilde{Q}}_{n-1,l}(z) \) is also in \( \pi_{kn-1} \). With these notations, a direct theorem for Hermite interpolation in the Walsh overconvergence theory is

**Theorem C ([1])**: For any \( f(z) \in A_\rho \) with \( \rho > 1 \), and for any positive integers \( r \) and \( l \),

\[
\lim_{n \to \infty} \{ h_{n-1}(z; f) - \tilde{\tilde{Q}}_{n-1,l}(z; f) \} = 0, \quad \text{for all} \quad |z| < \rho^{1+\gamma/l},
\]

(2.4)

the convergence being uniform and geometric on any closed subset of \( |z| < \rho^{1+\gamma/l} \). Moreover, the result is best possible.

A new result, a converse result to Theorem C, is the following. For notation, for each positive integer \( r \), let \( A_1 C^{(r-1)} \) denote the collection of all \( f(z) \) in \( A_1 \) for which \( f(z) \), \( f'(z) \), ..., and \( f^{(r-1)}(z) \) are all continuous on \( |z| = 1 \). For any \( f(z) \in A_1 C^{(r-1)} \) and for any \( n \geq 1 \), it is evident that the interpolatory polynomials \( h_{n-1}(z; f) \) and \( \tilde{\tilde{Q}}_{n-1,l}(z; f) \) of (2.1)-(2.2) are well-defined.

**Theorem 1**: Assume that \( f(z) \in A_1 C^{(r-1)} \). If \( \rho > 1 \), if \( l \) is a positive integer, and if the sequence

\[
\{ h_{n-1}(z; f) - \tilde{\tilde{Q}}_{n-1,l}(z; f) \}_{n=1}^{\infty}
\]

is uniformly bounded on every closed subset of \( |z| < \rho^{1+\gamma/l} \), then \( f(z) \in A_\rho \).

As the special case \( r = 1 \) of Theorem 1 reduces to Szabados' Theorem B, we remark that Theorem 1 then generalizes Theorem B.

The proof of Theorem 1 will be given in Section 3. Because it is needed in the proof of Theorem 1, we state, as in Theorem D below, a recent related result of Saff and Varga [3, theorem 2] on Hermite interpolation in the Walsh overconvergence theory.
THEOREM D ([3]) : For each \( f(z) \in A_p \), and for each pair of positive integers \( r \) and \( l \), the sequence (2.5) can be bounded in at most \( r + l - 1 \) distinct points in \( |z| > \rho^{1+(l/r)} \).

3. PROOF OF THEOREM 1

With the notations from Section 2, we begin with the following result which, for \( r = 1 \), reduces to Lemma 1 of [4].

**Lemma 1** : If \( f(z) := \sum_{k=0}^{\infty} a_k z^k \) is an element of \( A_1 C^{(r-1)} \), then for each positive integer \( l \),

\[
\begin{align*}
\mathcal{Q}_{r-1,l}(z; f) &= h_{r-1}(z; f) - \sum_{k=0}^{\infty} a_k z^k \\
&= h_{r-1}(z; f) - h_{r-1}(z; \sum_{k=0}^{\infty} a_k z^k) \\
&= h_{r-1}(z; \sum_{k=0}^{\infty} a_k z^k) - h_{r-1}(z; \sum_{k=0}^{(r+1)n-1} a_k z^k) \\
&= h_{r-1}(z; \sum_{k=0}^{rn-1} a_k z^k) + h_{r-1}(z; \sum_{k=rn}^{(r+l-1)n-1} a_k z^k) \\
&= \sum_{k=0}^{rn-1} a_k z^k + \sum_{k=rn}^{(r+l-1)n-1} a_k h_{r-1}(z; z^k) \\
&= \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \sum_{k=0}^{n-1} a_{k+(r+j-1)n} h_{r-1}(z; z^{k+(r+j-1)n}) \\
&= \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \sum_{k=0}^{n-1} a_{k+(r+j-1)n} h_{r-1}(z; z^{k+(r+j-1)n})
\end{align*}
\]

It is known (cf. [1, eq. (4.4)]) that

\[
\mathcal{Q}_{r-1,l}(z; z^{k+(r+j-1)n}) = z^k \beta_{j,r}(z^n), \quad \text{for } j = 1, 2, \ldots,
\]

where \( \beta_{j,r}(z) \) is defined in (2.3). Inserting the above identity into the previous display gives, with the definition of \( \mathcal{Q}_{r-1,l}(z; f) \) in (2.2), the desired result of (3.1). \( \square \)

Szabados [4] has pointed out that his special case \( r = 1 \) of Lemma 1 gives an elementary proof of Theorem A. We remark that Lemma 1 similarly gives an elementary proof of Theorem C. As its proof follows along the lines of the proof of Theorem 1, we omit the details.

Next, as \( \beta_{j,r}(z) \) from (2.3), is in \( \pi_{r-1} \), we can write

\[
\beta_{j,r}(z) := \sum_{v=0}^{r-1} C_{v,r}(j) z^v,
\]
where evidently
\[ C_{\nu,r}(j) := \sum_{k=0}^{r-1} (-1)^{k-\nu} \binom{r+j-1}{k} \binom{k}{\nu}, \quad \text{for } \nu = 0, 1, \ldots, r - 1. \quad (3.4) \]

**Lemma 2**: The polynomials
\[ C_{\nu,r}(x) := \sum_{k=0}^{r-1} (-1)^{k-\nu} \binom{x+r-1}{k} \binom{k}{\nu} = \sum_{k=0}^{r-1} (-1)^{k-\nu} \frac{(x+r-1) \cdots (x+r-k)}{(k-\nu)! \nu!}, \quad (3.5) \]

for \( \nu = 0, 1, \ldots, r - 1 \), form a Lagrangian basis for \( \pi_{r-1} \), i.e., for any \( p_{r-1}(x) \in \pi_{r-1} \),
\[ p_{r-1}(x) \equiv \sum_{j=0}^{r-1} p_{r-1}(j + 1 - r) C_{j,r}(x), \quad \text{for all } x. \quad (3.6) \]

In particular, choosing \( p_{r-1}(x) \equiv 1 \) in (3.5) gives
\[ 1 = \sum_{\nu=0}^{r-1} C_{\nu,r}(\lambda + l) \quad \text{for any integers } \lambda \text{ and } l. \quad (3.7) \]

**Proof**: It is evident from (3.5) that
\[ C_{\nu,r}(x + 1 - r) = \frac{x(x-1) \cdots (x-\nu+1)}{\nu!} x \]
\[ \times \left\{ 1 + \sum_{k=1}^{r-\nu-1} (-1)^k \frac{(x-\nu)(x-\nu-1) \cdots (x-k-\nu+1)}{k!} \right\}. \quad (3.8) \]

As the multiplier \( x(x-1) \cdots (x-(\nu-1)) \) in (3.8) vanishes for \( x = 0, 1, \ldots, \nu-1 \), then \( C_{\nu,r}(j + 1 - r) = 0 \) for \( j = 0, 1, \ldots, \nu-1 \), while for \( x = \nu \), (3.8) gives
\[ C_{\nu,r}(\nu + 1 - r) = 1. \]
Similarly, for \( x = \nu + l \) (where \( 1 \leq l \leq r - 1 \)), the quantity in braces in (3.8) reduces to
\[ \left\{ 1 + \sum_{k=1}^{l} (-1)^k \frac{l(l-1) \cdots (l-(k-1))}{k!} \right\} \]
which is the binomial expansion of \( (1 - l)^l = 0 \). Thus, we have shown that
\[ C_{\nu,r}(j + 1 - r) = \delta_{j,r}, \quad \text{for all } j = 0, 1, \ldots, r - 1. \]

Consequently, \( \left\{ C_{\nu,r}(x) \right\}_{\nu=0}^{r-1} \) forms a Lagrangian basis for \( \pi_{r-1} \), from which (3.6) and (3.7) directly follow. □

**Proof of Theorem 1**: Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be any element in \( A_1 C^{(r-1)} \)
satisfying the hypothesis of Theorem 1, and let \( R \) be any number satisfying
\[ 1 < R < p^{1+(l/r)}. \quad (3.9) \]
Now, the boundedness hypothesis of (2.5) implies, from (3.1) of Lemma 1, that there is a constant $M(R)$ such that

$$\max_{|z|=R} \left| h_{rs-1} \left( \sum_{k=(r+l-1)s}^{\infty} a_k z^k \right) \right| \leq M(R) < \infty, \quad (3.10)$$

for any $s \geq 1$. In particular, choosing $s = 2n$ in (3.10) gives

$$\max_{|z|=R} \left| h_{2rn-1} \left( z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) \right| \leq M(R). \quad (3.11)$$

Next, setting

$$h_{2rn-1} \left( z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) = \sum_{k=0}^{2rn-1} b_k z^k, \quad (3.12)$$

the bound from (3.11), along with Cauchy's formula, implies

$$|b_k| \leq M(R).R^{-k}, \quad k = 0, 1, ..., 2rn - 1. \quad (3.13)$$

Since the set of $2n$th roots of unity includes all $n$th roots of unity, we obtain (cf. (2.1)) the identity:

$$h_{rn}(z; g) = h_{rn-1}(z; h_{2rn-1}(z; g)), \quad (3.14)$$

for any $g(z) \in A_1 C^{(r-1)}$. Choosing $g(z) := \sum_{k=2(r+l-1)n}^{\infty} a_k z^k$, then $g(z)$ is just $f(z)$, minus a polynomial, and is hence in $A_1 C^{(r-1)}$, for any $n \geq 1$. Using in succession the identity of (3.14), the definition of (3.12), the fact that $h_{rn-1}$ is a linear operator which reproduces polynomials in $\pi_{rn-1}$, and the identity (3.2), we obtain the chain of equalities:

$$h_{rn-1} \left( z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) = h_{rn-1} \left( z ; h_{2rn-1} \left( z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) \right) =$$

$$= h_{rn-1} \left( z ; \sum_{k=0}^{2rn-1} b_k z^k \right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{rn-1} b_{k+n} h_{rn-1}(z; z^{k+n})$$

$$= \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{n-1} \sum_{\lambda_0=0}^{r-1} b_{k+(r+\lambda_0)n} h_{rn-1}(z; z^{k+(r+\lambda_0)n})$$

$$= \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda_0=0}^{r-1} \sum_{k=0}^{n-1} b_{k+(r+\lambda_0)n} z^k, \quad i.e.,$$

$$h_{rn-1} \left( z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda_0=0}^{r-1} \sum_{k=0}^{n-1} b_{k+(r+\lambda_0)n} z^k. \quad (3.15)$$
Now, it follows from the definition in (2.3) that
\[ |\beta_{\lambda+1,r}(z^n)| \leq 2^{r+\lambda}(|z|^n + 1)^{r-1} \quad \text{for all } z, \quad \text{and all } \lambda \geq 0, \]
from which it easily follows that
\[ \max_{|z|=R} |\beta_{\lambda+1,r}(z^n)| \leq 2^{2r+\lambda} R^{nr}, \quad \text{for all } \lambda \geq 0. \quad (3.16) \]

Applying the bounds of (3.16) and (3.13) to the terms of (3.15) gives, after an easy calculation, that
\[ \max_{|z|=R} \left| h_{rn-1}(z; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k) \right| \leq n 2^{3r} M(R). \quad (3.17) \]

This can be used as follows. By linearity again,
\[ h_{rn-1}(z; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k) = h_{rn-1}(z; \sum_{k=(r+l-1)n}^{\infty} a_k z^k) - h_{rn-1}(z; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k), \]
so that with (3.17) and (3.10) (for the case \( s = n \)),
\[ \max_{|z|=R} \left| h_{rn-1}(z; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k) \right| \leq (n 2^{3r} + 1) M(R). \quad (3.18) \]

Using in succession again the linearity of the operator \( h_{rn-1} \), the identity of (3.2), and (3.4), we obtain
\[ h_{rn-1}(z; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k) = \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r+l-2} a_k^{(r+\lambda+l-1)n} h_{rn-1}(z; z^k z^{(r+\lambda+l-1)n}) \]
\[ = \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r+l-2} a_k^{(r+\lambda+l-1)n} z^k \beta_l(z^n) \]
\[ = \sum_{k=0}^{n-1} \sum_{v=0}^{r-1} z^{k+v} \sum_{\lambda=0}^{r+l-2} C_{v,s}(\lambda + l) a_k^{(r+\lambda+l-1)n}. \]

Applying Cauchy’s formula and the bound of (3.18) to the above expression gives
\[ \left| \sum_{\lambda=0}^{r+l-2} C_{v,s}(\lambda + l) a_k^{(r+\lambda+l-1)n} \right| \leq \frac{(n 2^{3r} + 1) M(R)}{R^{k+v}}, \quad (3.19) \]
for all \( k = 0, 1, \ldots, n - 1; v = 0, 1, \ldots, r - 1 \).
Suppose we set
\[ \sum_{\lambda=0}^{r+1-2} C_{\nu,r}(\lambda + l) a_{k+(\lambda + l-1)n} := \mu_{k,v,n}, \] (3.20)
for \( k = 0, 1, ..., n - 1; \nu = 0, 1, ..., r - 1, \) where from (3.19),
\[ |\mu_{k,v,n}| \leq \frac{(n 2^{3r} + 1) M(R)}{R^{k+vn}}. \] (3.21)

On summing both sides of (3.20) with respect to \( v \) and using the identify of (3.7), we can write
\[ \sum_{j=r+1-1}^{2(r+1-1)-1} a_{k+jn} = \sum_{v=0}^{r-1} \mu_{k,v,n}, \]
so that
\[ \left| \sum_{j=r+1-1}^{2(r+1-1)-1} a_{k+jn} \right| \leq \sum_{v=0}^{r-1} |\mu_{k,v,n}|. \]

Applying the upper bound of (3.21) then gives
\[ \left| \sum_{j=r+1-1}^{2(r+1-1)-1} a_{k+jn} \right| \leq \frac{r(n 2^{3r} + 1) M(R)}{R^k}, \] (3.22)
for all \( k = 0, 1, ..., n - 1, \) all \( n \geq 1. \)

We now state a result which is implicit in the work of Szabados [4].

**Lemma 3 ([4])**: If \( g(z) = \sum_{k=0}^{\infty} \alpha_k z^k \) is an element of \( A_1 \), and if, for each positive integer \( s \) and each \( R \) with \( 1 < R < p^{l+1} \), there is a constant \( M(R) \) such that
\[ \left| \sum_{j=s}^{2s-1} \alpha_{k+jn} \right| \leq \frac{(2n + 1) M(R)}{R^k}, \] for all \( k = 0, 1, ..., n - 1, \) all \( n \geq 1, \) (3.23)
then
\[ \lim_{n \to \infty} |\alpha_n|^{1/n} \leq \begin{cases} R^{-1/2}, & \text{if } s = 1 \\ R^{-(3s^2+1)}, & \text{if } s > 1 \end{cases} < 1. \] (3.24)

Lemma 3 can be applied as follows. As \( f(z) \), by hypothesis an element in \( A_1 \) \( C^{(r-1)} \), is necessarily in \( A_1 \) \( C \), and as (3.22) holds, then (3.24) of Lemma 3 with \( s = r + l - 1 \) gives that
\[ \lim_{n \to \infty} |a_n|^{1/n} < 1. \] (3.25)
This last inequality ensures, as in [4], that $f(z)$ can be analytically continued from $|z| \leq 1$ into a larger circle. Let $\overline{\rho} > 1$ be the maximal radius for which $f(z)$ is analytic in $|z| < \overline{\rho}$, so that $f(z)$ has a singularity on $|z| = \overline{\rho}$. But, by Theorem D, the sequence (2.6) can be bounded in at most $r + l - 1$ distinct points in $|z| > \overline{\rho}^{1+\theta/r}$. As the hypothesis of Theorem 1 ensures that this sequence is uniformly bounded on every closed subset of $|z| < \rho^{1+\theta/r}$, it is evident that $\rho \leq \overline{\rho}$, showing that $f(z) \in A_{\rho}$.

To conclude, we mention some open questions. It would be interesting to see if similar converse results hold for lacunary interpolation in the roots of unity, or for Rivlin's case [2] of $l_2$-convergence. Moreover, the above proof of Theorem 1 depends on the use of Saff and Varga's Theorem D. Is it possible to prove Theorem 1 without the use of Theorem D?

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