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FINITE ELEMENT SOLUTION OF A NONLINEAR DIFFUSION PROBLEM WITH A MOVING BOUNDARY (*)

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Abstract — *The above problem is a boundary-value problem simulating the redistribution of impurities in semiconductor device structures The problem is formulated in a variational form. A fully discrete finite element solution is constructed. It is based on triangular elements varying in time Stability of the scheme is proved and an error estimate derived Some numerical results are introduced.*

Résumé — *Nous considérons un problème aux limites modélisant la distribution des impuretés dans des matériaux semi-conducteurs Le probleme est écrit sous forme variationnelle, la discrétisation totale utilise une méthode d'éléments finis associée à une triangulation dépendant du temps Nous montrons que le schéma obtenu est stable et donnons des estimations de l'erreur commise Nous présentons enfin des résultats numériques*

1. INTRODUCTION

In recent years two-dimensional process simulators for modelling and simulation in the design of VLSI semiconductor devices have appeared (see Chin, Kump and Dutton [1], Maldonado [4], Penumalli [5]). The underlying mathematical problem consists in solving numerically the following boundary value problem :

$$\frac{\partial u}{\partial t} = \nabla . [D(u) \nabla u] \quad \text{in} \quad \Omega(t), 0 < t < T, \Omega(t) = \{ (x, y) \mid \varphi(y, t) < x < L_0, \; 0 < y < B \}, \quad (1)$$

$$\left. \frac{\partial u}{\partial n} \right|_{\partial\Omega(t) - \Gamma(t)} = 0, 0 < t < T, \Gamma(t) = \{ (x, y) \mid x = \varphi(y, t), 0 < y < B \}, \quad (2)$$

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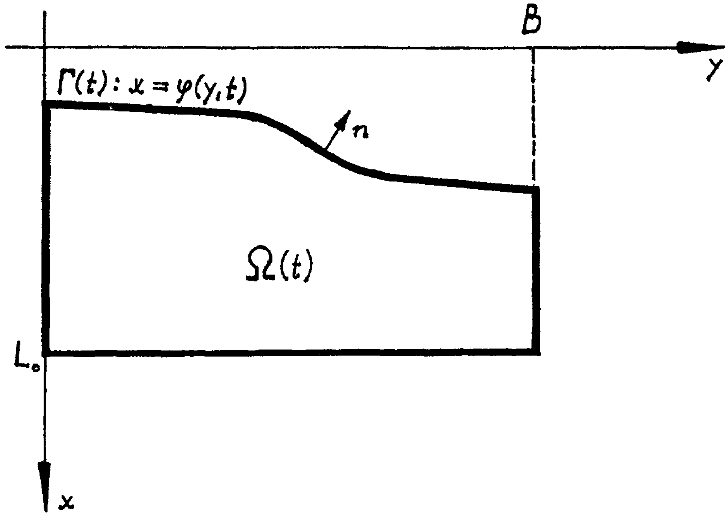

Figure 1.

$$D(u) \frac{\partial u}{\partial n} = \gamma \dot{\varphi}_n u \quad \text{on} \quad \Gamma(t), \quad 0 < t < T, \tag{3}$$

$$u(x, y, 0) = u^*(x, y) \quad \text{in} \quad \Omega(0). \tag{4}$$

Here u is the unknown concentration of an impurity, $D(u)$ is the concentration dependent diffusion coefficient and we assume that $D(u) \in C^0(< 0, \infty))$ and :

$$0 < d_0 \leqq D(u) \leqq d_0^{-1} \quad \forall u \geqq 0. \tag{5}$$

Further, φ is a given function of y and t belonging to $C^1(\langle 0, B \rangle \times \langle 0, T \rangle)$, $\frac{\partial u}{\partial n}$ is the derivative in the direction of the outward normal, γ is a constant, $\dot{\varphi}_n$ is the rate of the motion of $\Gamma(t)$ in the direction of the outward normal $\left(\dot{\varphi}_n = - \frac{\partial \varphi}{\partial t} \left[1 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right]^{-1/2} \right)$ and in the problem considered :

$$\dot{\varphi}_n \leqq 0, \quad 0 \leqq \varphi \leqq \frac{1}{2} L_0, \tag{6}$$

$u^*(x, y)$ is the given initial concentration. For more details we refer the reader to [4].

In the present paper the above boundary-value problem is formulated in a variational form. We construct a fully discrete finite element solution based on triangular elements varying in time. We prove stability and derive an error estimate. Finally, some numerical results are introduced.

2. A FINITE ELEMENT SOLUTION

Let Ω be a bounded open set in R^2. We denote by $H^{m,p}(\Omega)$, $m=0, 1, \ldots, 1 \leqq p \leqq \infty$, the Sobolev space $H^{m,p}(\Omega) = \{ v \mid D^\alpha v \in L^p(\Omega) \; \forall \mid \alpha \mid \leqq m \}$ normed by $\| v \|_{m,p,\Omega} = \sum_{|\alpha| \leqq m} \| D^\alpha v \|_{L^p(\Omega)}$. For $p = 2$ the index p will be omitted. The scalar product in $H^m(\Omega)$ is denoted $(.,.)_{m,\Omega}$. For $m = 0$ we have $H^0(\Omega) = L^2(\Omega)$ and instead of $\| . \|_{0,\Omega}$ and $(.,.)_{0,\Omega}$ we use the notation $\| . \|_\Omega$ and $(.,.)_\Omega$, respectively.

If u is a sufficiently smooth solution of (1)-(4) (we remark that we do not know any result from which existence of a solution of (1)-(4) follows) then by multiplying (1) by $v \in H^1(\Omega(t))$ and integrating over $\Omega(t)$ we get :

$$\forall t \in (0, T) \quad \left(\frac{\partial u}{\partial t}, v\right)_{\Omega(t)} + a(u, t; u, v) = 0 \quad \forall v \in V(t) = H^1(\Omega(t)); \tag{7}$$

here :

$$a(w, t; u, v) = \int_{\Omega(t)} D(w) \nabla u . \nabla v \, dx \, dy - \gamma \int_{\Gamma(t)} \dot{\varphi}_n \, uv \, d\Gamma . \tag{8}$$

The equation (7) will be used for defining a semidiscrete finite element solution. To this end we need to construct a suitable moving triangulation of the domain $\overline{\Omega}(t)$. We consider the one-to-one mapping of the rectangle $\overline{Q} = \langle 0, L_0 \rangle \times \langle 0, B \rangle$ on $\overline{\Omega}(t)$:

$$x = F(\alpha, \beta, t) = \varphi(\beta, t) + \alpha[1 - L_0^{-1} \varphi(\beta, t)], y = \beta . \tag{9}$$

We cover $\overline{\Omega}(0)$ by triangles completed along $\Gamma(0)$ by curved elements in a manner described in Zlámal [7]. Let $P_k = (x_k, y_k)$, $k = 1, \ldots, d$, be the nodes of this triangulation and let $Q_k = (\alpha_k, \beta_k)$ be their inverse images in the mapping (9), i.e. :

$$\alpha_k = \frac{x_k - \varphi(\beta_k, 0)}{1 - L_0^{-1} \varphi(\beta_k, 0)}, \quad \beta_k = y_k . \tag{10}$$

The triangulation $\mathcal{T}(t)$ of $\overline{\Omega}(t)$ is determined for $t > 0$ by the nodes :

$$P_k(t) = (x_k(t), y_k), \quad x_k(t) = F(\alpha_k, \beta_k, t), 0 < t \leqq T . \tag{11}$$

The elements of $\mathcal{T}(t)$ are again triangles or curved elements. Let $K(t)$ be an arbitrary element of $\mathcal{T}(t)$. At this moment we use a local notation $P_1(t)$, $P_2(t)$, $P_3(t)$ of the vertices of $K(t)$. We map $K(t)$ on the (time independent) reference element $\hat{K}$ with vertices $R_1 = (0, 0)$, $R_2 = (1, 0)$, $R_3 = (0, 1)$ in the ξ, η-plane. We have (see [7]) :

$$\left.\begin{aligned} x &= x(\xi, \eta, t) = \sum_{j=1}^{3} x_j(t)\, N_j(\xi, \eta) + (1 - \xi - \eta)\, \Phi(\eta, t)\,, \\ y &= y(\xi, \eta) = \sum_{j=1}^{3} y_j\, N_j(\xi, \eta) \end{aligned}\right\} \tag{12}$$

where :

$$N_1 = 1 - \xi - \eta\,, \quad N_2 = \xi\,, \quad N_3 = \eta$$

and :

$$\Phi(\eta, t) = 0 \qquad \text{for triangles}\,,$$

$$\Phi(\eta, t) = \frac{1}{1 - \eta}\{\, \varphi(y_1 + (y_3 - y_1)\,\eta, t) - x_1(t) - [x_3(t) - x_1(t)]\,\eta \,\}$$

for elements with a curved side $\widehat{P_1 P_3}$ $(x_1(t) = \varphi(y_1, t), x_3(t) = \varphi(y_3, t))$. The trial functions are on each element $K(t)$ of the form :

$$v(x, y, t) = \hat{v}(\xi, \eta) \equiv \sum_{j=1}^{3} v_j\, N_j(\xi, \eta)\,, \quad \xi = \xi(x, y, t)\,, \quad \eta = \eta(x, y, t) \tag{13}$$

where $\xi = \xi(x, y, t)$, $\eta = \eta(x, y, t)$ is the inverse mapping to the mapping (12). We denote by $V_h(t)$ the set of all trial functions. We have :

$$V_h(t) \subset V(t)\,, \qquad 0 \leqq t \leqq T\,. \tag{14}$$

We denote by $w_k(x, y, t)$, $k = 1, \ldots, d$, the basis functions of $V_h(t)$. w_k is uniquely determined by $w_k(x, y, t) \in V_h(t)$ and by

$$w_k(P_j(t), t) = \delta_k^j\,, \qquad 0 \leqq t \leqq T\,. \tag{15}$$

The notation $\hat{v}$ will be used also for functions v which are not trial functions. If v is defined on an element $K(t)$ then $\hat{v}(\xi, \eta, t)$ is defined on $\hat{K}$ by :

$$\hat{v}(\xi, \eta, t) = v(x(\xi, \eta, t), y(\xi, \eta), t)\,.$$

Remark : In [4] the mapping (9) was used to transform the equation (1) in an equation with independent variables α, β and to solve this new problem by the method of lines. We use (9) for constructing a moving triangulation $\mathcal{T}(t)$ of $\overline{\Omega}(t)$ without transforming (1). As input data only the coordinates of the nodes of $\mathcal{T}(0)$ are necessary.

The semidiscrete solution of the problem (1)-(4) is assumed in the form :

$$U(x, y, t) = \sum_{k=1}^{d} U_k(t)\, w_k(x, y, t) \tag{16}$$

and determined by :

$$\left.\begin{aligned} \forall t \in (0, T) \quad \left(\frac{\partial U}{\partial t}, v\right)_{\Omega(t)} + a(U, t; U, v) = 0 \quad \forall v \in V_h(t) \\ U(x, y, 0) = U^*(x, y)\,. \end{aligned}\right\} \tag{17}$$

$U^* \in V_h(0)$ is a suitable approximation of u^*, e.g. U^* is the interpolate of u^*.

If we denote by $\mathbf{U}(t)$ the d-dimensional vector $(U_1(t), \dots, U_d(t))^T$ (the superscript T means a transposition), by $\mathbf{U}^*$ the vector $(U^*(P_1), \dots, U^*(P_d))^T$ and by $M(t)$, $R(t)$ and $K(U, t)$ the $d \times d$ matrices :

$$M(t) = \left\{ (w_j, w_k)_{\Omega(t)} \right\}_{j,k=1}^{d}, \qquad R(t) = \left\{ \left(w_j, \frac{\partial w_k}{\partial t}\right)_{\Omega(t)} \right\}_{j,k=1}^{d},$$

$$K(U, t) = \left\{ a(U, t; w_j, w_k) \right\}_{j,k=1}^{d}$$

then the matrix form of (17) is :

$$\left.\begin{aligned} M(t)\,\dot{\mathbf{U}} + R(t)\,\mathbf{U} + K(U, t)\,\mathbf{U} = \mathbf{0}\,, \\ \mathbf{U}(0) = \mathbf{U}^*\,. \end{aligned}\right\} \tag{18}$$

Here $\dot{\mathbf{U}} = \dfrac{d}{dt}\mathbf{U}$ and the matrices M and K are standard mass and stiffness matrices, respectively. The matrix R is unsymmetric and we show later how to compute it.

We discretize (18) in time. For simplicity, we use a uniform partition of the interval $\langle 0, T \rangle$: $t_i = i\,\Delta t$, $i = 0, 1, \dots, q$ (hence $q\,\Delta t = T$). In the sequel $\mathbf{U}^i$, M^i, ... means $\mathbf{U}(t_i)$, $M(t_i)$, ... Now we set $t = t_{i+1}$ in (18), replace $\dot{\mathbf{U}}^{i+1}$ by $\Delta t^{-1}\,\Delta\mathbf{U}^i$, $\Delta\mathbf{U}^i = \mathbf{U}^{i+1} - \mathbf{U}^i$, and linearize the nonlinear term in (18). We get :

$$\left.\begin{aligned} M^{i+1}\,\Delta\mathbf{U}^i + \Delta t\left[R^{i+1} + K(\tilde{U}^i, t_{i+1})\right]\mathbf{U}^{i+1} = \mathbf{0}\,, \quad \tilde{U}^i = \sum_{k=1}^{d} U_k^i\, w_k^{i+1}\,, \\ i = 0, \dots, q-1\,, \end{aligned}\right\} \tag{19}$$

$$\mathbf{U}^0 = \mathbf{U}^*\,.$$

At first glance it is not clear that (19) determines uniquely $\mathbf{U}^i$, $i = 1, \ldots, q$. We show later that this is the case. Further, for practical computations it is necessary to do one more step : to replace curved elements by triangles (then, of course $V_h(t) \not\subset V(t)$) and to compute all matrices numerically. Also, we could use the Crank-Nicolson approach for solving (18) or, more generally, the Θ-method. Finally, the linearization of the nonlinear term need not give sufficiently acurate values. Better values of $\mathbf{U}^{i+1}$ can be won by iterating successively :

$$M^{i+1}(\mathbf{U}^{i+1,r} - \mathbf{U}^i) + \Delta t[R^{i+1} + K(U^{i+1,r-1}, t_{i+1})]\,\mathbf{U}^{i+1,r} = 0, \quad r = 1, \ldots,$$
$$\mathbf{U}^{i+1,0} = \mathbf{U}^i.$$

In the present paper we restrict ourselves to justify the procedure defined by (19).

Remark : The method proposed here can be applied as well for the solution of the parabolic system of nonlinear equations which governs the case of more impurities. This system is derived in [4].

3. PROPERTIES OF THE TRIANGULATION

We will consider a family $\{ \mathcal{T}_h^0 \}$ of the triangulations of $\bar{\Omega}^0$ from which a family $\{ \mathcal{T}_h(t) \}$ of the triangulations of $\bar{\Omega}(t)$ for $t \in (0, T \rangle$ is constructed as described in the preceding section.

Let h_{K^0} be the greatest side of an element $K^0 \in \mathcal{T}_h^0$ and :

$$h = \max_{K^0 \in \mathcal{T}_h^0} h_{K^0}. \tag{20}$$

We consider a family $\{ \mathcal{T}_h^0 \}$ such that :

$$h \to 0$$

and the minimum angle condition is satisfied (see Zlámal [8]), i.e. :

$$\vartheta_h \geqq \vartheta_0, \vartheta_0 = \text{const.} > 0 \tag{21}$$

where ϑ_h is the smallest angle of all elements of $\mathcal{T}_h^0$ (if the element is curved we mean by its angles the angles of the triangle with the same vertices). From $\{ \mathcal{T}_h^0 \}$ we construct $\{ \mathcal{T}_h(t) \}$ for all t by means of (11). Let $P_j(t)$, $j = 1, 2, 3$, be the vertices of an element $K(t)$ from $\mathcal{T}_h(t)$. First we introduce a lemma which is a counterpart of theorem 1 from [7]. Before, let us remark that the quantity h from the second section of [7] is not equal to h defined by (20).

In this paper, h_{K^0} will play the role of h from the second section of [7]. Further, the assertions of theorem 1 from [7] are true for triangles as well, i.e. they are true for the mapping (2) from [7].

LEMMA 1 : *Let the family* $\{ \mathcal{C}_h^0 \}$ *of triangulations satisfy the minimum angle condition* (21) *and let* $\frac{\partial^3 \varphi}{\partial y^3}$ *and* $\frac{\partial^3 \varphi}{\partial t \, \partial y^2} \in C^0(\langle 0, B \rangle \times \langle 0, T \rangle)$. *If* h *defined by* (20) *is sufficiently small,* $h \leq h_0$ *where* h_0 *does not depend on* $K(t)$, *then* (12) *maps* $\hat{K}$ *one-to-one on* $K(t)$ *for any* $t \in \langle 0, T \rangle$. *In case of curved elements the sides* $\overline{R_1 R_2}$ *and* $\overline{R_2 R_3}$ *are linearly mapped on the sides* $\overline{P_1(t)\, P_2(t)}$ *and* $\overline{P_2(t)\, P_3(t)}$, *respectively, the side* $\overline{R_1 R_3}$ *is mapped on the arc* $P_1\widehat{(t)\, P_3}(t)$. *The mapping as well as its inverse are of class* C^1. *Further, its Jacobian determinant* $J(\xi, \eta, t)$ *and both these mappings are bounded on* $\langle 0, T \rangle$ *in this way :*

$$C_0 \, h_{K^0}^2 \leqq | J(\xi, \eta, t) | \leqq C_0^{-1} \, h_{K^0}^2 , \quad \left| \frac{\partial J(\xi, \eta, t)}{\partial t} \right| \leqq C_0 \, h_{K^0}^2 , \tag{22}$$

$$\begin{aligned}
&| D^\alpha x(\xi, \eta, t) | \leqq C_0 \, h_{K^0}^{|\alpha|}, && | D^\alpha y(\xi, \eta) | \leqq C_0 \, h_{K^0}^{|\alpha|}, \quad | \alpha | = 1, 2 \\
&| D^\alpha \xi(x, y, t) | \leqq C_0 \, h_{K^0}^{-1} , && | D^\alpha \eta(x, y, t) | \leqq C_0 \, h_{K^0}^{-1} , \\
&\left| \frac{\partial}{\partial t} D^\alpha \xi(x, y, t) \right| \leqq C_0 \, h_{K^0}^{-1} , && \left| \frac{\partial}{\partial t} D^\alpha \eta(x, y, t) \right| \leqq C_0 \, h_{K^0}^{-1} .
\end{aligned} \quad \Bigg\} \; | \alpha | = 1 \tag{23}$$

In addition, the family $\{ \mathcal{C}_h(t) \}$ *satisfies the minimum angle condition uniformly on* $\langle 0, T \rangle$, *i.e. the minimum angle* $\vartheta_h(t)$ *of* $\mathcal{C}_h(t)$ *satisfies :*

$$\vartheta_h(t) \geqq \vartheta_1 \quad \forall t \in \langle 0, T \rangle . \tag{24}$$

Here, C_0 *and* ϑ_1 *are positive constants.*

Proof : In the sequel C denotes a positive constant independent on K^0 and not necessarily the same in any two places. $0(h_{K^0})$ means a quantity not greater in absolute value than $C h_{K^0}$ for $t \in \langle 0, T \rangle$. First, we state that as in [7] we can prove :

$$\begin{aligned}
&\Phi(\eta, t) = 0(h_{K^0}^2) , \quad \frac{\partial^j \Phi(\eta, t)}{\partial \eta^j} = 0(h_{K^0}^{j+1}) , \quad j = 1, 2 \\
&\frac{\partial \Phi(\eta, t)}{\partial t} = 0(h_{K^0}^2) , \quad \frac{\partial^2 \Phi(\eta, t)}{\partial t \, \partial \eta} = 0(h_{K^0}^2) .
\end{aligned} \tag{25}$$

Let :

$$J_0(t) = \det \begin{pmatrix} x_2(t) - x_1(t) & x_3(t) - x_1(t) \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} .$$

Evidently, $J_0 = J$ if the element $K(t)$ is a triangle. From (25) it follows easily :

$$J = J_0 + 0(h_{K0}^3), \quad \frac{\partial J}{\partial t} = \dot{J}_0 + 0(h_{K0}^3). \tag{26}$$

One verifies that :

$$J_0(t) = [1 - L_0^{-1} \varphi(\beta_1, t)] J^* + \Lambda(t) \tag{27}$$

where :

$$J^* = \det \begin{pmatrix} \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 \\ \beta_2 - \beta_1 & \beta_3 - \beta_1 \end{pmatrix},$$

$$\Lambda(t) = (1 - L_0^{-1} \alpha_2)(\beta_3 - \beta_1)[\varphi(\beta_2, t) - \varphi(\beta_1, t)] - \\ - (1 - L_0^{-1} \alpha_3)(\beta_2 - \beta_1)[\varphi(\beta_3, t) - \varphi(\beta_1, t)].$$

Using twice Taylor's theorem and the estimate $\alpha_k - \alpha_j = 0(h_{K0})$, $j, k = 1, 2, 3$, one proves :

$$\Lambda(t) = 0(h_{K0}^3), \quad \dot{\Lambda}(t) = 0(h_{K0}^3). \tag{28}$$

Now from theorem 1 of [7] it follows :

$$Ch_{K0}^2 \leqq J(\xi, \eta, 0) \leqq C^{-1} h_{K0}^2,$$

hence by (26) also :

$$Ch_{K0}^2 \leqq J_0(0) \leqq C^{-1} h_{K0}^2.$$

Setting $t = 0$ in (27) one gets, due to (6) and (28) :

$$Ch_{K0}^2 \leqq J^* \leqq C^{-1} h_{K0}^2$$

and from (27), (28) :

$$Ch_{K0}^2 \leqq J_0(t) \leqq C^{-1} h_{K0}^2, \quad \dot{J}_0(t) = 0(h_{K0}^2). \tag{29}$$

(29) and (26) give (22). The homeomorphism of $\hat{K}$ onto $K(t)$ as well as the estimates (23) can be proved in the same way as in [7] the homeomorphism of $\overline{T}_1$ onto $\overline{T}$ and the estimates (6) and (7) were proved. Finally, if $\gamma(t)$ is the smallest angle of $K(t)$ then :

$$J_0(t) = 2 \text{ area } K(t) = a(t)\, b(t) \sin \gamma(t).$$

Since $a(t) = 0(h_{K^0})$, $b(t) = 0(h_{K^0})$, it follows from (29) that $\sin \gamma(t) \geqq C$.

LEMMA 2 : *Let the assumptions of lemma* 1 *be satisfied and let* $u(x, y, t)$ *belong for each* $t \in \langle 0, T \rangle$ *to* $H^2(\Omega(t))$. *Then* :

$$\| u - u_I \|_{L^2(\Omega(t))} + h \| u - u_I \|_{H^1(\Omega(t))} \leqq C_0 h^2 \| u \|_{H^2(\Omega(t))}, \quad 0 \leqq t \leqq T. \tag{30}$$

Here u_I *is the interpolate of* u *and* h *is defined by* (20).

Proof : We use Bramble-Hilbert lemma and lemma 1 in a standard way.

We return to the matrix $R(t)$. From the definition of the trial functions and from (15) it follows that on an element $K(t)$ $w_k(x(\xi, \eta, t), y(\xi, \eta), t)$ is equal zero if $P_k(t)$ is not a vertex of $K(t)$ and it is equal to one of the shape functions $N_j(\xi, \eta)$, $j = 1, 2, 3$ if $P_k(t)$ is a vertex of $K(t)$. Therefore $\frac{\partial}{\partial t} w_k(x(\xi, \eta, t), y(\xi, \eta), t) = 0$. If we carry out the differentiation and set $\xi = \xi(x, y, t)$, $\eta = \eta(x, y, t)$ we obtain :

$$\frac{\partial w_k(x, y, t)}{\partial x} \frac{\partial x(\xi, \eta, t)}{\partial t}\bigg|_{\substack{\xi = \xi(x,y,t) \\ \eta = \eta(x,y,t)}} + \frac{\partial w_k(x, y, t)}{\partial t} = 0 .$$

We denote by $G(x, y, t)$ the function which on each $K(t)$ is defined as follows :

$$G(x, y, t)\,|_{K(t)} = \frac{\partial x(\xi, \eta, t)}{\partial t}\bigg|_{\substack{\xi = \xi(x,y,t) \\ \eta = \eta(x,y,t)}} . \tag{31}$$

Then $\frac{\partial w_k}{\partial t} = - G \frac{\partial w_k}{\partial x}$ and :

$$R(t) = - \left\{ \left(w_j, G \frac{\partial w_k}{\partial x} \right)_{\Omega(t)} \right\}_{j,k=1}^{d} \tag{32}$$

Assuming that $G \in L^\infty(\Omega(t))$ (19) is equivalent to this variational formulation :

$$\left.\begin{aligned} &(U^{i+1} - \tilde{U}^i, v)_{\Omega^{i+1}} - \Delta t \left(\frac{\partial U^{i+1}}{\partial x}, G^{i+1} v \right)_{\Omega^{i+1}} + \Delta t a(\tilde{U}^i, t_{i+1}; U^{i+1}, v) = 0 \\ &\qquad\qquad \forall v \in V_h^{i+1}, \quad i = 0, \dots, q-1, \\ &U^0 = U^* . \end{aligned}\right\} \tag{33}$$

For later purpose we need more than to show that $G \in L^{\infty}(\Omega(t))$. From (31) and (12) it follows that :

$$G|_{K(t)} = \left[\dot{x}_1(t) + (\dot{x}_2(t) - \dot{x}_1(t))\, \xi + (\dot{x}_3(t) - \dot{x}_1(t))\, \eta + \right. \tag{34}$$

$$\left. + (1 - \xi - \eta) \frac{\partial \Phi(\eta, t)}{\partial t} \right]_{\substack{\xi = \xi(x,y,t) \\ \eta = \eta(x,y,t)}} .$$

G is a continuous function on $\overline{\Omega}(t)$ assuming the values $\dot{x}_m(t)$ at the nodes $P_m(t)$, $m = 1, \ldots, d$, because the function $(1 - \xi - \eta) \dfrac{\partial \Phi(\eta, \mathrm{t})}{\partial \mathrm{t}}$ is equal zero on the sides $\overline{R_1 R_2}$ and $\overline{R_2 R_3}$ of the reference element $\hat{K}$. On each element it has continuous derivatives with respect to x and y. What we shall need is the estimate :

$$\| G \|_{H^{1,\infty}(\Omega(t))} \leqq C \qquad \forall t \in \langle 0, T \rangle . \tag{35}$$

To prove it we remark that:

$$\dot{x}_k(t) - \dot{x}_j(t) = (1 - L_0^{-1} \alpha_k) \left(\frac{\partial \varphi(\beta_k, t)}{\partial t} - \frac{\partial \varphi(\beta_j, t)}{\partial t} \right) -$$

$$- \frac{\partial \varphi(\beta_j, t)}{\partial t} (\alpha_k - \alpha_j) , \qquad j, k = 1, 2, 3 , \qquad j \neq k$$

from which we find out easily that :

$$\dot{x}_k - \dot{x}_j = 0(h_{K0}) \quad \forall t \in \langle 0, T \rangle .$$

(34), (25) and (23) give :

$$| G | \leqq C , \quad \left| \frac{\partial G}{\partial x} \right| \leqq C , \quad \left| \frac{\partial G}{\partial y} \right| \leqq C \quad \text{on} \quad K(t) .$$

4. STABILITY AND ERROR ESTIMATES

We introduce the notation :

$$b(w, t ; u, v) = a(w, t ; u, v) - \frac{1}{2} \int_{\Gamma(t)} \dot{\varphi}_n \, uv \, d\Gamma . \tag{36}$$

THEOREM 1 : *Let the assumptions of lemma* 1 *be satisfied and let* :

$$b(w, t; v, v) \geqq 0 \qquad \forall v \in V(t),\ t \in (0, T\rangle . \tag{37}$$

Then for Δt sufficiently small, $\Delta t \leqq \Delta t_0$ where Δt_0 does not depend on h and on i, the matrices $M^{\iota+1} + \Delta t[R^{\iota+1} + K(\tilde{U}^{\iota}, t_{\iota+1})]$, $\iota = 0, \ldots, q-1$, of the systems (19) *are regular so that $\mathbf{U}^{\iota}$, $i = 1, \ldots, q$, are uniquely determined. Furthermore, the scheme* (19) *is unconditionally stable in the L^2-norm, i.e. for $\Delta t \leqq \Delta t_0$ we have* :

$$\max_{1 \leqq \iota \leqq q} \| U^{\iota} \|_{L^2(\Omega^{\iota})} \leqq C \| U^0 \|_{L^2(\Omega^0)} \tag{38}$$

where C does not depend on Δt and on h.

Proof : We prove that if $\mathbf{U}^{\iota+1}$ satisfies (19) then for Δt sufficiently small we have :

$$\| U^{\iota+1} \|_{\Omega^{\iota+1}} \leqq (1 + C\,\Delta t) \| U^{\iota} \|_{\Omega^{\iota}} \tag{39}$$

where C depends neither on Δt nor on i. Taking $U^0 = 0$ we get $U^{\iota} = 0$, $i = 1, \ldots, 0$ hence the above matrices are regular. Further, from (39) it follows

$$\| U^{\iota+1} \|_{\Omega^{\iota+1}} \leqq (1 + C\,\Delta t)^{\iota+1} \| U^0 \|_{\Omega^0} \leqq e^{CT} \| U^0 \|_{\Omega^0} .$$

To prove (39) we choose $v = U^{\iota+1}$ in (33). We get :

$$(U^{\iota+1} - \tilde{U}^{\iota}, U^{\iota+1})_{\Omega^{\iota+1}} - \Delta t \left(\frac{\partial U^{\iota+1}}{\partial x}, G^{\iota+1} U^{\iota+1} \right)_{\Omega^{\iota+1}} + \\ + \Delta t\, a(\tilde{U}^{\iota}, t_{\iota+1}; U^{\iota+1}, U^{\iota+1}) = 0 . \tag{40}$$

For an arbitrary function $v \in V(t)$ we consider the integral $\int_{\Omega(t)} Gv \frac{\partial v}{\partial x}\, dx\, dy$.

Since $G \in H^{1,\infty}(\Omega(t))$ the function Gv belongs to $H^1(\Omega(t))$ and by Green's theorem we obtain :

$$\int_{\Omega(t)} Gv \frac{\partial v}{\partial x}\, dx\, dy = \int_{\partial\Omega(t)} Gv^2 n_x\, ds - \int_{\Omega(t)} \frac{\partial G}{\partial x} v^2\, dx\, dy - \int_{\Omega(t)} Gv \frac{\partial v}{\partial x}\, dx\, dy \tag{41}$$

where n_x is the x-component of the unit outward normal to $\partial\Omega(t)$. Since $n_x = 0$ on the parts $y = 0$ and $y = B$ of $\partial\Omega(t)$, since $G = 0$ when $x = L_0$ and since we have :

$$n_x = -\frac{1}{\sqrt{1 + \varphi_y^2}}, \qquad G = \frac{\partial \varphi}{\partial t} \quad \text{on} \quad \Gamma(t)$$

we get :

$$\int_{\partial\Omega(t)} G v^2 n_x \, ds = \int_{\Gamma(t)} \dot{\varphi}_n v^2 \, d\Gamma .$$

From (41) it follows :

$$\int_{\Omega(t)} G v \frac{\partial v}{\partial x} dx \, dy = \frac{1}{2} \int_{\Gamma(t)} \dot{\varphi}_n v^2 \, d\Gamma - \frac{1}{2} \int_{\Omega(t)} \frac{\partial G}{\partial x} v^2 \, dx \, dy$$

and we see that (40) is equivalent with :

$$(U^{\iota+1} - \tilde{U}^{\iota}, U^{\iota+1})_{\Omega^{\iota+1}} + \frac{1}{2} \Delta t \left(U^{\iota+1}, \frac{\partial G^{\iota+1}}{\partial x} U^{\iota+1} \right)_{\Omega^{\iota+1}} + \\ + \Delta t \, b(\tilde{U}^{\iota}, t_{\iota+1} ; U^{\iota+1}, U^{\iota+1}) = 0 . \quad (42)$$

The first term is bounded from bellow by $\frac{1}{2} \| U^{\iota+1} \|^2_{\Omega^{\iota+1}} - \frac{1}{2} \| \tilde{U}^{\iota} \|^2_{\Omega^{\iota+1}}$, the second term by $- C \Delta t \| U^{\iota+1} \|^2_{\Omega^{\iota+1}}$ due to (35) and the third term is nonnegative according to our assumptions. Therefore we get from (42) :

$$(1 - C \Delta t) \| U^{\iota+1} \|^2_{\Omega^{\iota+1}} \leqq \| \tilde{U}^{\iota} \|^2_{\Omega^{\iota+1}}$$

and (39) is proved if we show that :

$$\| \tilde{U}^{\iota} \|^2_{\Omega^{\iota+1}} \leqq (1 + C \Delta t) \| U^{\iota} \|^2_{\Omega^{\iota}} . \quad (43)$$

We have :

$$\| \tilde{U}^{\iota} \|^2_{\Omega^{\iota+1}} = \sum_{K^{\iota+1} \in \mathcal{C}^{\iota+1}} \int_{K^{\iota+1}} [\tilde{U}^{\iota}]^2 \, dx \, dy .$$

As $\hat{\tilde{U}}^{\iota} = \hat{U}^{\iota}$ it follows from (22) :

$$\int_{K^{\iota+1}} [\tilde{U}^{\iota}]^2 \, dx \, dy = \int_{\hat{K}} [\hat{U}^{\iota}]^2 \, | J^{\iota+1} | \, d\xi \, d\eta \leqq \int_{\hat{K}} [\hat{U}^{\iota}]^2 \, | J^{\iota} | \, d\xi \, d\eta + \\ + \int_{\hat{K}} [\hat{U}^{\iota}]^2 \frac{| J^{\iota+1} - J^{\iota} |}{| J^{\iota} |} | J^{\iota} | \, d\xi \, d\eta \leqq (1 + C \Delta t) \int_{K^{\iota}} [U^{\iota}]^2 \, dx \, dy .$$

Thus (43) is true.

Before introducing the error estimate we formulate the assumptions on the data φ and D and on the exact solution u :

$$A : \text{(i)}\ \frac{\partial^3 \varphi}{\partial y^3}, \frac{\partial^2 \varphi}{\partial t^2}, \frac{\partial^3 \varphi}{\partial t\, \partial y^2} \in C^0(\langle 0, B \rangle \times \langle 0, T \rangle),$$

(ii) $D(s)$ and $D'(s)$ are bounded for $s \in \langle 0, \infty)$,

$$\text{(iii)}\ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\, \partial y}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x\, \partial t}, \frac{\partial^2 u}{\partial y\, \partial t}, \frac{\partial^2 u}{\partial t^2} \in$$

$$\in L^\infty(\{ (x, y, t) \mid t \in (0, T), (x, y) \in \Omega(t) \}),$$

$$\text{(iv)}\ \| u \|_{2,\Omega(t)} + \left\| \frac{\partial u}{\partial x} \right\|_{2,\Omega(t)} + \left\| \frac{\partial u}{\partial t} \right\|_{2,\Omega(t)} \leqq C \quad \forall t \in (0, T).$$

THEOREM 2 : *Let the family* $\{ \mathcal{T}_h^0 \}$ *of triangulations satisfy the minimum angle condition* (21) *and the assumption A be fulfilled. Further, let the form $b(w, t; u, v)$ be uniformly $V(t)$-elliptic* :

$$b(w, t; v, v) \geqq b_0 \| v \|^2_{H^1(\Omega(t))} \quad \forall v \in V(t), t \in (0, T \rangle . \tag{45}$$

Then for Δt sufficiently small, $\Delta t \leqq \Delta t_0$ where Δt_0 does not depend on h, there holds :

$$\max_{1 \leqq \iota \leqq q} \| u^\iota - U^\iota \|_{L^2(\Omega^\iota)} + \left\{ \Delta t \sum_{\iota=1}^{q} \| u^\iota - U^\iota \|^2_{H^1(\Omega^\iota)} \right\}^{1/2} \leqq$$

$$\leqq C[\| u^0 - U^0 \|_{L^2(\Omega^0)} + h + \Delta t] . \tag{46}$$

Remark : The estimate in the H^1-norm is optimal with respect to h and Δt.

Proof : We use a technique which in case that the boundary does not move is essentially that of Wheeler [6], Dupont, Fairweather, Johnson [2] and Zlámal [9].

We begin with a modification of equations (7) and (33). We introduce the operator D_t defined on each $K(t)$ by :

$$D_t\, z(x, y, t) \Big|_{K(t)} = \frac{\partial \hat{z}}{\partial t} \Big|_{\substack{\xi = \xi(x,y,t) \\ \eta = \eta(x,y,t)}} . \tag{47}$$

Evidently :

$$D_t\, z = \frac{\partial z}{\partial x} G + \frac{\partial z}{\partial t} \tag{48}$$

so that we get :

$$(D_t\, u, v)_{\Omega(t)} - \left(\frac{\partial u}{\partial x}, Gv\right)_{\Omega(t)} + a(u, t\,;\, u, v) = 0 \quad \forall v \in V(t)\,. \tag{49}$$

For arbitrary functions v, $w \in V(t)$ we have by Green's theorem (see the proof of Theorem 1) :

$$\left(\frac{\partial w}{\partial x}, Gv\right)_{\Omega(t)} = \int_{\Gamma(t)} \dot{\varphi}_n\, wv\, d\Gamma - \left(w, \frac{\partial}{\partial x}[Gv]\right)_{\Omega(t)}\,. \tag{50}$$

The last two equalities give :

$$(D_t\, u, v)_{\Omega(t)} + \left(u, \frac{\partial}{\partial x}[Gv]\right)_{\Omega(t)} + d(u, t\,;\, u, v) = 0 \quad \forall v \in V(t) \tag{51}$$

where :

$$d(w, t\,;\, u, v) = b(w, t\,;\, u, v) - \frac{1}{2}\int_{\Gamma(t)} \dot{\varphi}_n\, uv\, d\Gamma\,. \tag{52}$$

The form $d(w, t\,;\, u, v)$ is uniformly $V(t)$-elliptic due to the assumptions (45) and (6). From (33) and (50) it follows :

$$\begin{aligned}(U^{\iota+1} - \tilde{U}^{\iota}, v)_{\Omega^{\iota+1}} + \Delta t\left(U^{\iota+1}, \frac{\partial}{\partial x}[G^{\iota+1} v]\right)_{\Omega^{\iota+1}} + {}&\\ + \Delta t\, d(\tilde{U}^{\iota}, t_{\iota+1}\,;\, U^{\iota+1}, v) = 0 \qquad \forall v \in V_h^{\iota+1}\,.&\end{aligned} \tag{53}$$

The equations (51) and (53) are the starting relations for deriving the error estimate. We decompose the exact solution u in $u = \zeta + e$, where $\zeta \in V_h(t)$ is the Ritz approximation defined by :

$$d(u, t\,;\, u - \zeta, v) = 0 \quad \forall v \in V_h(t)\,. \tag{54}$$

Later we show that there holds :

$$\| e \|_{1,\Omega(t)} + \| D_t\, e \|_{1,\Omega(t)} \leqq Ch \quad t \in (0, T\,\rangle\,. \tag{55}$$

Denoting :

$$u^{\iota} - U^{\iota} = u^{\iota} - \zeta^{\iota} + \zeta^{\iota} - U^{\iota} = e^{\iota} + \varepsilon^{\iota} \tag{56}$$

we see that with respect to (55) it is sufficient to find an estimate for ε^{ι}.

We introduce one more notation. If $z(x, y, t_\iota)$ is defined on Ω^ι we denote by $\tilde{z}^\iota$ the function defined on $\Omega^{\iota+1}$ by :

$$\tilde{z}^\iota(x, y)\,|_{K^{\iota+1}} = \hat{z}(\xi(x, y, t_{\iota+1}), \eta(x, y, t_{\iota+1}), t_\iota) .$$

This definition is in agreement with the notation $\tilde{U}^\iota$ introduced in (19) since according the new definition we have $\tilde{w}_k^\iota = \hat{w}_k(\xi(x, y, t_{\iota+1}), \eta(x, y, t_{\iota+1})) = w_k^{\iota+1}$, hence $\tilde{U}^\iota = \sum_{k=1}^{d} U^\iota\, w_k^{\iota+1}$ which is the same as in (19).

First we show that :

$$(\varepsilon^{\iota+1} - \tilde{\varepsilon}^\iota, v)_{\Omega^{\iota+1}} + \Delta t\left(\varepsilon^{\iota+1}, \frac{\partial}{\partial x}[G^{\iota+1} v]\right)_{\Omega^{\iota+1}} + \Delta t\, d(\tilde{U}^\iota, t_{\iota+1}; \varepsilon^{\iota+1}, v) = = \Delta t(\psi^{\iota+1}, v)_{1,\Omega^{\iota+1}} \quad \forall v \in V_h^{\iota+1} \tag{57}$$

where $\psi^{\iota+1}$ is a function such that :

$$\| \psi^{\iota+1} \|_{1,\Omega^{\iota+1}} \leqq C(\vartheta + \| \tilde{\varepsilon}^\iota \|_{\Omega^{\iota+1}}) , \qquad \vartheta = h + \Delta t . \tag{58}$$

We proceed as in [9]. With respect to the definition of ε^ι and to (53) it suffices to prove :

$$(\zeta^{\iota+1} - \tilde{\zeta}^\iota, v)_{\Omega^{\iota+1}} + \Delta t\left(\zeta^{\iota+1}, \frac{\partial}{\partial x}[G^{\iota+1} v]\right)_{\Omega^{\iota+1}} + + \Delta t\, d(\tilde{U}^\iota, t_{\iota+1}; \zeta^{\iota+1}, v) = \Delta t(\psi^{\iota+1}, v)_{1,\Omega^{\iota+1}} . \tag{59}$$

Multiplying (51) by Δt, choosing $t = t_{\iota+1}$ and adding to both sides the term $(\omega, v)_{\Omega^{\iota+1}}$, $\omega = u^{\iota+1} - \tilde{u}^\iota - \Delta t\, D_t\, u^{\iota+1}$ we obtain :

$$(u^{\iota+1} - \tilde{u}^\iota, v)_{\Omega^{\iota+1}} + \Delta t\left(u^{\iota+1}, \frac{\partial}{\partial x}[G^{\iota+1} v]\right)_{\Omega^{\iota+1}} + + \Delta t\, d(u^{\iota+1}, t_{\iota+1}; u^{\iota+1}, v) = \Delta t(\psi_1, v)_{1,\Omega^{\iota+1}} \tag{60}$$

where $\psi_1 \in V_h^{\iota+1}$ is defined by :

$$\Delta t(\psi_1, v)_{1,\Omega^{\iota+1}} = (\omega, v)_{\Omega^{\iota+1}} \quad \forall v \in V_h^{\iota+1} . \tag{61}$$

From Taylor's theorem it follows :

$$\hat{\omega} = \hat{u}^{\iota+1} - \hat{u}^\iota - \Delta t \frac{\partial \hat{u}^{\iota+1}}{\partial t} = \int_{t_\iota}^{t_{\iota+1}} (t_\iota - t) \frac{\partial^2 \hat{u}}{\partial t^2}\, dt .$$

Differentiating we get :

$$\frac{\partial^2 \hat{u}}{\partial t^2} = \widehat{\left(\frac{\partial^2 u}{\partial x^2}\right)} \hat{G}^2 + 2\widehat{\left(\frac{\partial^2 u}{\partial x\,\partial t}\right)} \hat{G} + \widehat{\left(\frac{\partial^2 u}{\partial t^2}\right)} + \widehat{\left(\frac{\partial u}{\partial t}\right)} \frac{\partial \hat{G}}{\partial t}.$$

We need also the estimate :

$$\left|\frac{\partial \hat{G}}{\partial t}\right| \leqq C, \qquad t \in \langle 0, T \rangle$$

which can be proved in a similar way as (35). Then choosing $v = \psi_1$ in (61) and using the last three relations and the estimate (35) we have :

$$\| \psi_1 \|_{1,\Omega^{\iota+1}} \leqq C\,\Delta t. \tag{62}$$

Since $u = \zeta + e$ it follows from (60) :

$$\left(\zeta^{\iota+1} - \tilde{\zeta}^{\iota}, v\right)_{\Omega^{\iota+1}} + \Delta t\left(\zeta^{\iota+1}, \frac{\partial}{\partial x}[G^{\iota+1} v]\right)_{\Omega^{\iota+1}} + \\ + \Delta t\, d(u^{\iota+1}, t_{\iota+1}\,; u^{\iota+1}, v) = \Delta t(\psi_2, v)_{1,\Omega^{\iota+1}} \tag{63}$$

where $\psi_2 \in V_h^{\iota+1}$ and there holds .

$$\left.\begin{aligned}\Delta t(\psi_2, v)_{1,\Omega^{\iota+1}} = -\left(e^{\iota+1} - \tilde{e}^{\iota}, v\right)_{\Omega^{\iota+1}} - \Delta t\left(e^{\iota+1}, \frac{\partial}{\partial x}[G^{\iota+1} v]\right)_{\Omega^{\iota+1}} + \\ + \Delta t(\psi_1, v)_{1,\Omega^{\iota+1}} \quad \forall v \in V_h^{\iota+1}.\end{aligned}\right\} \tag{64}$$

Using (22) we get :

$$\| e^{\iota+1} - \tilde{e}^{\iota} \|^2_{K^{\iota+1}} = \int_{\hat{K}} \left[\int_{t_\iota}^{t_{\iota+1}} \widehat{D_t e}\, dt\right]^2 | J^{\iota+1} |\, d\xi\, d\eta \leqq$$

$$\leqq C \int_{\hat{K}} \left[\int_{t_\iota}^{t_{\iota+1}} \widehat{D_t e}\, | J(t) |\, dt\right]^2 d\xi\, d\eta \leqq C[\Delta t]^2 \max_{t \in \langle t_\iota, t_{\iota+1} \rangle} \| D_t e \|^2_{K(t)}.$$

If we choose $v = \psi_2$ in (64) and use the preceding estimate and the relations (55), (35) and (62) we get :

$$\| \psi_2 \|_{1,\Omega^{\iota+1}} \leqq C\vartheta. \tag{65}$$

Now let us consider the term $d(u^{\iota+1}, t_{\iota+1}; u^{\iota+1}, v)$. We have :

$$d(u^{\iota+1}, t_{\iota+1}; u^{\iota+1}, v) = d(u^{\iota+1}, t_{\iota+1}; \zeta^{\iota+1}, v) = \\ = d(\tilde{U}^{\iota}, t_{\iota+1}; \zeta^{\iota+1}, v) + (\psi_3, v)_{1,\Omega^{\iota+1}} \quad (66)$$

where $\psi_3 \in V_h^{\iota+1}$ satisfies :

$$(\psi_3, v)_{1,\Omega^{\iota+1}} = d(u^{\iota+1}, t_{\iota+1}; \zeta^{\iota+1}, v) - d(\tilde{U}^{\iota}, t_{\iota+1}; \zeta^{\iota+1}, v) \quad \forall v \in V_h^{\iota+1} . \quad (67)$$

The estimate (2.4) from [9] holds in our case, too ; hence :

$$\max_{\bar{\Omega}(t)} | \nabla\zeta | \leqq C , \qquad t \in (0, T\rangle .$$

By (56) we have :

$$u^{\iota+1} - \tilde{U}^{\iota} = u^{\iota+1} - \tilde{u}^{\iota} + \tilde{e}^{\iota} + \tilde{\varepsilon}^{\iota} = \int_{t_\iota}^{t_{\iota+1}} D_t u \, dt + \tilde{e}^{\iota} + \tilde{\varepsilon}^{\iota} .$$

From the last two relations, from the estimate $\| \tilde{e}^{\iota} \|_{\Omega^{\iota+1}} \leqq C \| e^{\iota} \|_{\Omega^{\iota}}$ (can be proved in the same way as (43)) and from the estimate (55) we get :

$$\left| \int_{\Omega^{\iota+1}} [D(u^{\iota+1}) - D(\tilde{U}^{\iota})] \nabla\zeta^{\iota+1} . \nabla v \, dx \, dy \right| \leqq \forall C(\vartheta + \| \tilde{\varepsilon}^{\iota} \|_{\Omega^{\iota+1}}) \| v \|_{1,\Omega^{\iota+1}} .$$

Choosing $v = \psi_3$ in (67) we derive :

$$\| \psi_3 \|_{1,\Omega^{\iota+1}}^2 \leqq C(\vartheta + \| \tilde{\varepsilon}^{\iota} \|_{\Omega^{\iota+1}}) \| \zeta^{\iota+1} \|_{1,\Omega^{\iota+1}} \| \psi_3 \|_{1,\Omega^{\iota+1}} .$$

(59) with $\psi^{\iota+1} = \psi_2 + \psi_3$ follows from (63) and (66) and (58) is an easy consequence of the above inequality and of (65).

We return to the equation (57). It follows from (50) that for an arbitrary function $v \in V(t)$ it holds :

$$\left(v, \frac{\partial}{\partial x} [Gv] \right)_{\Omega(t)} = -\frac{1}{2} \int_{\Gamma(t)} \dot{\varphi}_n v^2 \, d\Gamma + \frac{1}{2} \left(v, \frac{\partial G}{\partial x} v \right)_{\Omega(t)} .$$

Therefore setting $v = \varepsilon^{\iota+1}$ in (57) we get :

$$(\varepsilon^{\iota+1} - \tilde{\varepsilon}^{\iota}, \varepsilon^{\iota+1})_{\Omega^{\iota+1}} + \frac{1}{2} \Delta t \left(\varepsilon^{\iota+1}, \frac{\partial G^{\iota+1}}{\partial x} \varepsilon^{\iota+1} \right)_{\Omega^{\iota+1}} + \\ + \Delta t \, b(\tilde{U}^{\iota}, t_{\iota+1}; \varepsilon^{\iota+1}, \varepsilon^{\iota+1}) = \Delta t(\psi^{\iota+1}, \varepsilon^{\iota+1})_{1,\Omega^{\iota+1}} .$$

Using (35) and (45) we have :

$$(1 - C\,\Delta t)\,\|\,\varepsilon^{i+1}\,\|^2_{\Omega^{i+1}} + b_0\,\Delta t\,\|\,\varepsilon^{i+1}\,\|^2_{1,\Omega^{i+1}} \leqq$$
$$\leqq \|\,\tilde{\varepsilon}^i\,\|^2_{\Omega^{i+1}} + b_0^{-1}\,\Delta t\,\|\,\psi^{i+1}\,\|^2_{1,\Omega^{i+1}}\,.$$

The term $\|\,\tilde{\varepsilon}^i\,\|^2_{\Omega^{i+1}}$ can be estimated in the same way as $\|\,\tilde{U}^i\,\|^2_{\Omega^{i+1}}$ in (43) :

$$\|\,\tilde{\varepsilon}^i\,\|^2_{\Omega^{i+1}} \leqq (1 + C\,\Delta t)\,\|\varepsilon^i\,\|^2_{\Omega^i}\,.$$

The last two inequalities and (58) give :

$$\|\,\varepsilon^{i+1}\,\|^2_{\Omega^{i+1}} + b_0\,\Delta t\,\|\,\varepsilon^{i+1}\,\|^2_{1,\Omega^{i+1}} \leqq (1 + C\,\Delta t)\,\|\,\varepsilon^i\,\|^2_{\Omega^i} + C\,\Delta t\,\vartheta^2\,.$$

Summing for $i = 0, \ldots, j - 1$ we come to the inequality :

$$\|\,\varepsilon^j\,\|^2_{\Omega^j} + b_0\,\Delta t \sum_{i=1}^{j} \|\,\varepsilon^i\,\|^2_{1,\Omega^i} \leqq C[\vartheta^2 + \|\,\varepsilon^0\,\|^2_{\Omega^0}] + C\,\Delta t \sum_{i=0}^{j-1} \|\,\varepsilon^i\,\|^2_{\Omega^i}$$

and using the discrete Gronwall inequality (see Lees [3]) we obtain :

$$\max_{1 \leqq i \leqq q} \|\,\varepsilon^i\,\|^2_{\Omega^i} + \Delta t\,b_0 \sum_{i=1}^{q} \|\,\varepsilon^i\,\|^2_{1,\Omega^i} \leqq C[\vartheta^2 + \|\,\varepsilon^0\,\|^2_{\Omega^0}]\,.$$

From this inequality and (55), (56) it follows the assertion (46) of Theorem 2.

It remains to prove (55).

The estimate :

$$\|\,e\,\|_{1,\Omega(t)} \leqq Ch \tag{68}$$

can be derived in a standard way using (30) and (45). In addition we need a uniform estimate from above of the form $d(w, t\,;\,u, v)$. To this end we have to show that :

$$\int_{\Gamma(t)} v^2\,d\Gamma \leqq C\,\|\,v\,\|^2_{1,\Omega(t)} \quad v \in V(t) \tag{69}$$

(where C does not depend on t). We have :

$$v(\varphi(y, t), y, t) = v(L_0, y, t) + \int_{L_0}^{\varphi(y,t)} \frac{\partial}{\partial x}\,v(\xi, y, t)\,d\xi$$

hence :

$$v^2(\varphi, y, t) \leqq 2\left[v^2(L_0, y, t) + L_0 \int_{\varphi}^{L_0} \left[\frac{\partial v}{\partial x}\right]^2 d\xi\right].$$

Multiplying both sides by $\left(1 + \left[\frac{\partial \varphi}{\partial y}\right]^2\right)^{1/2}$ and integrating from 0 to B with respect to y we obtain :

$$\int_{\Gamma(t)} v^2 \, d\Gamma \leqq C\left[\left\| \frac{\partial v}{\partial x} \right\|^2_{\Omega(t)} + \int_{x = L_0} v^2 \, dy\right].$$

Now :

$$\int_{x = L_0} v^2 \, dy \leqq C \| v \|^2_{H^1\left(\left(\frac{1}{2} L_0, L_0\right) \times (0, B)\right)} \leqq C \| v \|^2_{H^1(\Omega(t))}$$

which proves (69).

We come to the other part of (55). The function $\zeta(x, y, t)$ is of the form $\sum_{j=1}^{d} \zeta_j(t) \, w_j(x, y, t)$. The coefficients $\zeta_j(t)$ are continuously differentiable on $\langle 0, T \rangle$. This follows from the fact that the vector $\zeta(t)$ is a solution of a linear system with a regular and continuously differentiable matrix and with a continuously differentiable right-hand side on the interval $\langle 0, T \rangle$. Now we differentiate the equation $d(u, t; e, w_j) = 0$ with respect to t. We have :

$$\left.\begin{aligned} &\frac{d}{dt} \int_{K(t)} D(u) \, \nabla e . \nabla w_j \, dx \, dy = \frac{d}{dt} \int_{\hat{K}} D(\hat{u}) \, \widehat{\nabla e} . \widehat{\nabla w_j} \mid J \mid d\xi \, d\eta = \\ &\quad = \int_{\hat{K}} \frac{\partial D(\hat{u})}{\partial t} \widehat{\nabla e} . \widehat{\nabla w_j} \mid J \mid d\xi \, d\eta + \int_{\hat{K}} D(\hat{u}) \frac{\partial}{\partial t} \widehat{\nabla e} . \widehat{\nabla w_j} \mid J \mid d\xi \, d\eta \, + \\ &\quad + \int_{\hat{K}} D(\hat{u}) \, \widehat{\nabla e} \cdot \frac{\partial}{\partial t} \widehat{\nabla w_j} \mid J \mid d\xi \, d\eta + \int_{\hat{K}} D(\hat{u}) \, \widehat{\nabla e} \cdot \widehat{\nabla w_j} \frac{\partial}{\partial t} \mid J \mid d\xi \, d\eta \, . \end{aligned}\right\} \tag{70}$$

We consider the terms of the right-hand side of this equation. First there holds :

$$\frac{\partial}{\partial t} D(\hat{u}) = D'(\hat{u}) \, \widehat{D_t u} \, . \tag{71}$$

Further we need to compute $\frac{\partial}{\partial t} \widehat{\nabla e}$ and $\frac{\partial}{\partial t} \widehat{\nabla w_j}$. Hence we consider $\frac{\partial}{\partial t} \widehat{\nabla z}$ where z is any sufficiently smooth function defined on $K(t)$. Since

$$\widehat{\nabla z} = [\mathfrak{J}^{-1}]^T \, \nabla \hat{z}$$

where $\mathfrak{J}$ is the Jacobi matrix of the mapping (12), we have :

$$\left.\begin{aligned}\frac{\partial}{\partial t}\widehat{\nabla z} &= \frac{\partial}{\partial t}[\mathfrak{J}^{-1}]^T \nabla\hat{z} + [\mathfrak{J}^{-1}]^T \frac{\partial}{\partial t}\nabla\hat{z} = \frac{\partial}{\partial t}[\mathfrak{J}^{-1}]^T \mathfrak{J}^T[\mathfrak{J}^T]^{-1}\nabla\hat{z} + \\ &+ [\mathfrak{J}^{-1}]^T \nabla\widehat{D_t z} = \hat{S}\,\widehat{\nabla z} + \widehat{\nabla D_t z}\,, \quad \hat{S} = \left[\mathfrak{J}\frac{\partial}{\partial t}\mathfrak{J}^{-1}\right]^T .\end{aligned}\right\} \quad (72)$$

The elements $\hat{S}_{jk}$ of the matrix $\hat{S}$ are bounded :

$$|\hat{S}_{jk}| \leqq C\,, \qquad t \in \langle 0, T \rangle \tag{73}$$

(it is a consequence of (22), (23)). As $D_t\, w_j = 0$, we have :

$$\frac{\partial}{\partial t}\widehat{\nabla w_j} = \hat{S}\,\widehat{\nabla w_j}\,. \tag{74}$$

It follows from (22) that the function :

$$\hat{W} = \frac{\partial}{\partial t}|J|\,|J|^{-1} \tag{75}$$

is bounded,

$$|\hat{W}| \leqq C\,, \qquad t \in \langle 0, T \rangle\,. \tag{76}$$

From (70), (71), (72), (74) and (75) we get :

$$\left.\begin{aligned}\frac{\partial}{\partial t}\int_{K(t)} D(u)\,\nabla e.\nabla w_j\,dx\,dy &= \int_{K(t)} D'(u)\,D_t\,u\,\nabla e.\nabla w_j\,dx\,dy + \\ &+ \int_{K(t)} D(u)\,[S\,\nabla e + \nabla D_t\,e].\nabla w_j\,dx\,dy \\ &+ \int_{K(t)} D(u)\,\nabla e.[S\,\nabla w_j]\,dx\,dy + \int_{K(t)} D(u)\,\nabla e.\nabla w_j\,W\,dx\,dy\end{aligned}\right\}. \quad (77)$$

Now we differentiate the integral $\int_{K'(t)} \dot{\varphi}_n\, ew_j\, ds$ where $K'(t)$ is the curved side $P_1\widehat{(t)\,P_3}(t)$ of the element $K(t)$ lying on $\Gamma(t)$. Since :

$$\int_{K'(t)} \dot{\varphi}_n\, ew_j\, ds = -\int_0^1 \frac{\partial\hat{\varphi}}{\partial t}\,\hat{e}\hat{w}_j\,|y_3 - y_1|\,d\eta\,,$$

we have :

$$\frac{d}{dt}\int_{K'(t)} \dot{\varphi}_n \, e w_j \, ds = -\int_0^1 \left[\frac{\partial^2\hat{\varphi}}{\partial t^2}\hat{e} + \frac{\partial\hat{\varphi}}{\partial t}\widehat{D_t e}\right]\hat{w}_j \, | y_3 - y_1 | \, d\eta =$$

$$= (\gamma + 1)^{-1}\int_{K'(t)} Z e w_j \, ds + \int_{K'(t)} \dot{\varphi}_n \, D_t \, e w_j \, ds \quad (78)$$

where :

$$Z = -(\gamma + 1)\frac{\partial^2\varphi}{\partial t^2}\left(1 + \left[\frac{\partial\varphi}{\partial y}\right]^2\right)^{-1/2} \quad (79)$$

and it holds :

$$| Z | \leqq C, \quad t \in \langle 0, T \rangle . \quad (80)$$

From (52), (77) and (78) we get :

$$\frac{d}{dt} d(u, t; e, w_j) = 0 = d(u, t; D_t \, e, w_j) - f(w_j), \quad j = 1, \dots, d$$

where :

$$f(v) = -\int_{\Omega(t)} D'(u) \, D_t \, u \, \nabla e . \nabla v \, dx \, dy - \int_{\Omega(t)} D(u) \, [S \, \nabla e] . \nabla v \, dx \, dy -$$

$$- \int_{\Omega(t)} D(u) \, \nabla e . [S \, \nabla v] \, dx \, dy - \int_{\Omega(t)} D(u) \, \nabla e . \nabla v W \, dx \, dy - \int_{\Gamma(t)} e v Z \, d\Gamma .$$

From the last two equations we obtain :

$$d(u, t; D_t \, e, v) = f(v) \qquad \forall v \in V_h(t) . \quad (81)$$

It remains to prove :

$$\| D_t \, e \|_{1,\Omega(t)} \leqq Ch , \quad t \in (0, T \rangle . \quad (82)$$

First we estimate the term $f(v)$. From (68), (69), (73), (76) and (80) we get :

$$|f(v)| \leqq Ch \, \| v \|_{1,\Omega(t)} . \quad (83)$$

Using (45), (81) and (83) we derive :

$$\begin{aligned} b_0 \, \| v - D_t \, \zeta \|^2_{1,\Omega(t)} &\leqq d(u, t; v - D_t \, \zeta, v - D_t \, \zeta) = \\ &= d(u, t; v - D_t \, u, v - D_t \, \zeta) + f(v - D_t \, \zeta) \\ &\leqq C[\| v - D_t \, u \|_{1,\Omega(t)} + h] \, \| v - D_t \, \zeta \|_{1,\Omega(t)} \quad \forall v \in V_h(t) . \end{aligned}$$

Hence :

$$\| D_t e \|_{1,\Omega(t)} \leqq C[\| D_t u - v \|_{1,\Omega(t)} + h], \quad v \in V_h(t). \tag{84}$$

Let us denote :

$$d_t u = \frac{\partial u}{\partial x} g + \frac{\partial u}{\partial t}, \quad g = \frac{\partial}{\partial t} F(\alpha, \beta, t) \Big|_{\substack{\alpha = F^{-1}(x,y,t) \\ \beta = y}}.$$

Since $g = G$ at vertices of the element $K(t)$ we get, using (25) and (35) :

$$\| g - G \|_{0,\infty,K(t)} + h \| g - G \|_{1,\infty,K(t)} \leqq Ch^2$$

and consequently :

$$\| D_t u - d_t u \|_{1,\Omega(t)} \leqq Ch. \tag{85}$$

(82) follows from (84), (85) and (30) if we choose $v = (d_t u)_I$.

5. NUMERICAL RESULTS

The method was tested on an example where we succeeded to find the exact solution. The equation is linear ($D(u) = 1$), the domains $\overline{\Omega}^0$ and $\overline{Q}$ are the same and equal to the square $\langle 0, 1 \rangle \times \langle 0, 1 \rangle$ ($L_0 = B = 1$), the moving boundary is simple ($\varphi(y, t) = t$) and we had to add an inhomogeneous term in the equation (1). The example reads :

$$\frac{\partial u}{\partial t} = \Delta u + f(x, y, t) \quad \text{in} \quad \Omega(t), \ 0 < t < 0.5,$$

$$\Omega(t) = \{ (x, y) \mid t < x < 1, 0 < y < 1 \}$$

$$\frac{\partial u}{\partial x}\Big|_{x=1} = \frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\partial u}{\partial y}\Big|_{y=1} = 0$$

$$\frac{\partial u}{\partial x} = u \quad \text{for} \quad x = t$$

$$u(x, y, 0) = (\cos \pi y + 2)\left(x - \frac{1}{2}x^2 + 1\right).$$

Here

$$f(x, y, t) = (\cos \pi y + 2)(t - 1) + \pi^2 \cos \pi y\left(x - \frac{1}{2}x^2 + \frac{1}{2}t^2 - 2t + 1\right).$$

The exact solution is

$$u = (\cos \pi y + 2)\left(x - \frac{1}{2}x^2 + \frac{1}{2}t^2 - 2\,t + 1\right).$$

The L_2-scalar products are computed numerically using the formula

$$\int_T F(x, y)\,dx\,dy = \frac{1}{3}\,\text{area}(T)\,[F(P_1) + F(P_2) + F(P_3)]$$

for computation of an integral over a triangle T with vertices P_1, P_2, P_3. The line integral over the moving boundary $\Gamma(t)$ is computed piecewise by the trapezoidal rule. The triangulations are of the form given in figure 2. In the table there are given relative errors E in percents,

$$E = 100 \max_{1 \leq i \leq g} \max_{1 \leq j \leq d} \left| \frac{u_j^i - U_j^i}{u_j^i} \right|,$$

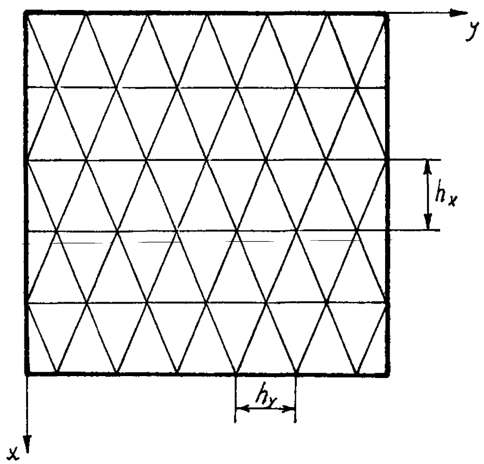

Figure 2. — Triangulation of the domain Ω_0 with $N_x = 5$, $\dot{N}_y = 6$.

for some N_x, N_y and Δt Here d is the number of nodes (equals to the number of unknowns), q is the number of time steps and $N_x = h_x^{-1}$, $N_y = h_y^{\ 1}$ (concerning h_x and h_y see *fig* 2)

N_x	5		7		9		11		13	
N_y	6		9		11		14		17	
d	45		84		125		186		259	
Δt	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{14}$	$\frac{1}{28}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{22}$	$\frac{1}{44}$	$\frac{1}{26}$	$\frac{1}{52}$
q	5	10	7	14	9	18	11	22	13	26
E	2 53	2 19	1 90	0 99	1 62	0 74	1 47	0 55	1 31	0 52

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