RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

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RAIRO – Modélisation mathématique et analyse numérique, tome 21, nº 1 (1987), p. 171-191.

<http://www.numdam.org/item?id=M2AN_1987__21_1_171_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 21, nº 1, 1987, p. 171 à 191)

HOW TO AVOID THE USE OF GREEN'S THEOREM IN THE CIARLET-RAVIART THEORY OF VARIATIONAL CRIMES (*)

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Communicated by R. TEMAM

Abstract. — *The paper generalizes the theory developed in* [1] *and* [2, *Section* 4.4] *to the case that the solution u of the given variational problem belongs to* $H^1(\Omega)$ *only. Mixed boundary conditions, approximation of a curved boundary and numerical integration are taken into account. The considerations are restricted to the two-dimensional case.*

Résumé. — *Dans cet article, nous généralisons la théorie développée dans* [1] *et* [2, *section* 4.4] *au cas où la solution u du problème variationnel se trouve dans* $H^1(\Omega)$ *seulement. Nous considérons des conditions aux limites mixtes, l'approximation de la frontière curviligne, et l'intégration numérique. Les considérations sont faites pour les problèmes de deux dimensions.*

The foundations of the theory mentioned in the title of this paper are given in Ciarlet, Raviart [1] and Ciarlet [2, Section 4.4]. Some extensions of this theory (which will be briefly denoted as the CR-theory) to the case of boundary value problems with various stable and unstable boundary conditions were done in Ženíšek [9], [10]. In all these papers the maximum rate of convergence is proved; thus the assumed smoothness of the exact solution u is unrealistic in the majority of problems appearing in applications. The smoothness of u allows us to use the Green's theorem in estimating the third term on the right-hand side of [2, (4.4.21)] — see also the first term on the right-hand side of (35). This simplifies very much considerations.

In this paper we consider the variational problem corresponding to a general elliptic boundary value problem with combined Dirichlet's and Neumann's boundary conditions. We assume only that the solution u of the variational

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problem exists, i.e. $u \in H^1(\Omega)$. Thus we cannot transform the term $\tilde{a}_m(\tilde{u}, w)$ (defined by (21)) to the form (50) by means of Green's theorem. Instead of it our main tool becomes Zlámal's ideal curved triangular element (see Zlámal [7]) which is considered simultaneously with the associate curved triangular element used in applications. As $u \in H^1(\Omega)$ the complete result of this paper will be only the proof of convergence (without any rate of convergence). The considerations of this paper are based on some results from [9]; thus we use some notions and symbols introduced in [9] without any deeper explanation and with reference to [9] only.

The notation of Sobolev spaces, their norms and seminorms is the same as in the book [2] and other references of this paper.

Let Ω be a bounded domain in E_2 with a Lipschitz-continuous boundary Γ. Let $a(v, w) : H^1(\Omega) \times H^1(\Omega) \to R$ be a bilinear form which is bounded and V-elliptic,

$$| a(v, w) | \leqslant M \| v \|_1 \| w \|_1 \quad \forall v, w \in H^1(\Omega), \tag{1}$$

$$\alpha \| v \|_1^2 \leqslant a(v, v) \quad \forall v \in V, \tag{2}$$

where α, M are positive constants and

$$V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1, \operatorname{mes}_1 \Gamma_1 > 0, \Gamma_1 \subset \Gamma \}, \tag{3}$$

and let $L(v) : H^1(\Omega) \to R$ be a bounded linear form,

$$| L(v) | \leqslant K \| v \|_1 \quad \forall v \in H^1(\Omega), \tag{4}$$

where K is a positive constant. (In (1)-(4) and in what follows we write for a greater simplicity $\| \cdot \|_1$ instead of $\| \cdot \|_{1,\Omega}$.)

Remark 1 : If $\operatorname{mes}_1 \Gamma_1 < \operatorname{mes}_1 \Gamma$ and Ω is a simply connected domain we consider only the case that Γ_1 consists of a finite number of disjoint arcs. The end-points of these arcs belong (by definition) to Γ_1. Thus $\Gamma_2 = \Gamma - \Gamma_1$ consists of a finite number of arcs without end-points. In the case of a multiply connected domain Ω the situation is similar.

Problem P : Let

$$W = \{ v \in H^1(\Omega) : v = \bar{u} \text{ on } \Gamma_1 \}, \tag{5}$$

where $\bar{u} \in H^{1/2}(\Gamma_1)$ is a given function. Find a function $u \in W$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{6}$$

The Lax-Milgram lemma implies that Problem P has just one solution $u \in W$.

In what follows we shall consider $a(v, w)$ and $L(v)$ of the forms

$$a(v, w) = \iint_{\Omega} k_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx \tag{7}$$

and

$$L(v) = L^{\Omega}(v) + L^{\Gamma}(v) \equiv \iint_{\Omega} vf \, dx + \int_{\Gamma_2} qv \, ds , \tag{8}$$

respectively, where $\Gamma_2 = \Gamma - \Gamma_1$. In (7) and in what follows a summation convection over repeated subscripts is adopted.

We assume that the following sufficient conditions for the validity of (1), (2), (4) hold :

k_{ij} are measurable and bounded functions on Ω, (9)

$$k_{ij}(x) \xi_i \xi_j \geqslant \mu_0 \xi_i \xi_i \quad \forall \xi_i, \xi_j \in R \quad \forall x \in \Omega , \tag{10}$$

where μ_0 is a positive constant,

$$f \in L_2(\Omega) , \quad q \in L_2(\Gamma_2) . \tag{11}$$

In the case of the use of numerical integration we shall have additional requirements concerning the smoothness of the functions k_{ij}, f and q.

Similarly as in [1], [2], [10] we shall consider three following variational crimes (the notion « variational crime » is due to Strang (see [4], [5])) :

1. Approximation of the space V and the manifold W by a finite dimensional space V_h and manifold W_h, respectively.

2. Approximation of the domain Ω by a domain Ω_h with a boundary Γ_h which is simpler than Γ.

3. Approximation of the forms $a(v, w)$, $L(v)$ by forms $a_h(v, w)$, $L_h(v)$ which are obtained by means of numerical integration.

Combining these three basic variational crimes we can obtain various situations ; we shall consider the most general case.

Let us choose a sequence $\{ h_m \}$ of real numbers with the following properties :

$$1 > h_m > 0 , \quad h_m > h_{m+1} , \quad \lim_{m \to \infty} h_m = 0 . \tag{12}$$

For every m let us construct an ideal triangulation τ_m^{id} of the domain Ω and its approximation τ_m^n (where n is a given integer) in the following way : Let us choose a finite number of nodal points on Γ; each corner of Γ (if any) and each end-point of arcs forming Γ_1 (if $\Gamma_1 \neq \Gamma$ — see Remark 1) belong to these points; the distance between two neighbouring nodal points is not greater than h_m. The triangulation τ_m^{id} is chosen in such a way that two different arcs, in which the boundary Γ is divided by the nodal points, are sides of two different boundary triangles. Further, the interior triangles of τ_m^{id} have only straight sides. Finally,

$$\hat{h}_m \leqslant h_m, \quad \bar{h}_m \geqslant c_0 h_m, \quad \vartheta_m \geqslant \vartheta_0, \tag{13}$$

where c_0, ϑ_0 are positive constants and

$$\hat{h}_m = \max_{T \in \tau_m} h_T, \quad \bar{h} = \min_{T \in \tau_m} h_T, \quad \vartheta_m = \min_{T \in \tau_m} \vartheta_T. \tag{14}$$

In (14) h_T and ϑ_T are the length of the greatest side of T and the magnitude of the smallest angle of T, respectively, and τ_m is the triangulation, which arises from τ_m^{id} if we substitute triangles with one curved side by triangles with straight sides. (If Ω has a polygonal boundary then $\tau_m = \tau_m^{id}$.)

If Ω has not a polygonal boundary we obtain the triangulation τ_m^n associated with τ_m^{id} in the following way: Let us choose an integer $n \geqslant 1$ and on each curved side of τ_m^{id} let us choose $n - 1$ nodal points with the coordinates

$$[\varphi(i/n), \psi(i/n)] \qquad (i = 1, \ldots, n-1),$$

where

$$x_1 = \varphi(t), \quad x_2 = \psi(t) \quad (0 \leqslant t \leqslant 1) \tag{15}$$

is the local parametric representation of the considered curved side (in more detail see [9, eqs. (6)], where the symbols $\bar{\varphi}$, $\bar{\psi}$ are used instead of φ, ψ). The side (15) is then approximated by the arc

$$x_1 = \varphi^*(t), \quad x_2 = \psi^*(t), \quad 0 \leqslant t \leqslant 1, \tag{16}$$

where $\varphi^*(t)$ and $\psi^*(t)$ are the Lagrange interpolation polynomials of degree n of the functions $\varphi(t)$ and $\psi(t)$, respectively, uniquely determined by the relations

$$\varphi^*(i/n) = \varphi(i/n) \quad (i = 0, 1, \ldots, n),$$
$$\psi^*(i/n) = \psi(i/n) \quad (i = 0, 1, \ldots, n).$$

The arcs of the type (16) form curved sides of the boundary triangles of the triangulation τ_m^n and the union of closed triangles $\bar{T} \in \tau_m^n$ forms the approximation $\bar{\Omega}_m^n$ of $\bar{\Omega}$.

Now we choose the remaining nodal points of τ_m^n and τ_m^{id}. If $n = 1$ then they are formed by the vertices of the triangles of τ_m^n or τ_m^{id}. If $n = 2$ then they are formed by the vertices of the triangles and by the mid-points of the straight sides. If $n = 3$ then they are formed by the vertices of the triangles, by the points dividing the straight sides of the triangles into thirds and by the « centres of gravity » P_T^0 of all triangles $T \in \tau_m^3$ (or $T \in \tau_m^{id}$). (In the case of a triangle T with straight sides the point P_T^0 is the center of gravity of T, in the case of a curved triangle T the point P_T^0 is the image of the point $(1/3, 1/3)$ in the transformation mapping the well-known standard triangle T_0 (*see* [6]-[10]) onto T).

On every triangle with straight sides function values prescribed at the nodal points determine uniquely a polynomial of degree n. On every curved triangle (both an ideal one and an approximating one) function values prescribed at the nodal points determine uniquely a function which is on both straight sides a polynomial of degree n in one variable. (Details are omitted; they can be found in [2], [6]-[8].)

Piecing together just mentioned finite elements we obtain N-dimensional spaces X_m^n and Y_m^n of continuous functions on τ_m^n and τ_m^{id}, respectively, where N is the number of nodal points in both triangulations τ_m^n and τ_m^{id}.

Let Γ_{m1} be the approximation of Γ_1 defined by the triangulation τ_m^n; we set

$$\begin{aligned} V_m &= \{ v \in X_m^n : v = 0 \text{ on } \Gamma_{m1} \} \\ &\equiv \{ v \in X_m^n : v(P_k) = 0,\ P_k \in \Gamma_{m1} \}, \end{aligned} \tag{17}$$

where P_k are the nodal points. In order to define suitably the finite element approximation W_m of W we shall assume that the function $\bar{u}$ is so smooth that there exists a function $z \in H^2(\Omega)$ such that $z = \bar{u}$ on Γ_1. Then we can set

$$W_m = \{ v \in X_m^n : v(P_k) = \bar{u}(P_k),\ P_k \in \Gamma_{m1} \}. \tag{18}$$

Remark 2 : In the definitions of V_m and W_m we need the space X_m^n only. The space Y_m^n will be used in (52)-(54).

In what follows we assume (similarly as in the CR-theory) that there exists a bounded domain $\tilde{\Omega}$ such that

$$\tilde{\Omega} \supset (\Omega \cup \Omega_m) \quad \forall m \tag{19}$$

and that k_{ij}, f are continuous and bounded functions on Ω having continuous and bounded extensions $\tilde{k}_{ij}$, $\tilde{f}$ onto $\tilde{\Omega}$. As to the functions $\tilde{k}_{ij}$ we further assume that there exists a constant $\tilde{\mu}_0$ such that

$$\tilde{k}_{ij}(x)\, \xi_i\, \xi_j \geqslant \tilde{\mu}_0\, \xi_i\, \xi_i \quad \forall \xi_i, \xi_j \in R \quad \forall x \in \tilde{\Omega}. \tag{20}$$

Thus every bilinear form

$$\tilde{a}_m(v, w) = \iint_{\Omega_m} \tilde{k}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx \tag{21}$$

has the property

$$\tilde{a}_m(v, v) \geqslant \tilde{\mu}_0 \mid v \mid^2_{1, \Omega_m} \quad \forall v \in H^1(\Omega_m). \tag{22}$$

Further we define

$$\tilde{L}_m(v) = \tilde{L}^{\Omega}_m(v) + \tilde{L}^{\Gamma}_m(v) \tag{23}$$

where

$$\tilde{L}^{\Omega}_m(v) = \iint_{\Omega_m} \tilde{f} v \, dx, \quad \tilde{L}^{\Gamma}_m(v) = \int_{\Gamma_{m2}} q_m \, v \, ds \tag{24}$$

with $\Gamma_{m2} = \Gamma_m - \Gamma_{m1}$. The symbol q_m denotes the function which is obtained by « transferring » the function q from Γ_2 onto Γ_{m2} (see [10]), i.e. if c is a part of Γ_2 with parametric representation (15) and c_m its approximation with parametric representation (16) then

$$\int_c qv \, ds = \int_0^1 q(\varphi(t), \psi(t)) \, v(\varphi(t), \psi(t)) \, \rho(t) \, dt, \tag{25}$$

$$\int_{c_m} q_m \, v \, ds = \int_0^1 q(\varphi(t), \psi(t)) \, v(\varphi^*(t), \psi^*(t)) \, \rho^*(t) \, dt \tag{26}$$

where

$$\rho(t) = [(\dot{\varphi}(t))^2 + (\dot{\psi}(t))^2]^{1/2}, \tag{27}$$

$$\rho^*(t) = [(\dot{\varphi}^*(t))^2 + (\dot{\psi}^*(t))^2]^{1/2}. \tag{28}$$

Using quadrature formulas on the triangles with integration points lying in $\bar{\Omega} \cap \bar{\Omega}_m$ we replace the forms $\tilde{a}_m(v, w)$ and $\tilde{L}^{\Omega}_m(v)$ by the forms $a_m(v, w)$ and $L^{\Omega}_m(v)$, respectively. (Details can be found in [1], [2] or [8].) Further, computing numerically the integral on the right-hand side of (26) for each $c_m \subset \Gamma_{m2}$ (see [10]) we obtain a linear form $L^{\Gamma}_m(v)$. Denoting

$$L_m(v) = L^{\Omega}_m(v) + L^{\Gamma}_m(v) \tag{29}$$

we can formulate the following discrete problem :

Problem P_m : Find a function $u_m \in W_m$ such that

$$a_m(u_m, v) = L_m(v) \quad \forall v \in V_m . \tag{30}$$

First we must prove the existence and uniqueness of the solution u_m of Problem P_m. This is solved (besides other problems) in Theorems 1-3.

THEOREM 1 : *Let the boundary* Γ *of the domain* Ω *be piecewise of class* C^{n+1}. *Then*

$$\| v \|_{1,\Omega_m}^2 \leqslant K \, | v |_{1,\Omega_m}^2 \quad \forall v \in V_m \quad \forall h_m < \tilde{h} \tag{31}$$

where $\tilde{h}$ *is sufficiently small fixed number and* K *a positive constant independent of* v *and* h_m.

Theorem 1 is proved in [9] in a more general form.

Remark 3 : If Γ is piecewise of class C^{n+1} then it has a finite number of points of C^{n+1}-discontinuity. These points are nodal points of all triangulations τ_m^n and τ_m^{id} $(m = 1, 2, \dots)$.

THEOREM 2 : *Let* $\tilde{k}_{ij} \in W_\infty^{(n)}(\tilde{\Omega})$ $(i, j = 1, 2)$ *and let the quadrature formula on the standard triangle* T_0 *used for calculation of* $a_m(v, w)$ *be of degree of precision* $d = \max(1, 2n - 2)$. *Then for all* $v, w \in V_m$ *we have*

$$|\tilde{a}_m(v, w) - a_m(v, w)| \leqslant C \tilde{B}_n \, h_m \, \| v \|_{1,\Omega_m} \, \| w \|_{1,\Omega_m} \tag{32}$$

where C *is a positive constant independent of* $\tilde{k}_{ij}$, v, w *and* h_m *and the constant* $\tilde{B}_n$ *is defined by*

$$\tilde{B}_n = \sum_{i,j=1}^{2} \| \tilde{k}_{ij} \|_{n,\infty,\tilde{\Omega}} . \tag{33}$$

Theorem 2 follows from [8, Theorem 7] (see also [2, Chapter 4]).

THEOREM 3 : *Let the assumptions of Theorems* 1 *and* 2 *be satisfied. Then for* $h_m \leqslant \hat{h}$ *the bilinear forms* $a_m(v, w)$ *are uniformly* V_m*-elliptic, i.e. there exists a positive constant* β *independent of* V_m *such that*

$$\beta \, \| v \|_{1,\Omega_m}^2 \leqslant a_m(v, v) \quad \forall v \in V_m \quad \forall h_m \leqslant \hat{h} , \tag{34}$$

and Problem P_m *has just one solution* u_m.

Proof : Relations (22) and (31) imply

$$\tilde{a}_m(v, v) \geqslant \tilde{\mu}_0 \, K^{-1} \, \| v \|_{1,\Omega_m}^2 \quad \forall v \in V_m .$$

Theorem 2 gives

$$a_m(v, v) - \tilde{a}_m(v, v) \geqslant - C\tilde{B}_n h_m \| v \|^2_{1,\Omega_m} \quad \forall v \in V_m .$$

Adding both inequalities up we obtain (34) with $\beta = \tilde{\mu}_0/(2K)$ for $h_m \leqslant \tilde{\mu}_0/(2 C\tilde{B}_n K)$.

Now we prove the existence and uniqueness of the solution of Problem P_m. As relation (30) represents a system of linear algebraic equations for the unknowns $u_m(P_k)$, where $P_k \notin \Gamma_{m1}$, it is sufficient to prove the uniqueness, i.e. to prove that the problem " find $u_m \in V_m$ such that $a_m(u_m, v) = 0 \ \forall v \in V_m$ " has only the trivial solution. This follows immediately from (34) if we set $v = u_m$. Theorem 3 is proved.

Now we are ready to formulate an abstract error theorem which is the starting point of the CR-theory and all its modifications.

THEOREM 4 : *Let the assumptions of Theorem* 1 *and* 2 *be satisfied. Then there exists a positive constant C independent of* V_m *and* W_m *such that for all* $h_m < \hat{h}$ *we have*

$$\| \tilde{u} - u_m \|_{1,\Omega_m} \leqslant C \left\{ \sup_{w \in V_m} \frac{| L_m(w) - \tilde{a}_m(\tilde{u}, w) |}{\| w \|_{1,\Omega_m}} + \right.$$

$$\left. + \inf_{v \in W_m} \left[\| \tilde{u} - v \|_{1,\Omega_m} + \sup_{w \in V_m} \frac{| \tilde{a}_m(v, w) - a_m(v, w) |}{\| w \|_{1,\Omega_m}} \right] \right\} \quad (35)$$

where $\tilde{u} \in H^1(\tilde{\Omega})$ *is the Calderon's extension of the solution* $u \in H^1(\Omega)$ *of Problem P from the domain* Ω *onto the domain* $\tilde{\Omega}$.

The proof of Theorem 4 follows the same lines as the proof of [2, Theorem 4.4.1] and thus it is omitted. (Of course, if $u \in H^k(\Omega)$, $k > 1$, then $\tilde{u} \in H^k(\tilde{\Omega})$ in Theorem 4.)

Before introducing the first application of Theorem 4 we remind two theorems from the theory of numerical integration in the finite element method and prove a theorem on approximations of $\tilde{u}$ in the sets W_m.

THEOREM 5 : *Let* $1 \leqslant r \leqslant n$. *Let* $\tilde{f} \in W^{(r)}_\infty(\tilde{\Omega})$ *and let the quadrature formula on the standard triangle* T_0 *used for calculation of* $L^\Omega_m(v)$ *be of degree of precision* $d = \max(1, r + n - 2)$. *Then for all* $v \in V_m$ *we have*

$$| L^\Omega_m(v) - \tilde{L}^\Omega_m(v) | \leqslant C h^r_m \| \tilde{f} \|_{r,\infty,\tilde{\Omega}} \| v \|_{1,\Omega_m} \quad (36)$$

where the constant C is independent of h_m, v *and* $\tilde{f}$.

The proof of Theorem 5 is very similar to the proofs of [2, Theorems 4.1.5 and 4.4.5].

THEOREM 6 : *Let the part Γ_2 of the boundary Γ be piecewise of class C^{n+1} and let the function $q(x_1, x_2)$ belong to the space $C^n(\overline{U})$, where U is a domain containing Γ_2. Let the quadrature formula used on the segment* [0, 1] *for calculation of $L_m^\Gamma(v)$ be of degree of precision $d = 2n - 1$. Then for sufficiently small h_m and for all $v \in V_m$ we have*

$$| L_m^\Gamma(v) - \tilde{L}_m^\Gamma(v) | \leqslant C h_m^n \| v \|_{1,\Omega_m}, \tag{37}$$

where the constant C does not depend on h_m and v.

Theorem 6 is proved in the proof of [10, Theorem 5].

THEOREM 7 : *Let $\bar{u}$ be so smooth that there exists a function $z \in H^2(\Omega)$ such that $z = \bar{u}$ on Γ_1. Then there exists a sequence $\{ v_m \}$, where $v_m \in W_m$, such that*

$$\lim_{m \to \infty} \| \tilde{u} - v_m \|_{1,\Omega_m} = 0. \tag{38}$$

Proof : According to [3], $C^\infty(\Omega) \cap V$ is dense in V. Let $\{ \varepsilon_k \}$ be an arbitrary sequence of real numbers with properties

$$\varepsilon_k > 0, \quad \varepsilon_k > \varepsilon_{k+1}, \quad \lim_{k \to \infty} \varepsilon_k = 0. \tag{39}$$

Let us set

$$w = u - z. \tag{40}$$

Then $w \in V$ and for every k there exists a function $w_{\varepsilon_k} \in C^\infty(\Omega) \cap V$ such that

$$\| w - w_{\varepsilon_k} \|_{1,\Omega} \leqslant \varepsilon_k/(3\tilde{C}), \tag{41}$$

where $\tilde{C}$ is the constant from inequality (42).

Let $\tilde{v}$ be the Calderon's extension of $v \in H^1(\Omega)$ into $H^1(E_2)$. Then we have

$$\| \tilde{v} \|_{1,E_2} \leqslant \tilde{C} \| v \|_{1,\Omega} \quad \forall v \in H^1(\Omega), \tag{42}$$

where the constant $\tilde{C}$ does not depend on v. Similarly, if $\tilde{v}^*$ denotes the Calderon's extension of $v \in H^2(\Omega)$ into $H^2(E_2)$ then

$$\| \tilde{v}^* \|_{2,E_2} \leqslant \tilde{C}^* \| v \|_{2,\Omega} \quad \forall v \in H^2(\Omega), \tag{43}$$

where the constant $\tilde{C}^*$ does not depend on v.

Relations (41), (42) imply

$$\| \tilde{w} - \tilde{w}_{\varepsilon_k} \|_{1,\tilde{\Omega}} \leqslant \tilde{C} \| w - w_{\varepsilon_k} \|_{1,\Omega} \leqslant \varepsilon_k/3. \tag{44}$$

Let $I_m v \in V_m$ be the interpolate of $v \in H^2(\Omega)$ (i.e. the function from V_m which has the same function values as v at the nodal points of τ_m^n). Owing to the definition of nodal points we have

$$I_m \tilde{v}^* = I_m v \quad \forall v \in H^2(\Omega). \tag{45}$$

Relations (43), (45) and the finite element interpolation theorems (see [2] or [8, Theorem 5]) imply

$$\| \tilde{w}_{\varepsilon_k}^* - I_m w_{\varepsilon_k} \|_{1,\Omega_m} \leqslant Ch_m \| \tilde{w}_{\varepsilon_k}^* \|_{2,\Omega_m} \leqslant \tilde{C}^* Ch_m \| w_{\varepsilon_k} \|_{2,\Omega} .$$

Thus, according to (12), there exists m_k^1 (depending on k) such that

$$\| \tilde{w}_{\varepsilon_k}^* - I_m w_{\varepsilon_k} \|_{1,\Omega_m} \leqslant \varepsilon_k/3 \quad \forall m \geqslant m_k^1 . \tag{46}$$

Finally, using the relation

$$\lim_{m \to \infty} \{ \operatorname{mes} (\Omega_m - \Omega) \} = 0$$

we find, according to the theorem on the absolute continuity of the Lebesgue integral,

$$\| \tilde{w}_{\varepsilon_k} - \tilde{w}_{\varepsilon_k}^* \|_{1,\Omega_m} = \| \tilde{w}_{\varepsilon_k} - \tilde{w}_{\varepsilon_k}^* \|_{1,\Omega_m - \Omega} \leqslant \varepsilon_k/3 \quad \forall m \geqslant m_k^2 . \tag{47}$$

Both inequalities (46) and (47) hold for $m \geqslant m_k = \max(m_k^1, m_k^2)$. It can be easily arranged that $m_k < m_{k+1}$ ($k = 1, 2, \ldots$).

Now we can construct a sequence $\{ w_m \}$, $w_m \in V_m$, such that

$$\lim_{m \to \infty} \| \tilde{w} - w_m \|_{1,\Omega_m} = 0 . \tag{48}$$

If $m_k \leqslant m < m_{k+1}$ then we set $w_m = I_m w_{\varepsilon_k} \in V_m$. The inequality

$$\| \tilde{w} - I_m w_{\varepsilon_k} \|_{1,\Omega_m} \leqslant \| \tilde{w} - \tilde{w}_{\varepsilon_k} \|_{1,\Omega_m} + \\ + \| \tilde{w}_{\varepsilon_k} - \tilde{w}_{\varepsilon_k}^* \|_{1,\Omega_m} + \| \tilde{w}_{\varepsilon_k}^* - I_m w_{\varepsilon_k} \|_{1,\Omega_m}$$

and relations (39), (44), (46), (47) imply then relation (48).

Now let us set

$$v_m = w_m + I_m z .$$

Then, according to (40), (45) and (48),

$$\| \tilde{u} - v_m \|_{1,\Omega_m} \leqslant \| \tilde{w} - w_m \|_{1,\Omega_m} + \| \tilde{z} - \tilde{z}^* \|_{1,\Omega_m - \Omega} + \\ + \| \tilde{z}^* - I_m z \|_{1,\Omega_m} \to 0 \quad \text{if} \quad m \to \infty .$$

Theorem 7 is proved.

In the case of a polygonal domain Ω the preceding theorems imply the following general result :

THEOREM 8 : *Let Ω be a bounded domain with a polygonal boundary Γ. Let the assumptions of Theorems 2, 5, 6 and 7 be satisfied, where $\tilde{\Omega} = \Omega$. Then*

$$\lim_{m \to \infty} \| u - u_m \|_{1,\Omega} = 0 , \tag{49}$$

where u and u_m are the solutions of Problems P and P_m, respectively.

Proof : As $\tilde{\Omega} = \Omega$ we have $\tilde{u} = u$ and $q_m = q$. Thus

$$\tilde{a}_m(\tilde{u}, w) \equiv a(u, w) = L^{\Omega}(w) + L^{\Gamma}(w) \equiv \tilde{L}_m^{\Omega}(w) + \tilde{L}_m^{\Gamma}(w) .$$

This result and Theorems 5, 6 imply

$$| L_m(w) - \tilde{a}_m(\tilde{u}, w) | \cdot \| w \|_{1,\Omega}^{-1} = \mathbf{O}(h_m^r) + \mathbf{O}(h_m^n) .$$

Thus the first term on the right-hand side of (35) tends to zero if $m \to \infty$.

As to the second term we have, according to Theorem 7,

$$\inf_{v \in W_m} \| \tilde{u} - v \|_{1,\Omega} \leqslant \| u - v_m \|_{1,\Omega} \to 0 .$$

Inspecting the proof of [8, Theorem 7] (and taking into account that we consider C^0-elements only) we see that relation (32) is valid for all $v, w \in X_m^n$. Thus we have

$$\inf_{v \in W_m} \sup_{w \in V_m} \{ | \tilde{a}_m(v, w) - a_m(v, w) | \cdot \| w \|_{1,\Omega}^{-1} \} \leqslant \\ \leqslant \inf_{v \in W_m} \{ C\tilde{B}_n h_m \| v \|_{1,\Omega} \} \leqslant C\tilde{B}_n h_m \| v_m \|_{1,\Omega} = \mathbf{O}(h_m)$$

because the sequence $\{ v_m \}$ is bounded, according to Theorem 7. Relation (49) follows now from Theorem 4. Theorem 8 is proved.

In the case of non-polygonal domains the situation is not so straightforward. Thus the CR-theory and its modifications assume the solution of Problem P sufficiently smooth and use the Green's theorem in order to find a more convenient expression for $\tilde{a}_m(\tilde{u}, w)$:

$$\tilde{a}_m(\tilde{u}, w) = \int_{\Gamma_{m2}} \left(\tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \nu_{mj} \right) w \, ds - \iint_{\Omega_m} \frac{\partial}{\partial x_j} \left(\tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \right) w \, dx \tag{50}$$

(see [2, p. 268] or [10]). The symbols ν_{m1}, ν_{m2} denote the components of the unit outward normal to Γ_m. The solution u is so smooth that it satisfies the equation

$$- \frac{\partial}{\partial x_j} \left(k_{ij} \frac{\partial u}{\partial x_i} \right) = f \quad \text{in } \Omega .$$

If $\Gamma_1 = \Gamma$ then (50) can be written in the form

$$\tilde{a}_m(\tilde{u}, w) = \iint_{\Omega_m} \tilde{f} w \, dx \equiv \tilde{L}_m^{\Omega}(w)$$

where the extension $\tilde{f}$ of f is defined by the relation

$$\tilde{f} = - \frac{\partial}{\partial x_j} \left(\tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \right) .$$

In this case the estimate of the first term on the right-hand side of (35) follows immediately from Theorem 5. (As to the case $\Gamma_1 \neq \Gamma$ see [10].)

Remark 4 : It should be noted that the sufficient smoothness of u enables the *CR*-theory to use finite element interpolation theorems instead of Theorem 7 and to obtain the optimum error estimates.

Our assumptions guarantee only $u \in H^1(\Omega)$ and the use of Green's theorem is forbidden for us. In order to estimate the first term on the right-hand side of (35) in the case of $u \in H^1(\Omega)$ let us define first some notions and notation.

The symbols ω_+ and ω_- have the following meaning :

$$\omega_+ = \Omega_m - \Omega , \quad \omega_- = \Omega - \Omega_m . \tag{51}$$

The symbols ω_+^1 and ω_+^2 denote the parts of ω_+ which lie along Γ_1 and Γ_2, respectively. (In other words, the boundary of ω_+^1 is formed by parts of Γ_1 and Γ_{m1} ; similarly the boundary of ω_+^2 is formed by parts of Γ_2 and Γ_{m2}.) The symbols ω_-^1 and ω_-^2 denote the parts of ω_- which lie along Γ_1 and Γ_2, respectively.

The symbol T^* will denote a triangle belonging to τ_m^n and approximating a corresponding ideal curved triangle $T^{id} \in \tau_m^{id}$. (In [9] the ideal curved triangles are denoted simply by T.)

The symbol $T^{id} : \Gamma_1$ denotes an ideal triangle $T^{id} \in \tau_m^{id}$ whose curved side lies on Γ_1. The symbol $T^* : \Gamma_1$ denotes a triangle $T^* \in \tau_m^n$ whose one side approximates a curved part of Γ_1.

Similarly as in [9], the symbol $\overline{w}$ denotes the natural extension of a function $w \in V_m$ from the domain Ω_m to the domain $\Omega_m \cup \omega_-$. (The definition of « natural extension » is given in [9, p. 271].)

Finally, the symbol $\hat{w}$ denotes a continuous function belonging to V which corresponds to a function $w \in V_m \subset X_m^n$ and is defined by the following relations :

$$\hat{w} = \overline{w} \quad \text{on} \quad \Omega - \Lambda , \quad \hat{w} = w^* \quad \text{on} \quad \Lambda , \tag{52}$$

where

$$\Lambda = \bigcup_{T^{id}:\Gamma_1} T^{id} \tag{53}$$

and where w^* is the function from Y_m^n which is uniquely determined by the values

$$w^*(P_k) = w(P_k) \quad (k = 1, \dots, N) , \tag{54}$$

$P_1, \dots, P_N$ being the nodal points of the triangulation τ_m^{id}. (The definition of the space Y_m^n is introduced in the text between relations (16) and (17).)

As $\hat{w} \in V$ we can write, according to (6) and $(24)_2$,

$$L_m(w) - \tilde{a}_m(\tilde{u}, w) = \int_{\Gamma_{m2}} q_m \, w \, ds - \tilde{L}_m^{\Gamma}(w) + \\ + \{ L_m(w) - L(\hat{w}) \} + \{ a(u, \hat{w}) - \tilde{a}_m(\tilde{u}, w) \} . \tag{55}$$

This relation is the starting point for estimating the first term on the right-hand side of (35) without using the Green's theorem. Now we express the terms in brackets in a suitable way. We have

$$L^{\Gamma}(\hat{w}) = \int_{\Gamma_2} q\overline{w} \, ds , \tag{56}$$

$$L^{\Omega}(\hat{w}) \equiv \iint_{\Omega} f\hat{w} \, dx = \iint_{\Omega_m} \tilde{f}w \, dx + \iint_{\omega_-^2} f\overline{w} \, dx - \\ - \iint_{\omega_+^2} \tilde{f}w \, dx + \sum_{T^*:\Gamma_1} \left\{ \iint_{T^{id}} f\hat{w} \, dx - \iint_{T^*} \tilde{f}w \, dx \right\} , \tag{57}$$

$$\iint_{T^{id}} f\hat{w}\,dx - \iint_{T^*} \tilde{f}w\,dx =$$

$$= \iint_{T^{id}} f\hat{w}\,dx - \iint_{T^*} \tilde{f}w\,dx - \iint_{T^{id}-T^*} f\overline{w}\,dx + \iint_{T^{id}-T^*} f\overline{w}\,dx$$

$$= \iint_{T^{id}} f\hat{w}\,dx - \iint_{T^{id}} f\overline{w}\,dx - \iint_{T^*-T^{id}} \tilde{f}w\,dx + \iint_{T^{id}-T^*} f\overline{w}\,dx$$

$$= \iint_{T^{id}} (\hat{w} - \overline{w}) f\,dx - \iint_{T^*-T^{id}} \tilde{f}w\,dx + \iint_{T^{id}-T^*} f\overline{w}\,dx . \tag{58}$$

Similarly

$$a(u, \hat{w}) - \tilde{a}_m(\tilde{u}, w) = \iint_{\omega^2_-} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_j} dx -$$

$$- \iint_{\omega^2_+} \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial w}{\partial x_j} dx + \sum_{T^* \Gamma_1} \left\{ \iint_{T^{id}} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(\hat{w} - \overline{w})}{\partial x_j} dx \right.$$

$$\left. - \iint_{T^*-T^{id}} \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial w}{\partial x_j} dx + \iint_{T^{id}-T^*} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_j} dx \right\} . \tag{59}$$

Relations (55)-(59) together with (24) and with the identities

$$\omega^1_+ = \bigcup_{T^* \Gamma_1} (T^* - T^{id}) , \quad \omega^1_- = \bigcup_{T^* \Gamma_1} (T^{id} - T^*) \tag{60}$$

imply the following lemma :

Lemma 1 : *We have*

$$| L_m(w) - \tilde{a}_m(\tilde{u}, w) | \leqslant | L^{\Omega}_m(w) - \tilde{L}^{\Omega}_m(w) | +$$

$$+ | L^{\Gamma}_m(w) - \tilde{L}^{\Gamma}_m(w) | + \left| \int_{\Gamma_{m2}} q_m w\,ds - \int_{\Gamma_2} q\overline{w}\,ds \right|$$

$$+ \left| \sum_{T^{id} \Gamma_1} \iint_{T^{id}} \left[- (\hat{w} - \overline{w}) f + k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(\hat{w} - \overline{w})}{\partial x_j} \right] dx \right|$$

$$+ \left| \iint_{\omega_+} \left\{ \tilde{f}w - \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial w}{\partial x_j} \right\} dx \right|$$

$$+ \left| \iint_{\omega_-} \left\{ - f\overline{w} + k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_j} \right\} dx \right| . \tag{61}$$

Now we can prove the main result of this paper :

THEOREM 9 : *Let the assumptions of Theorems* 1, 2, 5, 6 *and* 7 *be satisfied. Then*

$$\lim_{m \to \infty} \| \tilde{u} - u_m \|_{1,\Omega_m} = 0 \tag{62}$$

where $\tilde{u} \in H^1(\tilde{\Omega})$ is the Calderon's extension of the solution $u \in H^1(\Omega)$ of Problem P and u_m is the solution of Problem P_m.

Proof : Using Theorem 7 we find

$$\inf_{v \in W_m} \| \tilde{u} - v \|_{1,\Omega_m} \leqslant \| \tilde{u} - v_m \|_{1,\Omega_m} \to 0 \,. \tag{63}$$

Similarly as in the proof of Theorem 8 we have

$$\inf_{v \in W_m} \sup_{w \in V_m} \{ | \tilde{a}_m(v, w) - a_m(v, w) | \,.\, \| w \|_{1,\Omega_m}^{-1} \} = \mathbf{O}(h_m) \,. \tag{64}$$

It remains to prove

$$\sup_{w \in V_m} \{ | L_m(w) - \tilde{a}_m(\tilde{u}, w) | \,.\, \| w \|_{1,\Omega_m}^{-1} \} \to 0 \,. \tag{65}$$

Assertion (62) follows then from (63)-(65) and Theorem 4. The proof of (65) is divided into five parts A)-E) :

A) Using Theorems 5 and 6 we obtain

$$| L_m^{\Omega}(w) - \tilde{L}_m^{\Omega}(w) | + | L_m^{\Gamma}(w) - \tilde{L}_m^{\Gamma}(w) | \leqslant$$
$$\leqslant C h_m^r \| w \|_{1,\Omega_m} (1 \leqslant r \leqslant n) \quad \forall w \in V_m \,, \tag{66}$$

where the constant C does not depend on h_m and w.

B) Let us denote for the sake of brevity

$$D_1 = \sum_{T^{id}:\Gamma_1} \iint_{T^{id}} \left(k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(\hat{w} - \overline{w})}{\partial x_j} - (\hat{w} - \overline{w}) f \right) dx \,. \tag{67}$$

Using the assumptions $\tilde{f} \in W_{\infty}^{(r)}(\tilde{\Omega})$, $\tilde{k}_{ij} \in W_{\infty}^{(n)}(\tilde{\Omega})$ and the Cauchy inequalities we easily find

$$| D_1 | \leqslant (\| f \|_{0,\Omega} + \tilde{B}_n \| u \|_{1,\Omega}) \left\{ \sum_{T^{id}:\Gamma_1} \| \hat{w} - \overline{w} \|_{1,T^{id}}^2 \right\}^{1/2} . \tag{68}$$

According to (52)-(54), the function $\hat{w}$ is an ideal interpolate of the function $\overline{w}$

on T^{id} (because $\overline{w}(P_k) = w(P_k)$). Thus using the result proved in the proof of [7, Theorem 2] we obtain

$$\| \hat{w} - \overline{w} \|_{1,T^{id}} \leqslant Ch_m^n \| \overline{w} \|_{n+1,T^{id}} . \tag{69}$$

Inequalities (68) and (69) imply

$$| D_1 | \leqslant Ch_m^n \left\{ \sum_{T^*:\Gamma_1} (\| w \|_{n+1,T^*}^2 + \| \overline{w} \|_{n+1,T^{id}-T^*}^2) \right\}^{1/2} . \tag{70}$$

(In (69), (70) and in what follows the symbol C denotes a generic constant, not necessarily the same in any two places.) Let

$$x_1 = x_1^*(\xi_1, \xi_2), \quad x_2 = x_2^*(\xi_1, \xi_2) \tag{71}$$

be a mapping which maps one-to-one the curved triangle T^* onto the standard triangle T_0 lying in the ξ_1, ξ_2-plane and having the vertices (0, 0), (1, 0), (0, 1). According to the definition of the function $w \in X_m^n$ we have (see also [9, p. 269])

$$w |_{T^*} (x_1^*(\xi_1, \xi_2), x_2^*(\xi_1, \xi_2)) = p(\xi_1, \xi_2) , \tag{72}$$

where $p(\xi_1, \xi_2)$ is a polynomial of degree n. Using the theorem on transformation of multiple integrals and the properties of the mapping (71) (see [9, Lemma 1]) we find (because $| p |_{n+1,T_0} = 0$) :

$$\sum_{k=2}^{n+1} | w |_{k,T^*}^2 \leqslant Ch_m^{2-2n} \sum_{k=1}^{n} | p |_{k,T_0}^2 . \tag{73}$$

Using [8, Lemma 5] and the transformation from T_0 on T^* we obtain

$$| p |_{k,T_0}^2 \leqslant C | p |_{1,T_0}^2 \leqslant C \| w \|_{1,T^*}^2 \quad (k \geqslant 1) . \tag{74}$$

Relations (73), (74) imply

$$h_m^{2n} \| w \|_{n+1,T^*}^2 \leqslant Ch_m^2 \| w \|_{1,T^*}^2 . \tag{75}$$

The second term on the right-hand side of (70) can be estimated by the technique developed in [9]. Thus the proof is only sketched. Let N_1 be the number of curved triangles along Γ_1. Let us denote them by the symbols $T_1^*, T_2^*, ..., T_{N_1}^*$ and the corresponding ideal curved triangles by the symbols $T_1^{id}, T_2^{id}, ..., T_{N1}^{id}$. According to the properties of transformations (71) (see [9, Lemma 1]), we have

$$| \overline{w} |_{k,T_j^{id}-T_j^*}^2 \leqslant C \sum_{r=1}^{k} | p_j |_{r,\sigma}^2 \, h_m^{2-2r} \quad (k \geqslant 1) ,$$

where in accordance with (72) (see also [9, (27)])

$$p_j(\xi_1, \xi_2) = w\,|_{T_j^*}\,(x_1^*(\xi_1, \xi_2), x_2^*(\xi_1, \xi_2))$$

and where σ is a quadrilateral lying in the ξ_1, ξ_2-plane and having vertices $A_1(1-\delta, 0)$, $A_2(1+\delta, 0)$, $A_3(0, 1+\delta)$, $A_4(0, 1-\delta)$; δ is so small that (see [9, p. 275])

$$\operatorname{mes}\sigma = \mathbf{O}(h_m^n)\,. \tag{76}$$

As p_j is a polynomial of degree n the last inequality gives

$$\|\,\bar{w}\,\|_{n+1,T_j^{id}-T_j^*}^2 \leqslant C\left\{\,|\,p_j\,|_{0,\sigma}^2\,h_m^2 + \sum_{k=1}^{n}\,|\,p_j\,|_{k,\sigma}^2\,h_m^{2-2k}\right\}. \tag{77}$$

Each polynomial $p_j(\xi_1, \xi_2)$ can be written in the form

$$p_j(\xi_1, \xi_2) = \sum_{i=1}^{d} \alpha_i^j\, b_i(\xi_1, \xi_2)\,, \tag{78}$$

where $d = (n+1)(n+2)/2$, $b_i(\xi_1, \xi_2)$ are fixed basis functions and $\alpha_i^j = w(P_i^j)$, $P_i^j\,(i = 1, \ldots, d)$ being the nodal points of T_j^* in the local notation. Similarly as [9, (39)] we can prove

$$|\,w\,|_{0,\Omega_m}^2 \geqslant Ch_m^2\,A(\alpha_i^j)\,, \quad |\,w\,|_{1,\Omega_m}^2 \geqslant CB(\alpha_i^j)\,, \tag{79}$$

where

$$A(\alpha_i^j) = \sum_{j=1}^{N^*}\sum_{i=1}^{d}(\alpha_i^j)^2\,, \quad B(\alpha_i^j) = \sum_{j=1}^{N^*}\sum_{i=1}^{d}(\alpha_i^j - \alpha_0^j)^2\,, \tag{80}$$

$$\alpha_0^j = (\alpha_1^j + \alpha_2^j + \cdots + \alpha_d^j)/d; \tag{81}$$

$N^*(\geqslant N_1)$ denotes the total number of curved boundary triangles. (If $n = 1$ then N^* is the number of boundary triangles lying along the curved part of Γ.) As $|\,p_j\,|_{k,\sigma} = |\,p_j - \alpha_0^j\,|_{k,\sigma}\,(k \geqslant 1)$ we have, according to (77) and (79),

$$\sum_{T^*:\Gamma_1}\|\,\bar{w}\,\|_{n+1,T_j^{id}-T_j^*}^2\,\|\,w\,\|_{1,\Omega_m}^{-2} \leqslant C\left(\sum_{j=1}^{N_1}\,|\,p_j\,|_{0,\sigma}^2\right)\Big/A(\alpha_i^j) + $$

$$+\; C\left(\sum_{j=1}^{N_1}\sum_{k=1}^{n}\,|\,p_j - \alpha_0^j\,|_{k,\sigma}^2\,h_m^{2-2k}\right)\Big/B(\alpha_i^j)\,. \tag{82}$$

As $b_1 + \cdots + b_d = 1$ we can write

$$p_j - \alpha_0^j = \sum_{i=1}^{d}(\alpha_i^j - \alpha_0^j)\,b_i\,.$$

Thus

$$| p_j |_{0,\sigma}^2 \leqslant C \max_{i=1,\dots,d} | \alpha_i^j |^2 \operatorname{mes} \sigma , \tag{83}$$

$$| p_j - \alpha_0^j |_{k,\sigma}^2 \leqslant C \max_{i=1,\dots,d} | \alpha_i^j - \alpha_0^j |^2 \operatorname{mes} \sigma . \tag{84}$$

We see from (76), (80), (83) and (84) that the right-hand side of (82) is bounded by Ch_m^{2-n}. Using this result together with (75) and

$$\sum_{T^* : \Gamma_1} \| w \|_{1,T^*}^2 \leqslant \| w \|_{1,\Omega_m}^2$$

we obtain from (70) that

$$| D_1 | . \| w \|_{1,\Omega_m}^{-1} \leqslant C h_m . \tag{85}$$

C) Similarly as in part B) we have

$$\left| \iint_{\omega_+} \left(\tilde{f} w - \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial w}{\partial x_j} \right) dx \right| \leqslant$$

$$\leqslant (\| \tilde{f} \|_{0,\tilde{\Omega}} + \tilde{B}_n \| \tilde{u} \|_{1,\tilde{\Omega}}) \left\{ \sum_{T^* : \Gamma} \| w \|_{1,T^* - T^{id}}^2 \right\}^{1/2} , \tag{86}$$

$$\left| \iint_{\omega_-} \left(k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_j} - f \overline{w} \right) dx \right| \leqslant$$

$$\leqslant (\| f \|_{0,\Omega} + \tilde{B}_n \| u \|_{1,\Omega}) \left\{ \sum_{T^* : \Gamma} \| \overline{w} \|_{1,T^{id} - T^*}^2 \right\}^{1/2} , \tag{87}$$

$$\| w \|_{1,T_j^* - T_j^{id}}^2 \leqslant C h_m^2 | p_j |_{0,\sigma}^2 + C | p_j - \alpha_0^j |_{1,\sigma}^2 , \tag{88}$$

$$\| \overline{w} \|_{1,T_j^{id} - T_j^*}^2 \leqslant C h_m^2 | p_j |_{0,\sigma}^2 + C | p_j - \alpha_0^j |_{1,\sigma}^2 . \tag{89}$$

Relations (76), (79)-(84), (86)-(89) imply

$$\left| \iint_{\omega_+} \left(\tilde{f} w - \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial w}{\partial x_j} \right) dx \right| \cdot \| w \|_{1,\Omega_m}^{-1} \leqslant C h_m^{n/2} \tag{90}$$

$$\left| \iint_{\omega_-} \left(k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_j} - f \overline{w} \right) dx \right| \cdot \| w \|_{1,\Omega_m}^{-1} \leqslant C h_m^{n/2} \tag{91}$$

D) As $\overline{w} = w$ on $\overline{T}_j^*$ we have, according to (25), (26),

$$\int_{\Gamma_2} q(x_1, x_2)\,\overline{w}(x_1, x_2)\,ds - \int_{\Gamma_{m2}} q_m(x_1, x_2)\,w(x_1, x_2)\,ds =$$

$$= \sum_{j=1}^{N_2} \left\{ \int_0^1 q(\varphi_j(t), \psi_j(t))\,\overline{w}(\varphi_j(t), \psi_j(t))\,\rho_j(t)\,dt - \right.$$

$$\left. - \int_0^1 q(\varphi_j(t), \psi_j(t))\,\overline{w}(\varphi_j^*(t), \psi_j^*(t))\,\rho_j^*(t)\,dt \right\},$$

where N_2 is the number of boundary triangles lying along the curved part of Γ_2. Let us set for the sake of brevity

$$\Delta_{j1} = \varphi_j(t) - \varphi_j^*(t), \quad \Delta_{j2} = \psi_j(t) - \psi_j^*(t).$$

We have, according to [9, Lemma 2] :

$$\Delta_{j1} = \mathbf{O}(h_m^{n+1}), \qquad \Delta_{j2} = \mathbf{O}(h_m^{n+1}),$$

$$\rho_j(t) = \rho_j^*(t)\,[1 + \mathbf{O}(h_m^n)], \quad \rho_j^*(t) = \mathbf{O}(h_m).$$

Using Taylor's theorem we can write

$$\overline{w}(\varphi_j(t), \psi_j(t)) = \overline{w}(\varphi_j^*(t), \psi_j^*(t)) + \frac{\partial \overline{w}}{\partial x_1}(S_j)\,\Delta_{j1} + \frac{\partial \overline{w}}{\partial x_2}(S_j)\,\Delta_{j2},$$

where

$$S_j = (\varphi_j^*(t) + \vartheta_j\,\Delta_{j1}, \psi_j^*(t) + \vartheta_j\,\Delta_{j2}), \quad 0 < \vartheta_j < 1.$$

Thus $S_j \in \overline{T}_j^* \cup \overline{T}_j^{id}$. Using (72) and (78) we can find

$$\max_{t \in [0,1]} |\,\overline{w}(\varphi_j^*(t), \psi_j^*(t))\,| \leqslant C m_j$$

where

$$m_j = \max_{i=1,\dots,d} |\,\alpha_i^j\,|.$$

Finally, relations $\partial\overline{w}/\partial x_k = (\partial p_j/\partial \xi_i)(\partial \xi_i/\partial x_k)$ and $(13)_2$ together with [9, Lemma 1] give

$$\max_{t \in [0,1]} \left| \frac{\partial \overline{w}}{\partial x_i}(S_j) \right| \leqslant C h_m^{-1}\,m_j.$$

Combining all relations introduced here with $(79)_1$, $(80)_1$ and taking into account that $N_2 = \mathbf{O}(h_m^{-1})$ we obtain

$$\left| \int_{\Gamma_2} q\overline{w}\, ds - \int_{\Gamma_{m2}} q_m\, w\, ds \right| \cdot \| w \|_{1,\Omega_m}^{-1} \leqslant$$

$$\leqslant C h_m^n \sum_{j=1}^{N_2} m_j \{ A(\alpha_i^k) \}^{-1/2}$$

$$\leqslant C h_m^n \left\{ N_2 \sum_{j=1}^{N_2} m_j^2 \Big/ \sum_{j=1}^{N^*} \sum_{i=1}^{d} (\alpha_i^j)^2 \right\}^{1/2} \leqslant C h^{n-1/2}. \qquad (92)$$

E) Relations (66), (67), (85), (90), (91) and (92) together with Lemma 1 imply relation (65). Theorem 9 is proved.

Remark 5 : We proved more than relation (65) : Under the assumptions of Theorem 9 the rate of convergence of the first term on the right-hand side of (35) is $\mathbf{O}(h_m^{1/2})$ in the case $n = 1$ and $\mathbf{O}(h_m)$ in the case $n \geqslant 2$.

Remark 6 : For a greater simplicity we restricted our considerations to the case of triangular finite elements of the Lagrange type. Using results of [9] we can prove theorems analogous to Theorems 7 and 9 also in the case of triangular finite C^0-elements of the Hermite type. The proofs follows the same lines as the proofs of Theorems 7 and 9.

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