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THE *h-p* VERSION OF THE FINITE ELEMENT METHOD WITH QUASIUNIFORM MESHES (*)

by I. BABUŠKA ([1]), Manil SURI ([2])

Abstract. — *The classical error estimates for the h-version of the finite element method are extended for the h-p version. The estimates are expressed as explicit functions of h and p and are shown to be optimal. The estimates are given for the case where the solution* $u \in H^k$ *and the case when u has singularities at the corners of the domain.*

Résumé. — *Les estimations d'erreur classiques de la version h de la méthode des éléments finis sont étendues aux cas de la version h-p. Ces estimations sont exprimées explicitement en fonction de h et de p, et on montre qu'elles sont optimales.*
Ces estimations sont données dans le cas où u appartient à H^k *et dans le cas où u présente des singularités aux coins du domaine.*

1. INTRODUCTION

There are three versions of the finite element method : the *h-version*, the *p-version* and the *h-p version*. The *h*-version is the standard one, where the degree p of the elements is fixed, usually on low level, typically $p = 1, 2, 3$ and the accuracy is achieved by properly refining the mesh. The *p*-version, in contrast, fixes the mesh and achieves the accuracy by increasing the degree p of the elements uniformly or selectively. The *h-p* version is the combination of both.

The standard *h*-version has been thoroughly investigated theoretically (see e.g. [1, 9, 20] and others) and many program codes are available, both commercial and research. The *p*-version and the *h-p* version are new developments. There is only one commercial code, the system PROBE

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(Noetic Technologies, St. Louis) ([1]). Theoretical aspects have been studied only recently. The first theoretical paper appeared in 1981 (see [6]). See also [2, 5, 7, 10, 11, 14] for most recent results. For the numerical, computational, implementational and engineering aspects of the *h-p* version we refer to [3, 21-24].

The classical form of the error estimate for the *h*-version with quasiuniform mesh is

$$\|u_0 - u_{FE}\|_{H^1(\Omega)} \leqq C(p)\, h^{\eta - 1} \|u_0\|_{H^k(\Omega)} \tag{1.1a}$$

where

$$\eta = \min(k, p+1) \tag{1.1b}$$

and the constant $C(p)$ depends on p in an unspecified way. (See e.g. [1, 9, 20] and others.)

The main purpose of this paper is to analyze the *h-p* version with a quasiuniform mesh and uniform p and get an error estimate which is simultaneously optimal in both p and h. We show that the estimate (1.1) can be written in the form

$$\|u_0 - u_{FE}\|_{H^1(\Omega)} \leqq C \frac{h^{\eta - 1}}{p^{k-1}} \|u_0\|_{H^k(\Omega)} \tag{1.2}$$

with

$$\eta = \min(k, p+1)$$

and C independent of h, p and u_0. We will also prove estimates for the *h-p* version when the solution has singularities in the corners of the domain and in the case when essential (Dirichlet) conditions are prescribed but are not in the subspace of finite elements. Finally, we will present a numerical example illustrating the applicability of the developed (asymptotic) theory in a range of h and p used in practice.

2. THE NOTATION

For $\Omega \subset \mathbb{R}^2$ a polygonal domain, $x = (x_1, x_2) \in \mathbb{R}^2$, we let $L_2(\Omega) = H^0(\Omega)$, $H^k(\Omega)$, $H_0^k(\Omega)$, $k \geqslant 0$ integer, denote the usual Sobolev spaces. For $u \in H^k(\Omega)$ we denote by $\|u\|_{k,\Omega}$ and $|u|_{k,\Omega}$ the usual norm and seminorm, respectively. For $k \geqslant 0$ nonintegral, we define $H^k(\Omega)$ and $\|.\|_{k,\Omega}$ by the K-

([1]) In addition there is code FIESTA for solving 3 dimensional elasticity problems having p-version features but using only $1 \leqq p \leqq 4$.

method of the interpolation theory [8]. If I is an interval or a segment, then we define $H^k(I)$, $\|\cdot\|_{k,I}$, $k \geq 0$ analogously.

For $0 \leq t \leq 1$, we will denote by ${}_0\|\cdot\|_{t,I}$ the norm of the interpolation space $(H^0(I), H_0^1(I))_t$. This norm is equivalent to the $\|\cdot\|_{t,I}$ norm when $t \neq \frac{1}{2}$. For $t = \frac{1}{2}$, we obtain a norm

$$ {}_0\|\cdot\|_{\frac{1}{2},I} = \|\cdot\|_{H_{0,0}^{\frac{1}{2}}(I)} $$

which is not equivalent to the $\|\cdot\|_{\frac{1}{2},I}$ norm (see [17]). Moreover, if A is an end point of I, we may analogously define ${}_A\|\cdot\|_{t,I}$ to be the norm of the space $(H^0(I), H_A^1(I))_t$ where $H_A^1(I) = \{u \in H^1(I), u(A) = 0\}$.

Given $\rho > 0$, let

$$ R(\rho) = \{(x_1, x_2) \mid |x_1| < \rho,\ |x_2| < \rho\}\ . $$

For any $\Omega \subset \mathbb{R}^2$ we will denote $\rho_\Omega = \sup\ \{\text{diam}\ (B) \mid B \text{ a ball in } \Omega\}$.

The set of all algebraic polynomials of degree (total) less than or equal to p on Ω will be denoted by $\mathscr{P}_p^1(\Omega)$. By $\mathscr{P}_p^2(\Omega)$ we will denote the set of all polynomials of degree less than or equal to p in each variable on Ω. For $\Gamma \subset \mathbb{R}^2$ a straight segment, we define $\mathscr{P}_p(\Gamma)$ as the set of polynomials on Γ of degree less than or equal to p in s (s being the length parameter of Γ).

Let $\kappa > 0$. Then by $H_{PER}^k(R(\kappa)) \subset H^k(R(\kappa))$ we denote the space of all periodic functions with period 2κ. By $\mathcal{C}_p^1(R(\kappa))$ ($\mathcal{C}_p^2(R(\kappa))$) we denote the space of all trigonometric polynomials of (total) degree (degree in every variable) less than or equal to p.

3. THE MODEL PROBLEM

3.1. The formulation of the problem.

Consider the following model problem

$$ -\Delta u + u = f \quad \text{in} \quad \Omega \tag{3.1} $$

$$ u = g \quad \text{on} \quad \Gamma^1 \tag{3.2a} $$

$$ \frac{\partial u}{\partial n} = b \quad \text{on} \quad \Gamma^2 \tag{3.2b} $$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with vertices A_i, $i = 1, \dots, n+1$, $A_1 = A_{n+1}$,

$$ \Gamma^1 = \bigcup_{i = i_1, \dots, i_{n_1}} \bar{\Gamma}_i\,, \qquad \Gamma^2 = \bigcup_{j = j_1, \dots, j_{n_2}} \Gamma_j\,, \qquad \Gamma = \Gamma^1 \cup \bar{\Gamma}^2\,, $$

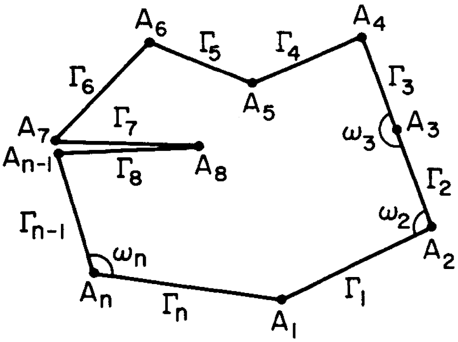

Figure 3.1. — The scheme and notation of the polygonal domain.

Γ is the boundary $\partial\Omega$ of Ω and Γ_j, $j = 1, \ldots, n$, are the open sides of the boundary $\partial\Omega$ (see *fig.* 3.1).

The internal angle at A_i is denoted by ω_i. We allow the possibility that $\omega_i = \pi$ or 2π. The case $\omega_i = 2\pi$ describes the slit (cracked) domain while the case $\omega_i = \pi$ is introduced to deal with the abrupt change of the type of the boundary condition or with nonsmoothness of g or b at the corresponding vertex. When Ω is stated to be a Lipschitz polygonal domain, then it will be assumed that $\omega_i < 2\pi$, $i = 1, 2, \ldots, n$.

Let $\tilde{H}_0^1(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma^1\}$. For $u, v \in H^1(\Omega)$ we let

$$(u, v)_{0,\Omega} = \int_\Omega uv\, dx\,, \qquad (u, v)_{1,\Omega} = \int_\Omega (\nabla u \,.\, \nabla v + uv)\, dx\,.$$

We interpret now (3.1) and (3.2) in the standard variational sense namely we seek $u \in H^1(\Omega)$ so that

$$u = g \quad \text{on} \quad \Gamma^1 \tag{3.3a}$$

and

$$(u, v)_{1,\Omega} = (f, v)_{0,\Omega} + \int_{\Gamma^2} bv\, ds \tag{3.3b}$$

holds for all $v \in \tilde{H}_0^1(\Omega)$.

We will assume that the solution u of (3.1) and (3.2) is

$$u = u_1 + u_2 + u_3 \tag{3.4}$$

where

$$u_1 \in H^{k_1}(\Omega) \cap \tilde{H}_0^1(\Omega)\,, \qquad k_1 > 1 \tag{3.4a}$$

(3.4b) $$u_2 \in H^{k_2}(\Omega), \qquad k_2 > 3/2$$

(3.4c) $$u_3 = \sum_{i=1}^{n} a_i\, u_{3,i} \in \tilde{H}_0^1(\Omega),$$

(3.4d) $$u_{3,i} = r_i^{\alpha_i}\, |\log r_i|^{\gamma_i}\, \varphi_i(\theta_i)\, \chi_i(r_i)$$

where r_i, θ_i are polar coordinates with respect to the origin located at the vertex A_i, $\alpha_i > 0$, $\gamma_i \geq 0$ integer, $\varphi_i(\theta_i)$ is an analytic function in θ_i and $\chi_i(r_i)$ is the C^∞ cut-off function so that $u_{3,i} = 0$ for $r_i \geq \rho_i > 0$, ρ_i sufficiently small.

The form (3.4) is the typical form of the solution of (3.1)-(3.2) (and of a system of second order) (see e.g. [4, 12, 16]). The assumption that $k > 3/2$ is usually satisfied in practice and hence is not a severe restriction.

3.2. The finite element method

Let $\mathcal{M} = \{\mathcal{T}^h\}$, $h \geq 0$ be a family of meshes $\mathcal{T}^h = \{S_i^h\}$ where $S_i^h \subset \Omega$ is an open triangle or parallelogram. Let $h_{S^h} = \text{diam}\,(S^h)$ and ρ_{S^h} be as defined in Section 2. We shall assume that the family $\{\mathcal{T}^h\}$ is regular in the sense that there exist positive constants σ, τ independent of h such that for all $S_i^h \in \mathcal{T}^h$, $\mathcal{T}^h \in \mathcal{M}$

(3.5a) $$\max h_{S_i^h} = h$$

(3.5b) $$\frac{h}{h_{S_i^h}} \leq \tau$$

(3.5c) $$\frac{h_{S_i^h}}{\rho_{S_i^h}} \leq \sigma .$$

(Condition (3.5b) is obviously the condition of quasiuniformity of the mesh). Further we assume that with $\mathcal{T}^h = \{S_i^h\}$, $i = 1, 2, \ldots, m_h$, $\bar{\Omega} = \bigcup_{i=1}^{m_h} \bar{S}_i^h$ and that each pair $\bar{S}_i^h$, $\bar{S}_j^h$, $i \neq j$ has either an entire side or a vertex in common, or has empty intersection.

Let F_j^h be an affine mapping with Jacobian having positive determinant which maps S_j^h onto the standard square $Q = (-1, 1) \times (-1, 1)$ when S_j^h is a parallelogram and onto the standard triangle

$$T = \{(x_1, x_2) \mid -1 < x_1 < 1,\ -1 < x_2 < x_1\}$$

when S_j^h is a triangle. Let now $\mathcal{V}_p^h(\Omega) \subset H^1(\Omega)$ denote the set of functions u such that if $u_{S_i^h}$ denotes the restriction of u to $S_i^h \in \mathcal{T}^h$ then

$u_{S_i^h} \circ (F_i^h)^{-1} \in \mathscr{P}_p^2(Q)$ if S_i^h is a parallelogram and $u_{S_i^h} \circ (F_i^h)^{-1} \in \mathscr{P}_p^1(T)$ if S_i^h is a triangle. We will then write $u_{S_i^h} \in \mathscr{P}_p(S_i^h)$ and $u \in \mathscr{V}_p^h(\Omega)$. Furthermore, we let $\mathring{\mathscr{V}}_p^h(\Omega) = \mathscr{V}_p^h(\Omega) \cap \tilde{H}_0^1(\Omega)$.

The mesh $\mathscr{T}^h$ on Ω induces a partition $\mathscr{L}_i^h = \{\gamma_{i,j}^h\}$, $j = 1, 2, \dots, m(i)$ of Γ_i, $i = 1, \dots, n$. Denote by $N_{i,j}^h$, $j = 0, 1, \dots, m(i)$ the nodal points of $\mathscr{L}_i^h$ (i.e. the end points of $\gamma_{i,j}^h$). We let $\mathscr{V}_p^h(\Gamma_i) \subset H^1(\Gamma_i)$ be the set of functions u such that the restriction $u_{\gamma_{i,j}^h}$ of u on $\gamma_{i,j}^h$ is a polynomial of degree $\leq p$. Moreover, $\mathring{\mathscr{V}}_p^h(\Gamma_i) \subset \mathscr{V}_p^h(\Gamma_i)$ will denote those polynomials that vanish on $N_{i,j}^h$, $j = 0, 1, \dots, m(i)$.

Let $g_p^h \in \bigcup_{\Gamma_i \subset \Gamma^1} \mathscr{V}_p^h(\Gamma_i)$ be the approximation of g (see (3.2)) described below. The h-p version of the finite element method consists now (for given p and h) of finding $u_p^h \in \mathscr{V}_p^h(\Omega)$ such that

$$u_p^h = g_p^h \quad \text{on} \quad \Gamma^1 \tag{3.6a}$$

$$(u_p^h, v)_{1,\Omega} = (f, v)_{0,\Omega} + \int_{\Gamma^2} bv \, ds \tag{3.6b}$$

holds for all $v \in \mathring{\mathscr{V}}_p^h(\Omega)$.

To define g_p^h we denote by g_{Γ_i} the restriction of g on $\Gamma_i \subset \Gamma^1$ and assume that $g_{\Gamma_i} \in H^r(\Gamma_i)$ with $r > 1$.

We define now g_p^h, $p \geq 1$ so that

$$g_{p,\Gamma_i}^h \in \mathscr{V}_p^h(\Gamma_i), \qquad \Gamma_i \subset \Gamma^1 \tag{3.7a}$$

$$g_{p,\Gamma_i}^h(N_{i,j}^h) = g(N_{i,j}^h), \quad j = 1, \dots, m(i), \quad i = i_1, \dots, i_{n_1} \tag{3.7b}$$

$$\int_{\Gamma_i} (g_{p,\Gamma_i}^h)' \, w' \, ds = \int_{\Gamma_i} g' \, w' \, ds, \qquad \Gamma_i \subset \Gamma^1 \tag{3.7c}$$

holds for all $w \in \mathring{\mathscr{V}}_p^h(\Gamma_i)$.

Remark : If we restrict (3.7*b*) to $j = 0, m(i)$ only ($N_{i,0}^h = A_i$, $N_{i,m(i)}^h = A_{i+1}$), then (3.7*b*) is satisfied as a consequence of (3.7*c*).

4. THE CONVERGENCE OF THE *h-p* VERSION : THE CASE OF THE SOLUTION $u \in H^k(\Omega)$

In this section we will analyze the rate of convergence of the h-p version when the solution of (3.1), (3.2) has the form (3.4) with $u_3 = 0$.

4.1. Basic approximation results

We present here some approximation results which will play an essential role later.

LEMMA 4.1 : *Let $S = Q$ or $S = T$ be the standard square or triangle. Then there exists a family of operators $\{\hat{\pi}_p\}$, $p = 1, 2, 3, \ldots$, $\hat{\pi}_p : H^k(S) \to \mathcal{P}_p(S)$ such that for any $0 \leq q \leq k$, $u \in H^k(S)$*

$$(4.1a) \qquad \|u - \hat{\pi}_p u\|_{q,S} \leq Cp^{-(k-q)}\|u\|_{k,S}, \qquad k \geq 0$$

$$(4.1b) \qquad |(u - \hat{\pi}_p u)(x)| \leq Cp^{-(k-1)}\|u\|_{k,S}, \qquad k > 1, \quad x \in S$$

where we denote $\mathcal{P}_p(S) = \mathcal{P}_p^2(S)$ for $S = Q$ and $\mathcal{P}_p(S) = \mathcal{P}_p^1(S)$ for $S = T$. The constant C in (4.1a), (4.1b) is independent of u and p but depends on k.

Moreover, if $u \in \mathcal{P}_p(S)$, then $\hat{\pi}_p(u) = u$.

Proof: The proof of this lemma is an adaptation of the proof given in [5]. Hence we will only outline the proof.

Let $r_0 > 1$ so that $\bar{S} \subset R(r_0)$. Since S is a Lipschitz domain, there exists an extension operator T mapping $H^k(S)$ into $H^k(R(2\,r_0))$ such that

$$(4.2a) \qquad Tu = 0 \quad \text{on} \quad R(2\,r_0) - R\left(\frac{3}{2}r_0\right)$$

$$(4.2b) \qquad \|Tu\|_{k,R(2\,r_0)} \leq C\,\|u\|_{k,S}$$

where C is independent of u. For a concrete construction of T we refer, for example, to [4, 19].

Let Φ be the one-to-one mapping of $R\left(\frac{\pi}{2}\right)$ onto $R(2\,r_0)$:

$$(4.3) \qquad \begin{aligned} R(2\,r_0) \ni x &= (x_1, x_2) = \Phi(\xi) \\ &= (2\,r_0 \sin \xi_1, 2\,r_0 \sin \xi_2) \end{aligned}$$

with $(\xi_1, \xi_2) = \xi \in R\left(\frac{\pi}{2}\right)$.

Further, we let

$$(4.4) \qquad \tilde{R} = \Phi^{-1}\left[R\left(\frac{3}{2}r_0\right)\right] \subset R\left(\frac{\pi}{2}\right)$$

where Φ^{-1} denotes the inverse mapping of Φ. Let $v = Tu$ and

$$(4.5) \qquad V(\xi) = v(\Phi(\xi)).$$

Because of (4.2*a*) we easily see that

$$\text{Supp } V(\xi) \subset \bar{\bar{R}} . \tag{4.6}$$

In addition it can be readily seen that

$$V \in H^k_{PER}(R(\pi)) , \tag{4.7a}$$

$$\| V \|_{k, R(\pi)} \leqslant C \| u \|_{k, S} , \tag{4.7b}$$

(4.7*c*) $V(\xi)$ is a symmetric function with respect to the lines $\xi_i = \pm \frac{\pi}{2}$, $i = 1, 2$.

Let us expand the function V in terms of its Fourier series

$$V(\xi_1, \xi_2) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{jl} e^{i(j\xi_1 + l\xi_2)} .$$

For any $p \geqslant 1$ we define

i) for $S = Q$:

$$\hat{\pi}_p V = \sum_{|j| \leqslant p} \sum_{|l| \leqslant p} a_{jl} e^{i(j\xi_1 + l\xi_2)} \tag{4.8a}$$

ii) for $S = T$:

$$\hat{\pi}_p V = \sum_{|j| + |l| \leqslant p} a_{jl} e^{i(j\xi_1 + l\xi_2)} . \tag{4.8b}$$

Then quite similarly as in [5] we have for $0 \leqslant q \leqslant k$

$$\| V - \hat{\pi}_p V \|_{q, R(\pi)} \leqslant C p^{-(k-q)} \| u \|_{k, S} \qquad k \geqslant 0 \tag{4.9a}$$

$$| (V - \hat{\pi}_p V(\xi)) | \leqslant C p^{-(k-1)} \| u \|_{k, S} , \qquad k > 1 . \tag{4.9b}$$

Because $(\hat{\pi}_p V)(\Phi^{-1}(x)) \in \mathscr{P}_p(S)$ and Φ is a regular mapping of $R(r_0)$ $\left(r_0 < \frac{\pi}{2} \right)$ on S, (4.9) yields the lemma immediately.

Let us quote now the following scaling result.

LEMMA 4.2 : *Let Ω and Ω^h be two open subsets of $\mathbb{R}^n$ such that there exists an affine mapping $F(x) = B(x) + b$ of Ω^h onto Ω and $F(\Omega^h) = \Omega$. Let* diam $(\Omega) = 1$, $\rho_\Omega = K$, diam $(\Omega^h) = h$, $\rho_{\Omega^h} = \bar{K} h$. *If the function $\hat{v} \in H^m(\Omega)$, $m \geqslant 0$ integer, then $v = \hat{v} \circ F \in H^m(\Omega^h)$ and*

$$| v |_{m, \Omega^h} \leqslant C h^{\frac{n}{2} - m} | \hat{v} |_{m, \Omega} \tag{4.10a}$$

(4.10*b*)
$$|\hat{v}|_{m,\Omega} \leq C h^{m-\frac{n}{2}} |v|_{m,\Omega^h}$$

where C depends on K and $\bar{K}$ but not on Ω, h, v.

For the proof see [9], Theorem 3.1.2.

The estimate of the error of the approximation of g by g_p^h is given in

LEMMA 4.3 : *Let $r > 1$, $0 \leq t \leq 1$, $p \geq 1$, then*

(4.11*a*)
$$_0\|g - g_p^h\|_{t,\gamma_{i,j}^h} \leq C \frac{h^{\nu - t}}{p^{r-t}} \|g\|_{r,\gamma_{i,j}^h}$$

(4.11*b*)
$$_0\|g - g_p^h\|_{t,\Gamma_i} \leq C \frac{h^{\nu - t}}{p^{r-t}} \|g\|_{r,\Gamma_i}$$

where

(4.11*c*)
$$\nu = \min\,(r, p + 1)$$

and C is independent of g, p and h.

The proof is given in [13]. The main idea is to expand g' in Legendre polynomials on every $\gamma_{i,j}^h$ of the partitioning of Γ_i induced by the mesh $\mathcal{T}^h$, prove (4.11) for r and t integral and by the interpolation argument obtain (4.11) in full generality.

Let us prove now :

LEMMA 4.4 : *Let S^h and S be the triangle or parallelogram satisfying the conditions of Lemma 4.2. Then for any $\hat{u} \in H^k(\Omega)$ corresponding to the function $u \in H^k(\Omega^h)$, $k \geq 0$ we have*

(4.12)
$$\inf_{\hat{p} \in \mathcal{P}_p(\Omega)} \|\hat{u} - \hat{p}\|_{k,\Omega} \leq C h^{\mu - 1} \|u\|_{k,\Omega^h}$$

where $\mu = \min\,(p + 1, k)$ and C depends on K, $\bar{K}$, k but is independent of p and u.

Proof: For $k = 0$ the result follows immediately from Lemma 4.2 taking $\hat{p} = 0$. Hence let $k > 0$. Assume first that k is an integer. Then

$$\inf_{\hat{p} \in \mathcal{P}_p(\Omega)} \|\hat{u} - \hat{p}\|_{k,\Omega} \leq \inf_{\hat{p} \in \mathcal{P}_p(\Omega)} \left\{ \|\hat{u} - \hat{p}\|_{\mu,\Omega} + \sum_{i=\mu+1}^{k} |\hat{u}|_{i,\Omega} + \sum_{i=\mu+1}^{k} |\hat{p}|_{i,\Omega} \right\}$$

where $\sum_{i=\mu+1}^{k} = 0$ for $k < \mu + 1$. Using Theorem 3.1.1 of [9], we see that

$$\inf_{\hat{p} \in \mathcal{P}_p(\Omega)} \|\hat{u} - \hat{p}\|_{k,\Omega} \leq C \sum_{i=\mu}^{k} |\hat{u}|_{i,\Omega}$$

$$\leq C \sum_{i=\mu}^{k} h^{i-1} |u|_{i,\Omega^h} \quad \text{(by 4.10}b\text{)}$$

$$\leq C h^{\mu-1} \|u\|_{k,\Omega^h}$$

and (4.12) is proven for k integer. For general k we use an interpolation argument.

Let us prove now :

LEMMA 4.5 : *Let S^h be a triangle or parallelogram with vertices A_i satisfying conditions* (3.5). *Let $u \in H^k(S^h)$. Then there exists a constant C depending on k, τ, σ but independent of u, p and h and a sequence $z_p^h \in \mathcal{P}_p(S^h)$, $p = 1, 2, \ldots$ (see def. of $\mathcal{P}_p(S^h)$ in Section* 3.2) *such that for any* $0 \leq q \leq k$

(4.13a) $$\|u - z_p^h\|_{q,S^h} \leq C \frac{h^{\mu-q}}{p^{k-q}} \|u\|_{k,S^h}, \qquad k \geq 0$$

(4.13b) $$|(u - z_p^h)(x)| \leq C \frac{h^{\mu-1}}{p^{k-1}} \|u\|_{k,S^h}, \qquad k > 1, \quad x \in S^h$$

(4.13c) $$\mu = \min(p+1, k).$$

If $k > 3/2$, then we can assume that $z_p^h(A_i) = u(A_i)$.

Further, for $t - \frac{1}{2}$

$$_0\|u - z_p^h\|_{t,\gamma} \leq C \frac{h^{(\mu-t)}}{p^{k-1}} \|u\|_{k,S^h}$$

where γ is any side of S_h.

Proof: Let $\hat{\pi}_p$ be the operator introduced in Lemma 4.1. Define now

$$\pi_p^h : H^k(S^h) \to \mathcal{P}_p(S^h)$$

so that

$$\pi_p^h u = (\hat{\pi}_p(u \circ F^{-1})) \circ F$$

where F is the linear mapping of S^h onto T, respectively Q (see Section 3.2). Denoting $\hat{u} = u \circ F^{-1}$ we get from Lemmas 4.1 and 4.4 for $q \leq k$

(4.14)
$$\begin{aligned}\|\hat{u}-\hat{\pi}_p\,\hat{u}\|_{q,S} &= \|(\hat{u}-\hat{p})-\hat{\pi}_p(\hat{u}-\hat{p})\|_{q,S} \\ &\leqslant Cp^{-(k-q)}\inf_{\hat{p}\in\mathscr{P}_p(S)}\|\hat{u}-\hat{p}\|_{k,S} \\ &\leqslant Cp^{-(k-q)}h^{\mu-1}\|u\|_{k,S^h}\,.\end{aligned}$$

Combining (4.14) with Lemma 4.2 we get for $0\leqslant m\leqslant q\leqslant k$

$$|u-\pi_p^h u|_{m,S^h}\leqslant Ch^{\mu-m}p^{-(k-q)}\|u\|_{k,S^h}$$

and hence

(4.15)
$$\|u-\pi_p^h u\|_{q,S^h}\leqslant Ch^{\mu-q}p^{-(k-q)}\|u\|_{k,S^h}\,.$$

Now analogously for $k>1$ and $\hat{x}\in S$

(4.16)
$$\begin{aligned}|(\hat{u}-\pi_p\,\hat{u})(\hat{x})| &\leqslant Cp^{-(k-1)}\inf_{\hat{p}\in\mathscr{P}_p(S)}\|\hat{u}-\hat{p}\|_{k,S} \\ &\leqslant Cp^{-(k-1)}h^{\mu-1}\|u\|_{k,S^h}\end{aligned}$$

and (4.13) is proven.

If $k>3/2$ then we modify z_p^h analogously as in Theorem 4.1 of [5]. We get $z_p^h(A_i)=u(A_i)$ and by interpolation

$$_0\|u-z_p^h\|_{t,\gamma}\leqslant Ch^{(\mu-t)}p^{-(k-t)}\|u\|_{k,S^h}$$

where γ is any side of S^h.

The proof of the following theorem is a modified version of Theorem 4.1 in [5].

THEOREM 4.6 : *Let u be the solution of* (3.1-3.2), $u\in H^k(\Omega)$, $k>3/2$ *and for* $\Gamma_i\subset\Gamma^1$ *let* $g_i\in H^r(\Gamma_i)$, $r\geqslant k-1/2$, *where* g_i *is the restriction of* g *to* Γ_i. *Then for each* $p\geqslant 1$ *and* $h>0$, *there exists* $\varphi_p^h\in\mathscr{V}_p^h(\Omega)$ *such that*

(4.17*a*)
$$\varphi_p^h=g_p^h \quad on \quad \Gamma^1$$

(4.17*b*)
$$\|u-\varphi_p^h\|_{1,\Omega}\leqslant C\,\frac{h^{\mu-1}}{p^{k-1}}\left(\|u\|_{k,\Omega}+\sum_i\|g\|_{r,\Gamma_i}\right)$$

(4.17*c*)
$$\mu=\min\,(p+1,k)$$

where g_p^h *is defined by* (3.7) *and* C *is independent of* u, p, h, *and* $\mathscr{T}^h$.

First we will introduce

LEMMA 4.7 : *Let* $S=Q$ *or* $S=T$ *and let* $\gamma=\overline{A_1A_2}$ *be a side of* S. *Let* $\psi\in\mathscr{P}_p(\gamma)$ *such that* $\psi(A_i)=0$, $i=1,2$. *Then there exists an extension*

$v \in \mathscr{P}_p(S)$, $v = \psi$ on γ, $v = 0$ on $\partial S - \gamma$ and

$$\|v\|_{1,S} \leqslant C\ {}_0\|\psi\|_{1/2,\gamma} \tag{4.18}$$

where the constant C is independent of p and ψ.

The lemma follows from Theorems 7.4 and 7.5 presented in the Appendix.

Proof of Theorem 4.6 : Let $\{S_i^h\} = \mathscr{C}^h$. Then by Lemma 4.5 there exists $z_{p,i}^h \in \mathscr{P}_p(S_i^h)$ such that $z_{p,i}^h = u$ at all vertices of S_i^h. Let now $\gamma^h = \bar{S}_j^h \cap \bar{S}_l^h$ and let N_1, N_2 be the end points of γ^h. Then $z_{p,j}^h - z_{p,l}^h = w_{j,l}^h$ is a polynomial on γ^h of degree at most p, and $w_{j,l}^h(N_i) = 0$, $i = 1, 2$. We now map $\bar{S}_j^h \cup \bar{S}_l^h$ onto $\bar{S}_j \cup \bar{S}_l$ by a continuous linear mapping F where S_j and S_l are congruent images of Q or T, suitably placed as shown in figure 4.1.

Using the notation used in the proof of Lemma 4.5 we get, by Lemma 4.5

$$\|\hat{u} - \hat{z}_{p,j}^h\|_{1,S_j} \leqslant C \frac{h^{\mu-1}}{p^{k-1}} \|u\|_{k,S_j^h}.$$

$\|\hat{u} - \hat{z}_{p,l}^h\|_{1,S_l}$ is analogously bounded. Also by Lemma 4.5,

$$\begin{aligned} {}_0\|\hat{w}_{j,l}^h\|_{1/2,\gamma} &\leqslant {}_0\|\hat{u} - \hat{z}_{p,j}^h\|_{1/2,\gamma} + {}_0\|\hat{u} - \hat{z}_{p,l}^h\|_{1/2,\gamma} \\ &\leqslant C \frac{h^{\mu-1}}{p^{k-1}} (\|u\|_{k,S_j^h} + \|u\|_{k,S_l^h}). \end{aligned}$$

Applying Lemma 4.7 there exists $\hat{\psi} \in \mathscr{P}_p(S_j)$ so that

$$\|\hat{\psi}\|_{1,S_j} \leqslant {}_0\|\hat{w}_{j,l}\|_{1/2,\gamma}$$

$$\hat{\psi} = \hat{w}_{j,l} \quad \text{on} \quad \gamma$$

and

$$\hat{\psi} = 0 \quad \text{on} \quad \partial S_j - \gamma.$$

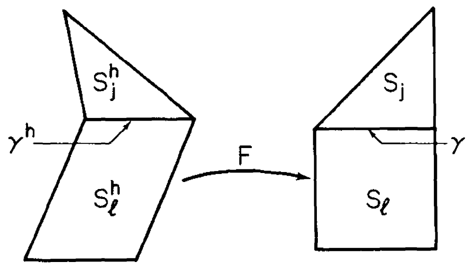

Figure 4.1. — Scheme for the map of two neighboring elements.

Hence we can modify $z_{p,j}^h$ to $\tilde{z}_{p,j}^h$ so that $\tilde{z}_{p,j}^h = z_{p,l}^h$ on γ^h and

$$\|\tilde{z}_{p,j}^h - u\|_{1, S_j^h} \leq C \frac{h^{\mu - 1}}{p^{k-1}} (\|u\|_{k, S_j^h} + \|u\|_{k, S_l^h}) .$$

Repeating this process we construct $\tilde{z}_{p,j}^h$ similarly on each S_j^h.

Defining $\tilde{\varphi}_p^h$ so that its restriction on S_j^h is $\tilde{z}_{p,j}^h$ we get $\tilde{\varphi}_p^h \in \mathscr{V}_p^h(\Omega)$

$$\|u - \tilde{\varphi}_p^h\|_{1,\Omega} \leq C \frac{h^{\mu - 1}}{p^{k-1}} \|u\|_{k,\Omega} .$$

Finally if $\partial S_j^h \cap \Gamma^1 = \gamma^h \neq \emptyset$, we have to modify $z_{p,j}^h$ so that $z_{p,j}^h = g_p^h$ on γ^h. Using (4.11*a*) and realizing that

$$\sum_j \|g\|_{r, \gamma_{i,j}^h}^2 \leq \|g\|_{r, \Gamma_i}^2$$

we can proceed quite analogously as before and complete the proof.

Remark : By the imbedding theorem we have $\|g\|_{k - 1/2, \Gamma_i} \leq \|u\|_{k,\Omega}$ and hence the second term in (4.17*b*) can be omitted.

4.2. The approximation results for $1 < k < 3/2$

In the previous section we analyzed the case when the solution u of (3.1)-(3.2) belongs to $H^k(\Omega)$, $k > 3/2$. We will now analyze the case when $u \in H^k(\Omega)$, $1 < k < 3/2$ and $g = 0$. In addition, we will assume that Ω is a Lipschitz domain.

As shown in [4], given any $t > 0$ and $k > 1$, the function u can be decomposed so that

(4.19)
$$u = v^t + \omega^t$$
$$v^t \in \tilde{H}_0^1(\Omega)$$
$$\omega^t \in H^k(\Omega) \cap \tilde{H}_0^1(\Omega)$$

and for any ~~$k > q > 1$~~

(4.20*a*)
$$\|v^t\|_{1,\Omega} \leq t^{q-1} \|u\|_{q,\Omega}$$

(4.20*b*)
$$\|\omega^t\|_{k,\Omega} \leq t^{q-k} \|u\|_{q,\Omega} .$$

Let $2 \geq k > 3/2$, and $1 \leq q \leq 3/2$. Then by Theorem 4.6 there exists $\varphi_p^h \in \mathscr{V}_p^h(\Omega)$ such that

$$\varphi_p^h = 0 \quad \text{on} \quad \Gamma^1$$

$$\|\omega^t - \varphi_p^h\|_{1,\Omega} \leq C \frac{h^{k-1}}{p^{k-1}} \|\omega^t\|_{k,\Omega}$$

since for $p \geq 1$, $\min(p+1, k) = k$. Hence

$$\|u - \varphi_p^h\|_{1,\Omega} \leq \|v^t\|_{1,\Omega} + \|\omega^t - \varphi_p^h\|_{1,\Omega}$$
$$\leq C\left(t^{q-1} + \frac{h^{k-1}}{p^{k-1}} t^{q-k}\right) \|u\|_{q,\Omega}.$$

Choosing $t = h/p$ we get

$$\|u - \varphi_p^h\|_{1,\Omega} \leq C\left(\frac{h}{p}\right)^{q-1} \|u\|_{q,\Omega} = C\frac{h^{\mu-1}}{p^{q-1}} \|u\|_{q,\Omega} \tag{4.21}$$

since

$$\min(p+1, q) = q, \qquad q \leq 3/2.$$

We remark that the assumption that Ω is a Lipschitz domain was used in the proof of decomposition (4.20). (4.21) shows that in Theorem 4.6 we can replace the restriction $k > 3/2$ by $k > 1$ provided that $g = 0$. In fact, we need less namely that $g|_{\Gamma_i} \in H^r(\Gamma_i)$, $\Gamma_i \subset \Gamma^1$, $r > 1$.

4.3. The rate of convergence of the *h-p* version of the finite element method

We will prove now

THEOREM 4.8: *Let* $u \in H^k(\Omega)$, $k > 1$ *be the solution of* (3.1)-(3.2). *Assume further that* g *is such that*

$$u = u_1 + u_2$$
$$u_1 \in H^{k_1}(\Omega) \cap \tilde{H}_0^1(\Omega)$$
$$u_2 \in H^{k_2}(\Omega), \qquad k_2 > 3/2$$

and that Ω *is a Lipschitz domain if* $k_1 \leq 3/2$. *Let* u_p^h *be the finite element solution of* (3.1)-(3.2) *as defined in Section* 3.2, *then*

$$\|u - u_p^h\|_{1,\Omega} \leq C(k)\frac{h^{\mu-1}}{p^{k-1}} \|u\|_{k,\Omega} \tag{4.22a}$$

$$k = \min(k_1, k_2) \tag{4.22b}$$

$$\mu = \min(p+1, k) \tag{4.22c}$$

where C *is independent of* u, h, p *but depends on* Ω, τ, σ.

Proof: If $g = 0$ then (4.22) follows immediately from Theorem 4.6 and (4.21).

If $g \neq 0$, then denote by U_p^h the exact solution of the problem (3.1)-(3.2) when replacing g by g_p^h. Denoting $\omega = u - U_p^h$ we see that

$$-\Delta\omega + \omega = 0$$
$$\frac{\partial\omega}{\partial n} = 0 \quad \text{on} \quad \Gamma^2$$
$$\omega = g - g_p^h \quad \text{on} \quad \Gamma^1 .$$

By Lemma 4.3 we have

$$_0\|\omega\|_{1/2,\Gamma^1} \leq C \frac{h^{\nu - 1/2}}{p^{r-1/2}} \|u\|_{r,\Gamma^1} \quad \text{where} \quad \nu = \min(r, p+1)$$

and $r = k - 1/2$ by the imbedding theorem. Because

$$\|\omega\|_{1,\Omega} = \inf \|v\|_{1,\Omega}$$

over all $v \in H^1(\Omega)$ such that $v = \omega$ on Γ^1, we have

$$\|\omega\|_{1,\Omega} \leq C\ {}_0\|\omega\|_{1/2,\Gamma^1} \leq C \frac{h^{\mu-1}}{p^{k-1}} \|u\|_{k,\Omega} .$$

By Theorem 4.6 and the basic properties of the finite element method we get for any $\phi_p^h \in \mathcal{V}_p^h(\Omega)$,

$$\begin{aligned}\|u_p^h - U_p^h\|_{1,\Omega} &\leq C \|\varphi_p^h - U_p^h\|_{1,\Omega} \\ &\leq C(\|u - \varphi_p^h\|_{1,\Omega} + \|u - U_p^h\|_{1,\Omega}) \\ &\leq C \frac{h^{\mu-1}}{p^{k-1}} \|u\|_{k,\Omega}\end{aligned}$$

and Theorem 4.8 is proven.

4.4. Optimality of the asymptotic rate of convergence

In this section we will prove that the estimate in Theorem 4.8 is optimal. To do so we will use the concept of the n-width. For details, see e.g. [18]. Denote

$$D_n(H^1(\Omega), H^k(\Omega)) = \inf_{\substack{S_n \subset H^1(\Omega) \\ \dim S_n = n}} \ \sup_{\substack{u \in H^k \\ \|u\|_{k,\Omega} = 1}} \ \inf_{v \in S_n} \|u - v\|_{1,\Omega}$$

the n-width in the sense of Kolmogrov. Then by Theorem 2.5.1 and 2.5.2 of [1] we have

$$D_n(H^1(\Omega), H^k(\Omega)) \geq C n^{-(k-1)/2} . \tag{4.23}$$

Let us now compute the dimension of the space $\mathcal{V}_p^h(\Omega)$ in terms of p and h. The number of elements is of order $O\left(\frac{1}{h^2}\right)$. Over each element we have $O(p^2)$ polynomial basis functions. Hence, $n = \dim \mathcal{V}_p^h(\Omega) \leqslant C \frac{p^2}{h^2}$. Hence for $p + 1 \geqslant k$ we have

$$(4.24)\quad \|u - u_p^h\|_{1,\Omega} \leqslant C(k)\left(\frac{h}{p}\right)^{k-1} \|u\|_{k,\Omega} \leqslant C(k)\, n^{-\left(\frac{k-1}{2}\right)} \|u\|_{k,\Omega}.$$

Comparing (4.24) with (4.23) we see that the estimate is optimal.

5. THE CONVERGENCE RATE OF THE *h-p* VERSION. THE CASE OF THE SINGULAR SOLUTION

In Section 4 we analyzed the rate of the *h-p* version when the solution of (3.1)-(3.2) has the form (3.4) with $u_3 = 0$. Now we will analyze the rate of convergence in the case $u = u_3$. For simplicity and without a loss of generality we will assume that $n = 1$ in (3.4*c*).

5.1. An approximation result

Consider the square $R = R(h)$ defined in Section 2. Let (r, θ) denote the polar coordinates with the origin at 0 (see *fig.* 5.1). For $\kappa > 1$ let S_κ be the subset of R bounded by the lines $L_\kappa^1 : x_2 + h = \kappa(x_1 + h)$ and

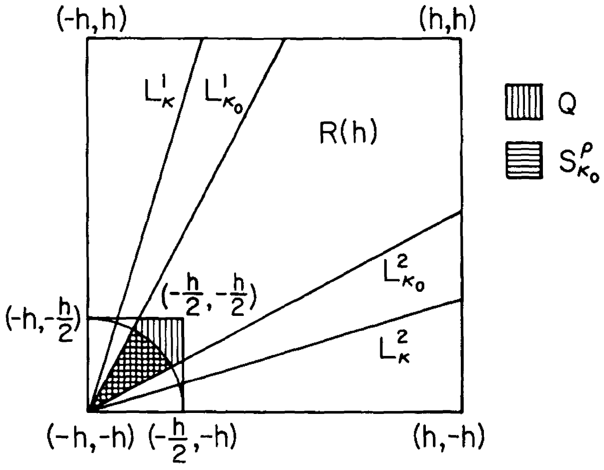

Figure 5.1. — Scheme of $R(h)$, Q, $S^p_{\kappa_0}$.

$L_\kappa^2 : x_1 + h = \kappa(x_2 + h)$. Let S_κ^ρ be the region $S_\kappa \cap \{(r, \theta) | r < \rho\}$ $\left(0 < \rho < \frac{h}{2}\right)$.

We will consider the approximation of a function u with support in $S_{\kappa_0}^\rho$ for some $\kappa_0 > \kappa$ which vanishes on the lines L_κ^1, L_κ^2. We will assume that the function u has the form

$$u(r, \theta) = r^\alpha |\log r|^\gamma \chi_0\left(\frac{r}{h}\right) \Phi(\theta) \tag{5.1}$$

where Φ and χ_0 are sufficiently smooth functions (e.g. C^∞ functions) such that $0 \leq \chi_0 \leq 1$, $\chi_0(r) = 1$ for $0 \leq r \leq \frac{\bar\rho}{3}$, $\chi_0(r) = 0$ for $r \geq \frac{2\bar\rho}{3}$, $0 < \bar\rho < 1/2$ and $\Phi(\theta_1) = \Phi(\theta_2) = 0$ where θ_1, θ_2 are polar coordinates of the lines L_κ^1 and L_κ^2.

Let Q be the region bounded by the lines $L_{\kappa_0}^1$, $L_{\kappa_0}^2$, and $x_1 = -\frac{h}{2}$, $x_2 = -\frac{h}{2}$. We will estimate the approximation error $\|u - u_p^h\|_{1,Q}$, $u_p^h \in \mathscr{P}_p^1(R)$.

We first map $R = R(h)$ onto the square $\hat R = R(1)$ by the transformation $\hat x_i = \frac{x_i}{h}$ or equivalently $(\hat r, \hat\theta) = \left(\frac{r}{h}, \theta\right)$. This maps Q into $\hat Q$. Then, if $\hat u(\hat r, \hat\theta) = u(r, \theta)$ we have

$$\hat u(\hat r, \hat\theta) = h^\alpha \hat r^\alpha |\log h\hat r|^\gamma \chi_0(\hat r) \Phi(\hat\theta) \tag{5.2}$$

where $\hat u = 0$ on the lines $\tilde L_\kappa^1$ and $\tilde L_\kappa^2$, the maps of L_κ^1 and L_κ^2. Since γ is by assumption a positive integer, we have for $h, \hat r < 1$

$$\hat u(\hat r, \hat\theta) = \sum_{l=0}^{\gamma} C(l) h^\alpha \hat r^\alpha |\log h|^l |\log \hat r|^{\gamma - l} \chi_0(\hat r) \Phi(\theta) = \sum_{l=0}^{\gamma} \hat u_l . \tag{5.3}$$

By Theorem 5.1 of [5] there exists $\hat z_p^l \in \mathscr{P}_{p+2}^2(\hat R)$ such that $\hat z_p^l = 0$ on the lines $\hat L_\kappa^1$ and $\hat L_\kappa^2$ and

$$\|\hat u_l - \hat z_p^l\|_{1,\hat Q} \leq C \frac{h^\alpha}{p^{2\alpha}} |\log h|^l \, |\log p|^{\gamma - l} .$$

Hence, we see that

$$\hat z_p = \sum_{l=0}^{\gamma} \hat z_p^l \in \mathscr{P}_{p+2}^2(\hat R) ,$$

$\hat{z}_p = 0$ on $\hat{L}_\kappa^1$ and $\hat{L}_\kappa^2$ and

$$\|\hat{u} - \hat{z}_p\|_{1,Q} \leq C \frac{h^\alpha}{p^{2\alpha}} \sum_{l=0}^{\gamma} |\log h|^l \, |\log p|^{\gamma - l} \leq C \frac{h^\alpha}{p^{2\alpha}} \max(|\log h|^\gamma, |\log p|^\gamma). \tag{5.4}$$

By suitably changing the constant in (5.4), we see that we may obtain a $\hat{z}_p \in \mathscr{P}_p^1(\hat{R})$ satisfying (5.4). By Lemma 4.2 the same estimate holds for $\|u - z_p\|_{1,Q}$ so that we have

LEMMA 5.1 : *Let u be given by* (5.1). *Then there exists* $z_p \in \mathscr{P}_p^1(R)$ *such that* $z_p = 0$ *on the lines* L_κ^1 *and* L_κ^2 *and*

$$\|u - z_p\|_{1,Q} \leq C g(h, p, \gamma) \frac{h^\alpha}{p^{2\alpha}} \tag{5.5a}$$

where

$$g(h, p, \gamma) = \max(|\log h|^\gamma, |\log p|^\gamma) \tag{5.5b}$$

and C is a constant independent of p and h.

5.2. The rate of convergence of the *h-p* version

We now return to the problem of approximating function u_3 given in (3.4*d*). To this end let

$$u_3 = u_{3,1} + u_{3,2}$$

where

$$u_{3,1} = u_3 \, \chi_0\left(\frac{r}{h}\right) \tag{5.6a}$$

$$u_{3,2} = u_3\left(1 - \chi_0\left(\frac{r}{h}\right)\right). \tag{5.6b}$$

Obviously $u_{3,2} = 0$ in the neighborhood of the origin.

Our first goal is to approximate $u_{3,1}$ over the set of triangles or parallelograms having a vertex at the origin as shown in figure 5.2.

We will assume that $OB_1 \subset \Gamma^1$ and $OB_{m+1} \subset \Gamma^2$. Let $\tilde{\Gamma} = \bigcup_{i=0}^{m} B_i B_{i+1}$.

Then Lemma 5.1 yields the following result, the proof of which may be found in [5].

LEMMA 5.2 : *Let u be given by* (5.6*a*) *with* $\bar{\mathfrak{p}}$ *(in the definition of*

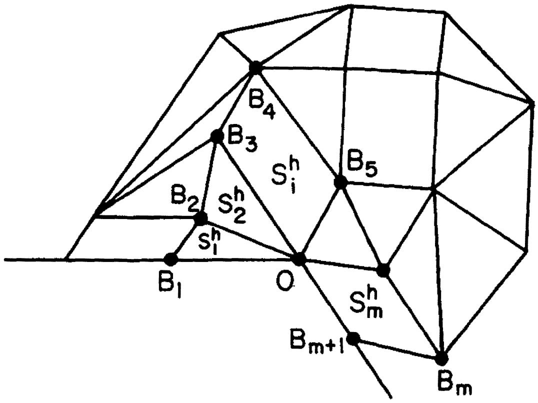

Figure 5.2. — Scheme of the mesh in the neighborhood of the singularity.

χ_0) *sufficiently small (depending on* τ *and* σ *only), then there exists* $z_p \in H^1(\Omega)$, $z_p \in \mathscr{P}_p(S_i^h)$, $z_p = 0$ *on* OB_1 *and on* $\tilde{\Gamma}$ *such that*

$$\|u - z_p\|_{H^1(\Omega)} \leq C g(h, p, \gamma) \frac{h^\alpha}{p^{2\alpha}} \tag{5.7a}$$

$$g(h, p, \gamma) = \max\left(|\log h|^\gamma, |\log p|^\gamma\right) \tag{5.7b}$$

where C *depends on* σ, τ *but is independent of* p *and* h.

Let us consider now the function $u = u_{3,2}$ given by (5.6b). We have $u = 0$ for $r \leq \bar{\rho} h$. Further,

$$|D^\beta u| \leq C(\beta)\, r^{\alpha - |\beta|}\, |\log r|^\gamma$$

where $\beta = (\beta_1, \beta_2)$, $\beta_i \geq 0$, $\beta_1 + \beta_2 = |\beta|$ and

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1}\, \partial x_2^{\beta_2}}.$$

Hence we have

$$\|u\|_{k,\Omega} \leq C(k) |\log h|^\gamma \max\left(1, h^{\alpha + 1 - k}\right). \tag{5.8}$$

Denoting by u_p^h the finite element approximation of u, we get by Theorem 4.8

$$\begin{aligned}(5.9)\qquad \|u-u_p^h\|_{1,\Omega} &\leq C(k)\frac{h^{\eta-1}}{p^{k-1}}\|u\|_{k,\Omega}\\ &\leq C(k)\frac{h^{\eta-k+\alpha}}{p^{k-1}}|\log h|^{\gamma}\end{aligned}$$

with $k>1$ arbitrary and $\eta=\min(p+1,k)$. Let us take $k=2\alpha+1$ in (5.9). Then $\eta-k+\alpha=\eta-\alpha-1=\min(\alpha,p-\alpha)$ so that

$$(5.10)\qquad \|u-u_p^h\|_{1,\Omega}\leq C\frac{h^{\min(\alpha,p-\alpha)}}{p^{2\alpha}}|\log h|^{\gamma}.$$

If p is small with respect to α, we can select k so that $C(k)h^{\eta-k+\alpha}/p^{k-1}$ will be minimal. For example, with $k=2$ we get

$$(5.11)\qquad \|u-u_p^h\|_{1,\Omega}\leq Ch^{\alpha}|\log h|^{\gamma}.$$

Combining the estimates for $u_{3,1}$ and $u_{3,2}$ we get :

THEOREM 5.3 : *Let u be given by* (3.4*d*). *Then there exists* $\varphi_p^h\in\mathring{\mathscr{V}}_p^h(\Omega)$ *such that*

$$(5.12a)\qquad \|u-\phi_p^h\|_{1,\Omega}\leq Cg(h,p,\gamma)\min\left(h^{\alpha},\frac{h^{\min(\alpha,p-\alpha)}}{p^{2\alpha}}\right)$$

$$(5.12b)\qquad g(h,p,\gamma)=\max(|\log h|^{\gamma},|\log p|^{\gamma})$$

and C depends on σ, τ *but is independent of p and h.*

Remark 1 : When α is an integer and $\gamma=0$, the estimate (5.12*a*) is very pessimistic, since the solution u given by (3.4*d*) is smooth. When α is an integer and $\gamma>0$, then the estimate (5.12*a*) is a correct one.

Let us now summarize in one theorem the error estimate for the *h-p* version with quasiuniform mesh and uniform p.

THEOREM 5.4 : *Let* Ω *be a polygonal domain as introduced in Section* 2. *Suppose that u, the solution of* (3.1)-(3.2) *can be written in the form* (3.4). *Assume further if* $1<k_1\leq 3/2$ *that* Ω *is a Lipschitz domain. Assume that* u_p^h *is the finite element solution with triangular and parallelogram elements satisfying* (3.6) *and the boundary condition on* Γ^1 *defined by* (3.7). *Then*

$$(5.13a)\qquad \|u-u_p^h\|_{1,\Omega}\leq C\max_i(\xi_1^i,\xi_2)R$$

$$(5.13b)\qquad \xi_1^i=g(h,p,\gamma_i)\min\left(h^{\alpha_i},\frac{h^{\min(\alpha_i,p-\alpha_i)}}{p^{2\alpha_i}}\right)$$

(5.13*c*) $$g(h,p,\gamma_i) = \max\left(|\log h|^{\gamma_i}, |\log p|^{\gamma_i}\right)$$

(5.13*d*) $$\xi_2 = \frac{h^{\min(k_1-1,\,k_2-1,\,p)}}{p^{\min(k_1-1,\,k_2-1)}}$$

(5.13*e*) $$R = \|u_1\|_{k_1,\Omega} + \|u_2\|_{k_2,\Omega} + \sum_i |a_i|$$

and C depends on τ, σ *in* (3.5), Ω, k_i, γ_i, α_i *but is independent of* $\mathcal{T}^h$, h, p, u.

Remark 2 : We formulated Theorem 5.4 only in the frame of Sobolev spaces. By interpolation arguments, it is also possible to formulate the theorem in the frame of Besov spaces.

Remark 3 : We addressed only the case of the polygonal domain and elements which are triangles or parallelograms. By the standard mapping approach, the results are also valid for curvilinear elements.

6. APPLICATIONS

In this section we will study the consequences of Theorem 5.4 in connection with computations.

First let us mention that although we discussed the *h-p* version in connection with the problem (3.1)-(3.2), all conclusions are valid also for the elasticity problem. In (3.4*d*) we assumed that α_i are real. In the case of the elasticity problem, α_i are in general complex with $\operatorname{Re}\alpha_i > 0$. The estimate (5.13) is still valid with $\alpha_i = \operatorname{Re}\alpha_i$.

Our theory is of asymptotic character. Hence it is important to see the applicability of Theorem 5.4 in the range of practical parameters. To this end let us consider the plane strain elasticity problem when Ω is an L-shaped domain shown in figure 6.1.

Let us assume that on $\partial\Omega$ tractions are prescribed, i.e. $\Gamma^1 = \emptyset$. The solution of this problem is the displacement vector (u_1, u_2) where

(6.1*a*) $$u_1 = \frac{1}{2G} r^\alpha[(\kappa - Q(\alpha+1))\cos\alpha\theta - \alpha\cos(\alpha-2)\theta]$$

(6.1*b*) $$u_1 = \frac{1}{2G} r^\alpha[(\kappa + Q(\alpha+1))\sin\alpha\theta - \alpha\cos(\alpha-2)\theta]$$

where

$$\alpha = 0.544\,483\,737$$
$$Q = 0.543\,075\,579$$

G is the modulus of rigidity and $\kappa = 3 - 4\nu$ where ν is Poisson's ratio which we assume to be $\nu = 0.3$. The solution has a typical singularity at O. The

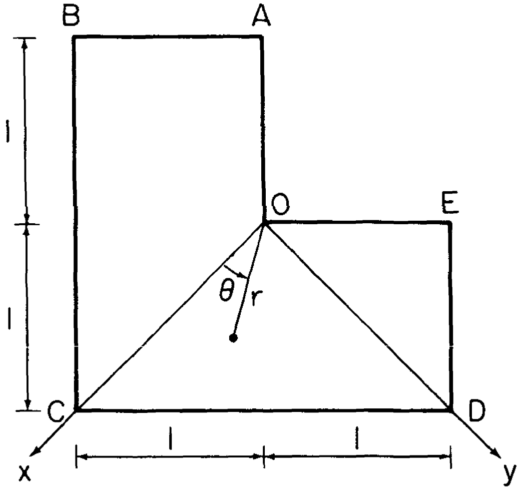

Figure 6.1. — *L*-shaped plane elastic body.

sides OA and OE are traction free. Instead of the norm $\| \cdot \|_{1,\Omega}$ we will be interested in the energy norm $\| \cdot \|_E$ which is equivalent to the $\| \cdot \|_{1,\Omega}$ norm. Denoting $W(u)$, respectively $W(u_p^h)$, to be the strain energy of the exact, respectively the finite element solution, we have

$$\|u - u_p^h\|_E = (W(u) - W(u_p^h))^{1/2} \tag{6.2}$$

and we define the relative error in the energy norm as

$$\|e\|_{E,R} = \left[\frac{W(u) - W(u_p^h)}{W(u)} \right]^{1/2} . \tag{6.3}$$

In the next figures we will present the results of computations which were performed with a computer program called PROBE [21, 23] developed by Noetic Technologies Corporation, St Louis.

We will consider a uniform mesh with square elements as shown in figure 6.2.

The solution $u \in H^{1+\alpha-\varepsilon}(\Omega)$, $\varepsilon > 0$ arbitrary.

Theorem 5.4 gives for $p \geq 1$ the estimate :

$$\|u - u_p^h\|_E \leq C \min \left[h^\alpha, \frac{h^{\min(\alpha, p-\alpha)}}{p^{2\alpha}} \right] \tag{6.4}$$

where C depends on α but is independent of h and p. Figure 6.3 shows the relative error in the energy norm $\|e\|_{E,R}$ (for different degrees p) in dependence on h. We also show the slope h^{α} in the figure. We see that with respect to h the error is in the asymptotic range also for moderate p and h.

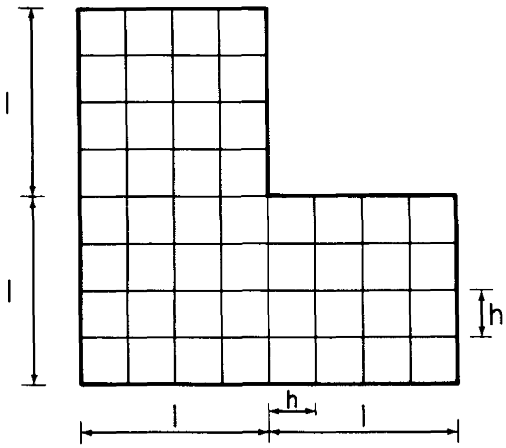

Figure 6.2. — The scheme of the uniform mesh.

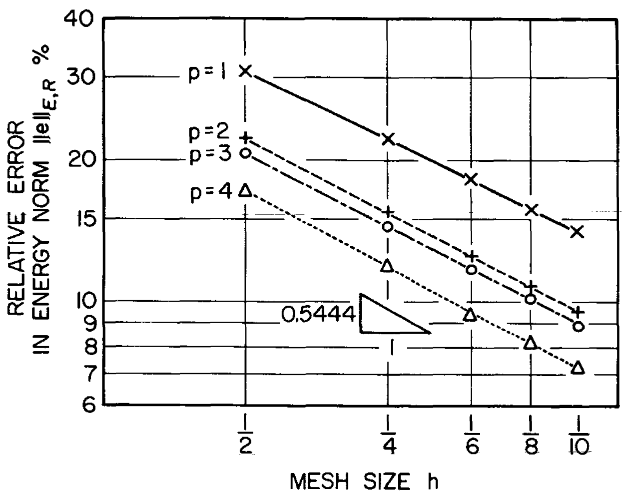

Figure 6.3. — The relative error in the energy norm in dependence on *h*.

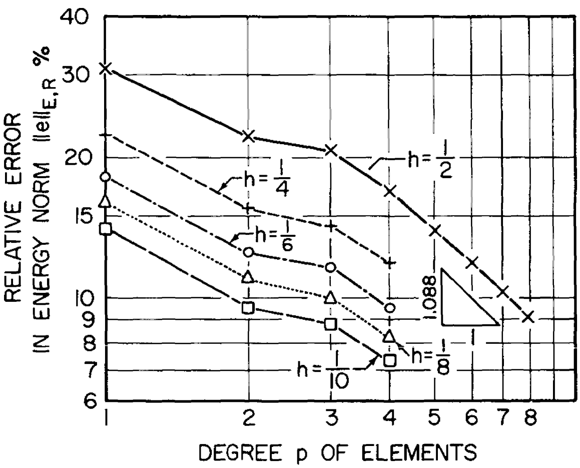

Figure 6.4. — The relative error in the energy norm in dependence on *p*.

Figure 6.4 shows the error in dependence on p and different h. Because of the size of computations, only in the case $h = 1/2$ is the error given for $p > 4$. (For $p = 4$ and $h = 1/10$, the number of degrees of freedom $N = 5\,119$). Estimate 6.4 gives the rate $p^{-2\alpha}$ which appears only for $p > 3$. For large p and small h we have $N = p^2/h^2$ and hence

$$\|u - u_p^h\|_E \leq C \frac{N^{-\frac{\alpha}{2}}}{p^\alpha}. \tag{6.5}$$

(6.5) shows that if the measure of computational work is N, then the use of higher p is preferable.

Figure 6.5 shows the dependence of the relative error in the energy norm on the number of degrees of freedom N for various p. In addition, the performance of the p-version for $h = 1/2$ is shown in the figure. We see that $p = 2$ is more effective than $p = 3$, and asymptotically for $p \to \infty$, the higher p are more effective as follows from (6.4). The p-version has a rate which is twice that of the h-version (see also [5]).

We addressed in this paper only the case of the quasiuniform mesh. If the mesh is strongly refined, then its performance is different. Figure 6.6 shows the strongly refined mesh with n layers $(n = 2)$. The mesh is a geometric one with the ratio 0.15. The ratio 0.15 leads to nearly optimal convergence. See [13, 14].

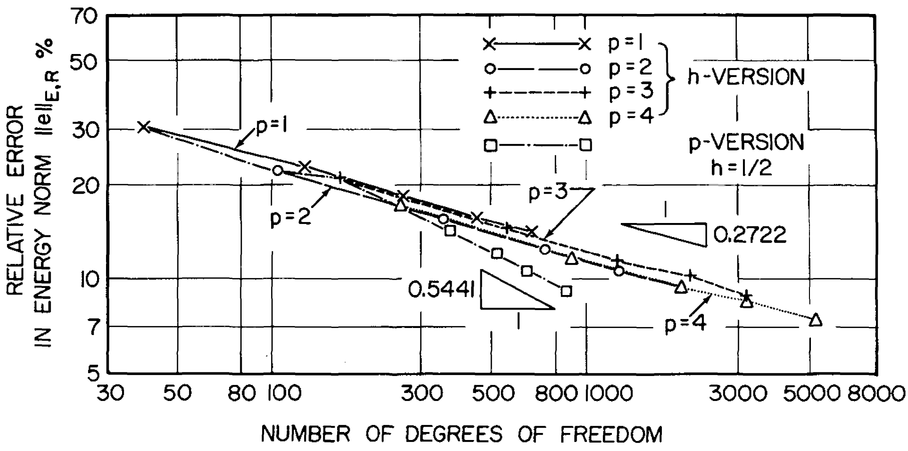

Figure 6.5. — The relative error in the energy norm in dependence on *N*.

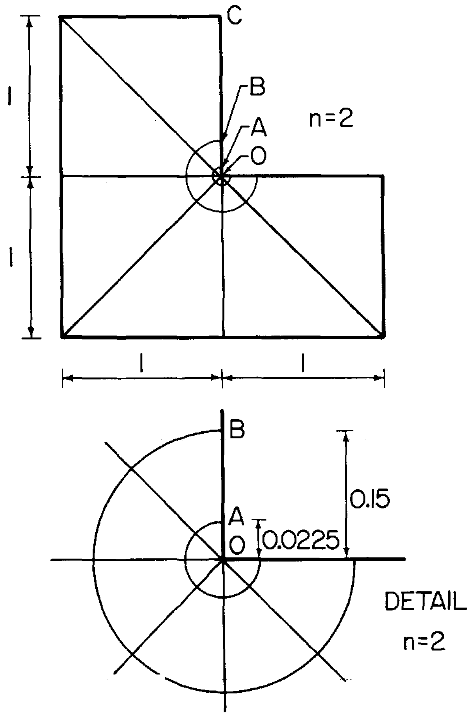

Figure 6.6. — The strongly refined mesh with $n = 2$ layers.

Figure 6.7 compares the performance of the h, p versions for the uniform and strongly refined mesh for our example. The performance of the p-version on strongly refined meshes is in practice very similar to the general h-p version, leading to an exponential rate of convergence. We see that the p-version performance depends very strongly on the mesh.

For more about the comparison between the h, p and h-p version we refer to [3].

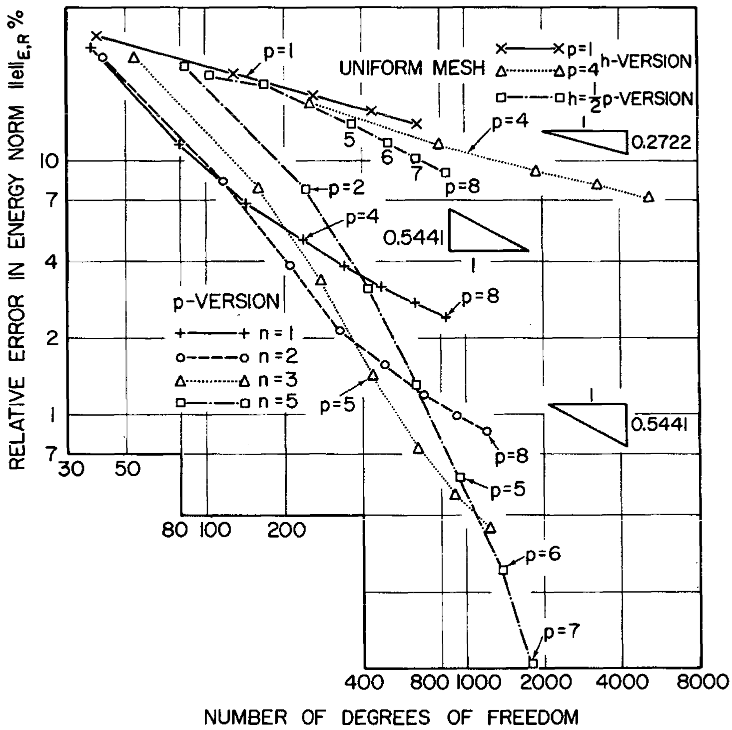

Figure 6.7. — The error in the energy norm in dependence on N for various meshes.

7. APPENDIX

Theorems 7.4 and 7.5 proven in this section are slightly generalized forms of Lemma 4.7 and are of interest by themselves.

Let us consider the equilateral triangle $T = ABC$ as shown in figure 7.1.

We denote

$$\gamma_1 = \gamma_1^A \cup \gamma_1^B = \overline{AP_1} \cup \overline{P_1B} = \overline{AB} ,$$
$$\gamma_2 = \gamma_2^A \cup \gamma_2^C = \overline{AP_2} \cup \overline{P_2C} = \overline{AC} ,$$
$$\gamma_3 = \gamma_3^B \cup \gamma_3^C = \overline{BP_3} \cup \overline{P_3C} = \overline{BC} .$$

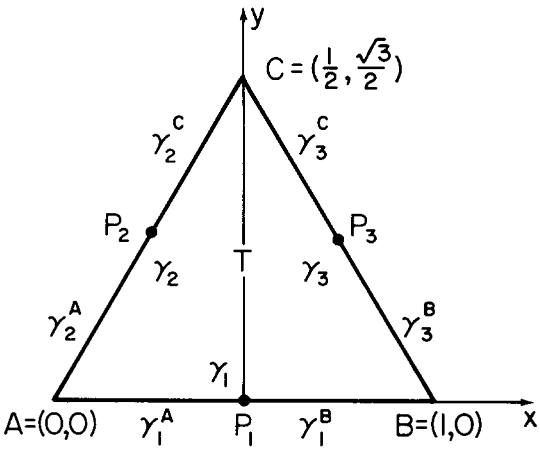

Figure 7.1. — The scheme of the equilateral triangle.

The notation is also shown in figure 7.1. Let $f \in \mathscr{P}_p(\gamma_1)$. Then we define

(7.1) $$F_1^{[f]}(x, y) = \frac{\sqrt{3}}{2\,y} \int_{x - \frac{y}{\sqrt{3}}}^{x + \frac{y}{\sqrt{3}}} f(t)\, dt\,.$$

The value of F_1 at a point $P \in T$ depends only on the values f along the segment $\overline{Q_1 Q_2}$, $Q_1 = \left(x - \frac{y}{\sqrt{3}}, 0\right)$, $Q_2 = \left(x + \frac{y}{\sqrt{3}}, 0\right)$. We prove now the following lemma.

LEMMA 7.1 : *Let $f \in \mathscr{P}_p(\gamma_1)$ with $f(A) = f(B) = 0$ and let $F_1^{[f]}(x, y)$ be defined by* (7.1). *Then*

(7.2)

$$\begin{aligned}
&a)\ \ F_1^{[f]}(x, y) \in \mathscr{P}_p^1(T)\\
&b)\ \ F_1^{[f]}(x, 0) = f(x)\\
&c)\ \ \|F_1^{[f]}\|_{1,T} \leqslant C_0 \|f\|_{1/2,\gamma_1}\\
&d_1)\ \ {}_A\|F_1^{[f]}\|_{k,\gamma_2^A} \leqslant C\ {}_A\|f\|_{k,\gamma_1^A} \quad 0 \leqslant k \leqslant 1\\
&d_2)\ \ {}_B\|F_1^{[f]}\|_{k,\gamma_3^B} \leqslant C\ {}_B\|f\|_{k,\gamma_1^B} \quad 0 \leqslant k \leqslant 1\\
&d_3)\ \ \|F_1^{[f]}\|_{k,\gamma_2^C} \leqslant C\,\|f\|_{0,\gamma_1} \quad 0 \leqslant k \leqslant 1\\
&d_4)\ \ \|F_1^{[f]}\|_{k,\gamma_3^C} \leqslant C\,\|f\|_{0,\gamma_1} \quad 0 \leqslant k \leqslant 1
\end{aligned}$$

where the constant C is independent of p and f.

Proof: It is immediate that (7.2*b*) holds. Let $f = x^n$ with $0 \leqslant n \leqslant p$ integer. Then

$$\begin{aligned} F(x, y) &= \frac{\sqrt{3}}{2y} \int_{x - \frac{y}{\sqrt{3}}}^{x + \frac{y}{\sqrt{3}}} t^n \, dt \\ &= \frac{\sqrt{3}}{2y(n+1)} \left[\left(x + \frac{y}{\sqrt{3}} \right)^{n+1} - \left(x - \frac{y}{\sqrt{3}} \right)^{n+1} \right] \\ &= \frac{\sqrt{3}}{2y(n+1)} \left[\left(x + \frac{y}{\sqrt{3}} \right) - \left(x - \frac{y}{\sqrt{3}} \right) \right] P_n(x, y) \\ &= \frac{1}{(n+1)} P_n(x, y) \in \mathcal{P}_p^1(T). \end{aligned}$$

Hence (7.2*a*) holds.

To prove (7.2*c*) we first extend f by zero to a function defined on the entire x-axis $\mathbb{R}$ so that (see [18])

$$\| f \|_{1/2, \mathbb{R}} \leqslant C_0 \| f \|_{1/2, \gamma_1} \tag{7.3}$$

where we have used the same notation f to denote the extended function as well. Then by (7.1) $F_1(x, y)$ is well defined on the entire half plane $\Omega = \{(x, y) \mid y > 0\}$. For $(x, y) \in \Omega$ we have

$$F_1(x, y) = \int_{-\infty}^{+\infty} f(t) H(x - t, y) \, dt = (f * H(., y))(x) \tag{7.4}$$

where

$$H(x, y) = \frac{\sqrt{3}}{2y}, \quad -\frac{y}{\sqrt{3}} \leqslant x \leqslant \frac{y}{\sqrt{3}} \tag{7.5}$$
$$= 0 \text{ otherwise.}$$

Let $\tilde{g}(\xi)$ represent the Fourier transform of the function $g(x)$ in the x direction. Then by (7.4)

$$\tilde{F}_1(\xi, y) = \tilde{f}(\xi) \tilde{H}(\xi, y) \tag{7.6}$$

where

$$\tilde{H}(\xi, y) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{2y} \int_{-y/\sqrt{3}}^{y/\sqrt{3}} e^{-i\xi x} \, dx = \frac{1}{\sqrt{2\pi}} \frac{\sin(\xi y/\sqrt{3})}{\xi y/\sqrt{3}}. \tag{7.7}$$

Let $\tilde{\Omega} = \{(\xi, y) \mid y > 0\}$ and calculate the $H^1(\Omega)$ norm of $F_1(x, y)$. By Parseval's equality, we have using (7.6)

$$\| F_1 \|_{H^1(\Omega)}^2 = \| \tilde{F}_1 \|_{H^1(\tilde{\Omega})}^2 = \iint_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\xi \tilde{H}(\xi, y)|^2 \, d\xi \, dy + \\ + \iint_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 \left| \frac{\partial}{\partial y} \tilde{H}(\xi, y) \right|^2 d\xi \, dy + \iint_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\tilde{H}(\xi, y)|^2 d\xi \, dy .$$

Now letting $z = y\xi/\sqrt{3}$ we get, by (7.7),

$$\int_0^\infty |\tilde{H}(\xi, y)|^2 \, dy = \frac{1}{2\pi} \int_0^\infty \sqrt{3} \, \frac{\sin^2 z}{z^2} \frac{dz}{|\xi|} \leq \frac{C}{|\xi|} . \tag{7.8}$$

Hence

$$\iint_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\xi \tilde{H}(\xi, y)|^2 \, d\xi \, dy \leq \tag{7.9}$$

$$\leq C \int_{-\infty}^{\infty} |\xi| \, |\tilde{f}(\xi)|^2 \, d\xi \leq C \, \|f\|^2_{1/2, \mathbb{R}} \leq C \, {}_0\|f\|^2_{1/2, \gamma_1} .$$

Also

$$\frac{\partial}{\partial y} \tilde{H}(\xi, y) = \frac{\xi}{\sqrt{3}} \left[\frac{\cos z}{z} - \frac{\sin z}{z^2} \right]$$

which is bounded at $z = 0$. Hence

$$\int_0^\infty \left| \frac{\partial}{\partial y} \tilde{H}(\xi, y) \right|^2 dy \leq C \, |\xi| ,$$

so that

$$\iint_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 \left| \frac{\partial}{\partial y} \tilde{H}(\xi, y) \right|^2 d\xi \, dy \leq \tag{7.10}$$

$$\leq C \int_{-\infty}^{\infty} |\xi| \, |\tilde{f}(\xi)|^2 \, d\xi \leq C \, {}_0\|f\|^2_{1/2, \gamma_1} .$$

The third term can be bounded analogously. Using (7.8)-(7.10), (7.2c) follows. Inequalities (7.2d_3), (7.2d_4) follow immediately for $k = 0$, $k = 1$ and hence by an interpolation argument (see [8]) they hold for all $0 \leq k \leq 1$.

We prove now (7.2d_1). Let the variable x be used to represent both the distance from A along γ_1 and the distance from A along γ_2. Denoting

$$G(x) = \frac{1}{x} \int_0^x f(t) \, dt \tag{7.11}$$

it is readily seen that

$${}_A\|F_1^{[f]}\|_{k, \gamma_2^A} = {}_A\|G(x)\|_{k, I} \qquad I = (0, 1/2) . \tag{7.12}$$

Using (9.9.1) of [15], p. 244 we get

$$\|G(x)\|_{0, I} \leq C \, \|f\|_{0, \gamma_1^A} . \tag{7.13}$$

Further, integrating (7.11) by parts we have

$$G(x) = f(x) - f(0) - \frac{1}{x}\int_0^x t f'(t)\,dt \tag{7.14}$$

and hence

$$\begin{aligned} G'(x) &= f'(x) + \frac{1}{x^2}\int_0^x t f'(t)\,dt - f'(x) \\ &= -\frac{1}{x^2}\int_0^x (x-t) f'(t)\,dt + \frac{1}{x}\int_0^x f'(t)\,dt\,. \end{aligned}$$

Using 9.9.5 of [15], p. 245 with $r = 2$ we get

$$\left\| \frac{1}{x^2}\int_0^x (x-t) f'(t)\,dt \right\|_{0,I} \leq C \|f'\|_{0,I}$$

and by 9.9.1 of [15], p. 244 we get

$$\left\| \frac{1}{x}\int_0^x f'(t)\,dt \right\|_{0,I} \leq C \|f'\|_{0,I}\,.$$

Hence

$$\|G'(x)\|_{0,I} \leq C \|f'\|_{0,I}\,. \tag{7.15}$$

Combining (7.13) and (7.15) we get $(7.2d_1)$ for $k = 0$ and $k = 1$ and hence by the interpolation argument $(7.2d_1)$ holds for all $0 \leq k \leq 1$. The inequality $(7.2d_2)$ is essentially the same as $(7.2d_1)$ and Lemma 7.1 is completely proven.

Let now $f = f_i \in \mathscr{P}_p(\gamma_i)$, $i = 1, 2, 3$. Then we denote by $F_i^{[f_i]}(x, y)$ the polynomial extension of f_i into T, defined for $i = 1$ by (7.1) and for $i = 2, 3$ by (7.1) after properly rotating the coordinates. Obviously Lemma (7.1) is applicable for $i = 1, 2, 3$ when properly interpreted through the rotation of the coordinates.

We now prove

LEMMA 7.2 : *Let T be the triangle as in figure 7.1 and f satisfy $f(A) = f(B) = f(C) = 0$ and $f_i = f|_{\gamma_i} \in \mathscr{P}_p(\gamma_i)$, $i = 1, 2, 3$ where by $f|_{\gamma_i}$ we denote the restriction of f on γ_i. Then there exists $\Phi_i \in \mathscr{P}_p(\gamma_i)$, $i = 1, 2$ such that*

(7.16) *a)* $U = F_1^{[\Phi_1]} + F_2^{[\Phi_2]} \in \mathscr{P}_p(T)$

b) $U = f_i$ *on* γ_i , $\quad i = 1, 2$

c) $\|U\|_{1,T} \leq C\,[{}_0\|f_1\|_{1/2,\gamma_1} + {}_0\|f_2\|_{1/2,\gamma_2}]$

d_1) $${}_A\|\Phi_i\|_{k,\gamma_i} \leq C\left[\sum_{j=1}^{2} {}_0\|f_j\|_{k,\gamma_j}\right], \qquad i = 1, 2, \quad 0 \leq k \leq 1$$

d_2) $${}_B\|\Phi_1\|_{k,\gamma_1^B} \leq C\left[{}_B\|f_1\|_{k,\gamma_1^B} + \sum_{j=1}^{2} \|f_j\|_{0,\gamma_j}\right],$$

d_3) $${}_C\|\Phi_2\|_{k,\gamma_2^C} \leq C\left[{}_C\|f_2\|_{k,\gamma_2^C} + \sum_{j=1}^{2} \|f_j\|_{0,\gamma_j}\right], \qquad 0 \leq k \leq 1$$

where C is a constant independent of p and f.

Proof: Let $\Phi_i \in \mathcal{P}_p(\gamma_i)$. Then as in Lemma 1 we define

$$G_i(x) = \frac{1}{x}\int_0^x \Phi_i(t)\,dt\,, \qquad i = 1, 2\,. \tag{7.17}$$

Condition (7.16*b*) will be satisfied if

$$\Phi_1(x) + G_2(x) = \Phi_1(x) + \frac{1}{x}\int_0^x \Phi_2(t)\,dt = f_1(x) \tag{7.18a}$$

$$\Phi_2(x) + G_1(x) = \Phi_2(x) + \frac{1}{x}\int_0^x \Phi_1(t)\,dt = f_2(x) \tag{7.18b}$$

hold for all $x \in \mathfrak{I} = (0, 1)$. Since $f_i \in \mathcal{P}_p(\mathfrak{I})$ it is easy to see that $\Phi_i \in \mathcal{P}_p(\mathfrak{I})$ satisfying (7.18) exist. Due to the assumption on f we have $f_1(0) = f_2(0) = 0$. Φ_i are uniquely determined up to a constant K with $\Phi_1(0) = K$, $\Phi_2(0) = 0 - K$.

We now define

$$\begin{aligned} \psi_1(x) &= \Phi_1(x) + \Phi_2(x)\,, \quad & \psi_2(x) &= \Phi_1(x) - \Phi_2(x) \\ h_1(x) &= f_1(x) + f_2(x)\,, \quad & h_2(x) &= f_1(x) - f_2(x) \end{aligned} \tag{7.19}$$

so that (7.18) yields

$$\psi_1(x) + \frac{1}{x}\int_0^x \psi_1(t)\,dt = h_1(x) \tag{7.20a}$$

$$\psi_2(x) - \frac{1}{x}\int_0^x \psi_2(t)\,dt = h_2(x)\,. \tag{7.20b}$$

Here $\psi_1(x)$ is unique, $\psi_1(0) = 0$, while $\psi_2(x)$ is unique up to the constant K such that $\psi_2(0) = 2K$.

We first analyze (7.20*a*). By differentiation we obtain

$$\psi_1' - \frac{1}{x^2}\int_0^x \psi_1(t)\,dt + \frac{1}{x}\psi_1 = h_1'\,.$$

Using (7.20*a*) we get

$$\psi_1' + \frac{2\,\psi_1}{x} = h_1' + \frac{h_1}{x}. \tag{7.21}$$

The homogeneous solution of (7.21) is $1/x^2$. A particular solution can be found by using the method of variation of constants. Hence, substituting $\psi_1(x) = \dfrac{T(x)}{x^2}$ into (7.21) we get

$$T'(x) = h_1' x^2 + h_1 x$$

from which

$$\psi_1(x) = \frac{1}{x^2}\int_0^x t^2 h_1'(t)\,dt + \frac{1}{x^2}\int_0^x t h_1(t)\,dt.$$

Integrating by parts we get

$$\psi_1(x) = h_1(x) - \frac{1}{x^2}\int_0^x t h_1(t)\,dt. \tag{7.22}$$

the unique solution of (7.20*a*).

We now show that

$$_A\|\psi_1\|_{k,3} \leq C\,_A\|h_1\|_{k,3}, \qquad 0 \leq k \leq 1. \tag{7.23}$$

Let

$$F(x) = \int_0^x t h_1(t)\,dt = -\int_0^x (x-t)\,h_1(t)\,dt + x\int_0^x h_1(t)\,dt.$$

Then

$$-\frac{F(x)}{x^2} = \frac{G(x)}{x^2} - Q(x)$$

where

$$G(x) = \int_0^x (x-t)\,h_1(t)\,dt$$

$$Q(x) = \frac{1}{x}\int_0^x h_1(t)\,dt.$$

Using (9.9.4) of [15], p. 245 with $r = 2$ and (9.9.1) of [15], p. 244 we obtain

$$\|F(x)/x^2\|_{0,3} \leq \|G(x)/x^2\|_{0,3} + \|Q\|_{0,3} \leq C\,\|h_1\|_{0,3}$$

which yields (7.23) for $k = 0$. Next, differentiating (7.22) we get

$$\psi_1' = h_1' + \frac{2}{x^3}\int_0^x t h_1(t)\,dt - \frac{h_1}{x} = h_1' - \frac{1}{x^3}\int_0^x t^2 h_1'(t)\,dt\,. \tag{7.24}$$

Let

$$F(x) = \int_0^x t^2 h_1'(t)\,dt = \int_0^x (x-t)^2 h_1'(t)\,dt$$
$$- x^2 \int_0^x h_1'(t)\,dt + 2x\int_0^x t h_1'(t)\,dt\,.$$

We have then

$$\frac{F(x)}{x^3} = \frac{G(x)}{x^3} - Q(x) + R(x)$$

where

$$G(x) = \int_0^x (x-t)^2 h_1'(t)\,dt$$
$$Q(x) = \frac{1}{x}\int_0^x h_1'(t)\,dt$$
$$R(x) = \frac{2}{x^2}\int_0^x t h_1'(t)\,dt\,.$$

This gives

$$\|F(x)x^3\|_{0,\mathfrak{J}} \leq \|G(x)x^3\|_{0,\mathfrak{J}} + \|Q(x)\|_{0,\mathfrak{J}} + \|R(x)\|_{0,\mathfrak{J}}\,.$$

The first two terms can be bounded once more by $\|h_1'\|_{0,\mathfrak{J}}$ using (9.9.4) of [15], p. 245 and (9.9.1), p. 244. Moreover,

$$R(x) = \frac{2}{x^2}\left[-\int_0^x (x-t)\,h_1'(t)\,dt + x\int_0^x h_1'(t)\,dt\right]$$

so that $\|R\|_{0,\mathfrak{J}}$ can also be bounded by $\|h_1'\|_{0,\mathfrak{J}}$. This yields (7.23) for $k = 1$. By the interpolation argument (see [8]) we get immediately (7.23). Let us consider now (7.20*b*). Differentiating it and using once more (7.20*b*) we get

$$\psi_2' = h_2' + \frac{h_2}{x}\,. \tag{7.25}$$

Integrating we get

$$\psi_2(x) = h_2(x) - \int_x^1 \frac{h_2(t)}{t}\,dt\,. \tag{7.26}$$

(7.26) is that solution of (7.20b) with $\psi_2(1) = h_2(1)$.

Once more we wish to show

$$(7.27) \qquad {}_A\|\psi_2\|_{k,\mathfrak{J}} \leq C\ {}_A\|h_2\|_{k,\mathfrak{J}}, \qquad 0 \leq k \leq 1 .$$

Using (7.26) and (9.9.9) from [15], p. 245 with $\alpha = 0$ we get

$$\|\psi_2\|_{0,\mathfrak{J}} \leq C\ \|h_2\|_{0,\mathfrak{J}} .$$

Since $h_2(0) = f_1(0) - f_2(0) = 0$, (7.25) yields

$$\psi_2' = h_2' + \frac{1}{x}\int_0^x h_2'(t)\,dt$$

and by (9.9.1) of [15], p. 244 we get

$${}_A\|\psi_2\|_{1,\mathfrak{J}} \leq C\,{}_A\|h_2\|_{1,\mathfrak{J}} .$$

An interpolation argument leads immediately to (7.27). Hence we have constructed solutions of (7.20a, b) such that (7.23) and (7.27) hold.

Coming back to (7.19), using $k = 1/2$ we see that for $i = 1, 2$

$${}_A\|\Phi_i\|_{1/2,\gamma_i} \leq C\,[{}_A\|f_1\|_{1/2,\gamma_1} + {}_A\|f_2\|_{1/2,\gamma_2}]$$

and applying Lemma 7.1 we get immediately (7.16c) and also (7.16d_1). Returning to (7.20) we see that with $\mathfrak{J}^* = (1/2, 1)$

$${}_B\|\psi_i\|_{k,\mathfrak{J}^*} \leq C\,[{}_B\|h_i\|_{k,\mathfrak{J}^*} + \|h_i\|_{0,\mathfrak{J}}], \qquad i = 1, 2 .$$

Hence also

$${}_B\|\Phi_i\|_{k,\mathfrak{J}^*} \leq C\left[{}_B\|f_i\|_{k,\mathfrak{J}^*} + \sum_{j=1}^{2}\|f_j\|_{0,\mathfrak{J}}\right], \qquad i = 1, 2$$

wich immediately leads to (7.16d_2), (7.16d_3).

The following lemma is taken from [6].

LEMMA 7.3 : *Let T be the triangle as before, f be continuous on ∂T, $f_2 = f_3 = 0$ and $f_1 \in \mathcal{P}_p(\gamma_1)$. Then there exists a polynomial $v \in \mathcal{P}_p^1(T)$ such that*

$$\|v\|_{1,T} \leq C\ \|f_1\|_{1,\gamma_1}$$

$$v = f_1 \quad on \quad \gamma_1$$

$$v = 0 \quad on \quad \gamma_2, \gamma_3$$

where C is a constant independent of f and p.

THEOREM 7.4 : *Let T be the equilateral triangle shown in figure 7.1 and f satisfy $f(A) = f(B) = f(C) = 0$ and $f_i = f|_{\gamma_i} \in \mathcal{P}_p(\gamma_i)$, $i = 1, 2, 3$. Then there exists $U \in \mathcal{P}_p^1(T)$ such that $U = f$ on ∂T and*

$$\|U\|_{1,T} \leq C\left[\sum_{i=1}^{3} {}_0\|f\|_{1/2,\gamma_i}\right]$$

where the constant C is independent of p and f.

Proof: Without loss of generality we can assume that $f_2 = f_3 = 0$.

Let $f_1 \neq 0$, $f_2 = 0$. By Lemma 7.2 we construct Φ_1, Φ_2 and $U = F_1^{[\Phi_1]} + F_2^{[\Phi_2]}$. Then $U \in \mathcal{P}_p(T)$, $U = f_i$ on γ_i, $i = 1, 2$ and

$$\|U\|_{1,T} \leq C\ {}_0\|f_1\|_{1/2,\gamma_1}. \tag{7.28}$$

Denote by g_3 the trace of U on γ_3. Then we have $g_3(B) = g_3(C) = 0$ and

$${}_0\|g_3\|_{1/2,\gamma_3} \leq C\ {}_0\|f_1\|_{1/2,\gamma_1} \tag{7.29}$$

by applying Lemmas 7.2 and 7.1.

Because of (7.16d_3) $\|\Phi_2\|_{1,\gamma_2^C} \leq C\ {}_0\|f_1\|_{1/2,\gamma_1}$ and hence using Lemma 7.1 we have also

$$\|g_3\|_{1,\gamma_3^C} \leq C\ {}_0\|f_1\|_{1/2,\gamma_1}. \tag{7.30}$$

Let now analogously as before

$$U_1 = F_3^{[\Phi_3^{[1]}]} + F_1^{[\Phi_1^{[1]}]},$$

so that

$$U_1 \in \mathcal{P}_p^1(T), \quad U_1 = g_3 \text{ on } \gamma_3, \quad U_1 = 0 \text{ on } \gamma_1$$

and

$$\|U_1\|_{1,T} \leq C\ {}_0\|g_3\|_{1/2,\gamma_3} \leq C\ {}_0\|f_1\|_{1/2,\gamma_1} \tag{7.31}$$

Denote by $g_2^{[1]}$ the trace of U_1 on γ_2. Then $g_2^{[1]}(A) = g_2^{[1]}(C) = 0$. Because of (7.30), applying Lemma 7.1 and Lemma 7.2 analogously as before we conclude that

$$\|g_2^{[1]}\|_{1,\gamma_2} \leq C\left[\|g_3\|_{1,\gamma_3^C} + {}_0\|g_3\|_{1/2,\gamma_3}\right] \leq C\ {}_0\|f_1\|_{1/2,\gamma_1}.$$

Now applying Lemma 7.3 there is $U_2 \in \mathcal{P}_p^1(T)$ such that

$$\|U_2\|_{1,T} \leq C\ \|g_2^{[1]}\|_{1,\gamma_2} \leq C\ {}_0\|f_1\|_{1/2,\gamma_1} \tag{7.32}$$

and

$$U_2 = g_2^{[1]} \text{ on } \gamma_2, \qquad U_2 = 0 \text{ on } \gamma_1, \gamma_3 .$$

Let now

$$V = U - U_1 + U_2 .$$

Then it is easy to see that $V \in \mathscr{P}_p^1(T)$, $V = f_1$ on γ_1, $V = 0$ on γ_2, γ_3 and because of (7.28), (7.31) and (7.32) we get

$$\| V \|_{1,T} \leqslant C \, {}_0\| f_1 \|_{1/2, \gamma_1}$$

which concludes the proof of Theorem 7.4.

Let $S = (x, y \mid |x| < 1, |y| < 1)$ be a square and γ_i its sides as shown in figure 7.2

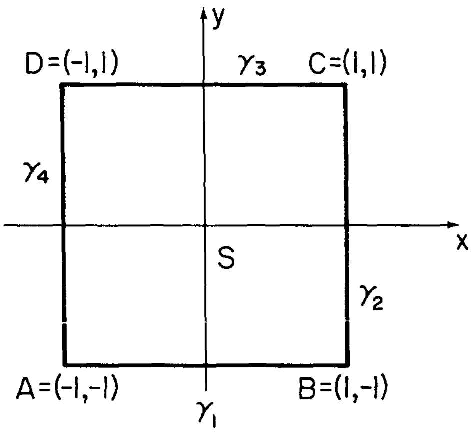

Figure 7.2. — The scheme of the square.

THEOREM 7.5 : *Let S be the square shown in figure 7.2 and f satisfy $f(A) = f(B) = f(C) = f(D) = 0$ and $f_i = f|_{\gamma_i} \in \mathscr{P}_p(\gamma_i)$, $i = 1, \ldots, 4$. Then there exists $U \in \mathscr{P}_p^2(S)$ such that $U = f$ on ∂S and*

$$\| U \|_{1,S} \leqslant C \left(\sum_{i=1}^{4} {}_0\| f_i \|_{1/2, \gamma_i} \right)$$

where the constant C is independent of p and f.

Proof: Let T be triangle shown in figure 7.1 and

$$Q = \left\{ \xi, \eta \mid (\xi, \eta) \in T, \eta < \frac{3\sqrt{3}}{8} \right\}$$

be the trapezoid shown in figure 7.3.

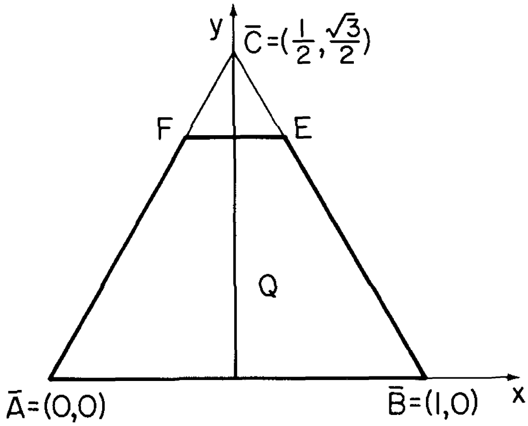

Figure 7.3. — Scheme of the trapezoid.

The mapping

$$(7.33)\qquad \xi=\frac{1}{2}+\frac{3x}{16}\left(-y+\frac{5}{3}\right)\qquad \eta=(1+y)\frac{3\sqrt{3}}{16}$$

maps S onto Q. The mapping is obviously one-to-one and the Jacobian and its inverse are bounded.

Let us first prove the theorem in the case that $f_i=0$, $i=2,3,4$. Denote $\bar{f}_1(\xi)=f_1(2\xi-1)$, $0<\xi<1$.

Obviously $\bar{f}_1(\bar{A})=f_1(\bar{B})=0$ and

$$_0\|\bar{f}_1\|_{1/2,\bar{\gamma}_1}\leq C_0\|f_1\|_{1/2,\gamma_1}.$$

Let $\bar{U}\in\mathscr{P}_p^1(T)$ such that $\bar{U}=f_1$ on $\bar{A}\bar{B}$ and $\bar{U}=0$ on $\bar{A}\bar{C}$ and $\bar{B}\bar{C}$. By Theorem 7.4, $\bar{U}(\xi,\eta)$ exists and

$$\|\bar{U}\|_{1,T}\leq C\,_0\|\bar{f}_1\|_{1/2,\bar{\gamma}_1}\leq C\,_0\|f_1\|_{1/2,\gamma_1}.$$

Because $\bar{U}\in\mathscr{P}_p^1(T)$ we have

$$\begin{aligned}\bar{U}(\xi,\eta)&=\sum_{0\leq k+j\leq p}a_{k,j}\,\xi^k\eta^j\\&=\sum_{0\leq k+j\leq p}a_{k,j}\left(\frac{1}{2}+\frac{3x}{16}\left(-y+\frac{5}{3}\right)\right)^k\left((1+y)\frac{3\sqrt{3}}{16}\right)^j\\&=U(x,y)\in\mathscr{P}_p^2(S)\end{aligned}$$

and

$$\|U\|_{1,S} \leq C_0 \|f\|_{1/2,\gamma_1} .$$

Because $\bar{f}_2 = \bar{f}_3 = 0$ we have $U(\pm 1, y) = 0$, $U(x, -1) = f_1$ and using Lemma 7.1, Lemma 7.2 we conclude by similar arguments as used in the proof of Theorem 7.4 that

$$\|U(x, 1)\|_{1,\gamma_3} < C_0 \|f_1\|_{1/2,\gamma_1} .$$

Of course $U(x, 1) \in \mathscr{P}_p(\gamma_3)$ and $U(\pm 1, 1) = 0$. Hence with

$$V = U(x, 1)(y + 1)/2$$

we see that

$$\|V\|_{1,S} \leq C \|U(x, 1)\|_{1,\gamma_3} \leq {}_0\|f_1\|_{1/2,\gamma_1}$$

and $V(x, 1) = U(x, 1)$. Hence $W = U - V \in \mathscr{P}_p^2(S)$, $W = f$ on ∂S and

$$\|W\|_{1,S} \leq C_0 \|f_1\|_{1/2,\gamma_1} .$$

The theorem is therefore proven in the case that $f = 0$ on three sides of S and hence it holds also if f is general but $f = 0$ at the vertices $ABCD$.

It remains to prove that in the general case there exist $\Phi \in \mathscr{P}_p^2(S)$ such that Φ has the same traces at $ABCD$ as f and

$$\|\Phi\|_{1,S} \leq C \left[\sum_{i=1}^{4} {}_0\|f_i\|_{1/2,\gamma_i} \right] . \tag{7.34}$$

To this end we define $F_1^{[f_1]}(\xi, \eta)$ by (7.1) and define $F_1^{[f_1]}(x, y)$ by inserting (7.33) for (ξ, η). Then $\left\|F_1^{[f_1]}(x, +1)\right\|_{1,\gamma_3} \leq C \|f_1\|_{0,\gamma_1}$ and hence analogously as above we can change $F_1^{[f_1]}(x, y)$ to $\tilde{F}_1$ so that $\tilde{F}(x, +1) = 0$ and $\left\|\tilde{F}_1\right\|_{1,S} \leq C_0 \|f_1\|_{1/2,\gamma_1}$. Changing the role of γ_1 and γ_3 we can analogously construct $\tilde{F}_3 \in \mathscr{P}_p^2(S)$ so that

$$\left\|\tilde{F}_3\right\|_{1,S} \leq C_0 \|f_3\|_{1/2,\gamma_3}, \quad \tilde{F}_3(x, 1) = f_3, \quad \tilde{F}_3(x, -1) = 0 .$$

Hence $\Phi = \tilde{F}_1 + \tilde{F}_3 \in \mathscr{P}_p^2(Q)$ has the same traces at $ABCD$ as f and (7.34) holds. This completes the proof of Theorem 7.5.

Remark : Theorems 7.4, 7.5 also hold when f is not a polynomial. This is known from the theory of Sobolev spaces. The importance of Theorems 7.4 and 7.5 lies in the fact that if f_i are polynomials, then there exists a polynomial extension.

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