RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

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RAIRO – Modélisation mathématique et analyse numérique, tome 21, nº 2 (1987), p. 239-260.

<http://www.numdam.org/item?id=M2AN_1987__21_2_239_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 21, n° 2, 1987, p. 239 à 260)

MATHEMATICAL FORMULATION OF FLUID-STRUCTURE INTERACTION PROBLEMS (*)

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Communiqué par R. TEMAM

Abstract. — *In this article we study three different types of fluid-structure interaction problems. A general mathematical framework is given and then we show how the general results can be applied to the three specific problems ; evolutionary problems, as well as vibration (eigenvalue) problems are considered.*

Résumé. — *Dans cet article nous étudions trois types différents de problèmes d'interaction fluide-structure. Un cadre mathématique général est donné et nous montrons ensuite comment les résultats généraux peuvent être appliqués aux trois problèmes spécifiques. On considère des problèmes d'évolution et aussi des problèmes de vibration (valeurs propres).*

INTRODUCTION

Our object here is to derive the mathematical formulation of three fluid-structure interaction problems from the point of view of existence and uniqueness of solution and that of the existence of eigenmodes. For the three problems the assumptions leading to linear problems are made : the displacements are small and under the considered assumptions the geometry is fixed (even if a free boundary exists), the solid obeys to the law of linear elasticity and the fluid is either compressible and at rest or is incompressible and subjected to an irrotational motion. Under these hypotheses the evolution of the structure is governed by an evolution equation of the second order in time and the vibrations (the eigenmodes) are solutions of a linear eigenvalue problem.

(*) Received in March 1986.

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A general mathematical framework which applies to the three physical problems is described in § 1. Then § 2 to 4 are devoted to the study of the following situations :

i) *Case I : Hydroelasticity* (§ 2)

This example corresponds for instance to the motion of the fuel of a rocket. The solid is the container and the fluid (the fuel) is limited by the container and a free surface.

ii) *Case II : Elastoacoustics* (§ 3)

In this case a bounded domain is completely filled by the fluid and limited by the solid. The fluid is compressible and at rest and we study the noise generated in the fluid by the vibrations of the vessel.

iii) *Case III : Closed shell imbedded in an unlimited fluid* (§ 4)

In this case the solid is a closed shell with an empty interior and this shell is imbedded in an incompressible unlimited fluid and we study the interactions of this shell with the surrounding fluid. This case leads to an exterior problem.

PLAN

1. THE MATHEMATICAL SETTING

1.1. The general assumptions

We are given two Hilbert spaces $\mathscr{V}$ and $\mathscr{H}$, with $\mathscr{V} \subset \mathscr{H}$, the injection being continuous. We are also given a closed subspace V of $\mathscr{V}$ and we denote by H the closure of V in $\mathscr{H}$. Then V and H are both Hilbert spaces when equipped with the scalar product of $\mathscr{V}$ and $\mathscr{H}$, V is included in H and the imbedding of V into H is continuous. We denote by $((.,.))$ and $(.,.)$ the scalar products in V and H, and

$$\|X\| = ((X, X))^{1/2}, \qquad |X| = (X, X)^{1/2},$$

represent the corresponding norms.

We consider now a and b

(1.1) a is a bilinear symmetric continuous form on V

(1.2) b is a bilinear symmetric continuous form on H

(1.3) b is coercive on H, i.e. There exists $\alpha_0 > 0$ such that

$$b(X, X) \geqslant \alpha_0 |X|^2, \quad \forall X \in H.$$

Due to (1.2) and (1.3), $X \mapsto b(X, X)^{1/2}$ defines a norm on H which is equivalent to the given one. We also assume that

(1.4) for every $\lambda > 0$, $a + \lambda b$ is coercive on V, i.e.

there exists $\alpha_1 = \alpha_1(\lambda) > 0$ such that

$$(a + \lambda b)(X, X) \geqslant \alpha_1 \|X\|^2, \quad \forall X \in V.$$

It follows from (1.4) that

$$a(X, X) \geqslant 0, \quad \forall X \in V \tag{1.5}$$

and that, for every $\lambda > 0$,

$$X \mapsto \{a(X, X) + \lambda b(X, X)\}^{1/2}$$

defines a norm on V which is equivalent to the given one.

1.2. The spectral problem

Let V' be the dual of V; we note that H can be identified to a subspace of V' so that we have

$$V \subset H \subset V' \tag{1.6}$$

where each space is dense in the following one and the injections are continuous. We will assume furthermore that

(1.7) The injection of V into H is compact.

The bilinear form b allows us to define a linear continuous operator in H, B, by setting

$$(BX, Y) = b(X, Y), \quad \forall X, Y \in H; \tag{1.8}$$

B is self-adjoint and invertible. Similarly we define A, linear continuous from V into V' by setting

$$(1.9) \qquad (AX, Y) = a(X, Y), \quad \forall X, Y \in V ;$$

A is self-adjoint and due to (1.4), for every $\lambda > 0$, $A + \lambda B$ is invertible linear continuous from V into V'. Note also that due to (1.7), since $(A + \lambda B)^{-1}$ is continuous from V' into V and hence from H into V,

$$(1.10) \quad (A + \lambda B)^{-1} \text{ is a self-adjoint compact operator in } H, \ \forall \lambda > 0 .$$

By standard results of spectral theory (see for instance R. Courant and D. Hilbert [5]) there exists a basis in H, $\{Y_j\}_{j \in \mathbb{N}}$ which is orthonormal in H for the scalar product $b(X, Y)$ and orthogonal in V for $a(X, Y)$, and there exists a sequence of numbers λ_j, such that

$$(1.11) \qquad AY_j = \lambda_j BY_j, \quad \forall j = 1, \dots ,$$

$$(1.12) \qquad 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lambda_j \to \infty, \quad j \to \infty .$$

The property $\lambda_1 \geqslant 0$ follows obviously from (1.5).

The spectral problem (1.11) will provide the eigenmodes of the vibration problems that we will consider hereafter.

1.3. The abstract evolution problem

Let $T > 0$ be given and let us assume that we are given a family of linear continuous forms on V, $L(t) \in V'$, $0 \leqslant t \leqslant T$, such that $L \in L^2(0, T; V')$ (1) ; as it will appear in the sequel these forms L are related to applied external forces.

In the evolution problem we are looking for a function X from $(0, T)$ into V such that

$$(1.13) \qquad B\ddot{X}(t) + AX(t) = L(t), \quad t \in (0, T)$$

(1) If W is a Banach space, $L^2(0, T; W)$; $1 \leqslant p \leqslant \infty$, is the space of L^p functions from $(0, T)$ into W, which is Banach for the norm

$$\left(\int_0^T \| f(t) \|_W^p \, dt \right)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty$$

$$\operatorname*{Sup\,ess}_{t \in (0, T)} \| f(t) \|_W \quad \text{if} \quad p = \infty .$$

If $p = 2$ and W is a Hilbert space, we obtain of course a Hilbert space ; $\mathscr{C}([0, T])$ is the space of continuous functions from $[e, T]$ into W, which is Banach for the usual norm $\operatorname*{Max}_{t \in [0, t]} \| f[t] \|$.

or alternatively

$$(1.14)\quad \frac{d^2}{dt^2}\cdot b(X(t), Y) + a(X(t), Y) = (L(t), Y),\ \forall t \in (0, T),\ \forall Y \in V$$

where $\dot{X} = dX/dt$. This evolution equation is completed by the initial data,

$$(1.15)\qquad X(0) = X_0\,, \qquad \dot{X}(0) = X_1\,.$$

By application of general results on linear evolution equations of the second order in time (see for instance J. L. Lions and E. Magenes [7]), we have the following

THEOREM 1.1 : *Under the above assumptions, for L given in $L^2(0, T ; V')$, X_0, X_1 given in V and H, there exists a unique function X satisfying* (1.13)-(1.15) *and*

$$(1.16)\qquad X \in \mathscr{C}([0, T] ; V)\,, \qquad \dot{X} \in \mathscr{C}([0, T] ; H)\,.$$

Remark 1.1 :

i) Of course we can obtain a more regular solution of (1.13)-(1.15) if we assume more regularity on the data L, X_0, X_1 ; for the details see J. M. Guidaglia and R. Teman [6].

ii) The solution X satisfies also $\ddot{X} \in L^2(0, T ; V')$ or $\ddot{X} \in \mathscr{C}([0, T] ; V')$ if we assume that $L \in \mathscr{C}([0, T] ; V')$ and we have an energy equality

$$(1.17)\qquad \frac{1}{2}\frac{d}{dt}\cdot b(X(t), X(t)) + a(X(t), X(t)) = (L(t), X(t))\,.$$

This equation is obvious if, as mentionned in i), X is more regular ; it is otherwise proved by approximation (*cf.* [6]).

Remark 1.2 : As usual the eigenmodes of the problem are obtained by looking for the unforced solutions $(L(t) \equiv 0)$ of the evolution equation of the form ;

$$X(t) = e^{i\omega t}\, Y$$

hence we find

$$AY = \omega^2\, BY$$

which is exactly (1.11) with $Y = Y_j$, $\omega^2 = \lambda_j$.

Sketch of the proof of Theorem 1.1

i) The existence of solution is obtained by the Galerkin method based for instance on the eigenvectors Y_j.

As usual, for every $l \in \mathbb{N}$ we define an approximate solution $X_l(t)$ taking its values in the space spanned by $Y_1, \ldots, Y_l$:

$$X_l(t) = \sum_{j=1}^{l} g_{lj}(t) Y_j$$

such that

$$b(\ddot{X}_l(t), Y_j) + a(X_l(t), Y_j) = (L(t), Y_j), \quad 1 \leqslant j \leqslant l, \tag{1.18}$$

$$(X_l(0), Y_j) = (X_0, Y_j), \quad 1 \leqslant j \leqslant l, \tag{1.19}$$

$$(\dot{X}_l(0), Y_j) = (X_1, Y_j), \quad 1 \leqslant j \leqslant l. \tag{1.20}$$

The equation (1.18) amounts to a linear evolution system for the g_{lj} with initial data provided by (1.19)-(1.20) and the existence of X_l follows. We obtain then *a priori* estimates on the X_l by using the analogue of (1.17) which show that the X_l belong to a bounded set of $L^\infty(0, T; V)$ and the $\dot{X}_l$ belong to a bounded set of $L^\infty(0, T; H)$. These estimates and appropriate compactness results allow us to pass to limit $l \to \infty$ and to obtain a solution $X \in L^\infty(0, T; V)$ with $\dot{X} \in L^\infty(0, T; H)$.

ii) For the uniqueness we use the energy equality (1.17) satisfied with $L = 0$ by the difference $X = \bar{X} - \tilde{X}$ of two possible solutions $\bar{X}, \tilde{X}$ of (1.13)-(1.15). Finally the continuity properties in (1.16) are proved (see for instance [6, 7]).

2. APPLICATIONS IN MECHANICS

We first describe the general framework for our fluid-structure interaction problems (Section 2.1) and then in Sections 2.2 to 2.4 we describe the applications.

2.1. The general framework

We denote by Ω_s an open set of $\mathbb{R}^3$ which will represent the shape of a thick shell ; another open set Ω_f of $\mathbb{R}^3$ represents the domain filled by the fluid. It is assumed that Ω_s and Ω_f have smooth boundaries $\partial\Omega_s$ and $\partial\Omega_f$ and that $\partial\Omega_s$ and $\partial\Omega_f$ have in common some part $\Gamma(\subset \partial\Omega_s \cup \partial\Omega_f)$. The rest of the boundary of Ω_s (the free one) is denoted by γ ; whence $\partial\Omega_s = \Gamma \cup \gamma$. The vector ν with components ν_1, ν_2, ν_3 will represent the unit outward normal on $\partial\Omega_s$ and the unit inward normal on $\partial\Omega_f$.

In a linear theory [2] under the assumption of small displacements, the unknown are the following ones :

[2] One can of course consider non linear theories as well ; *cf.* J. Mathieu [8].

— The potential $\varphi = \varphi(x, t)$ of the fluid displacements ($x \in \Omega_f$, $x = (x_1, x_2, x_3)$, $t \in (0, T)$), $u = u(x, t)$ with components $u_1(x, t)$, $1 \leq i \leq 3$.

— The Cauchy stress tensors $\sigma = \sigma(x, t)$ with components $\sigma_{ij}(x, t)$, in Ω_s. The shell is furthermore subjected to surface forces $F = F(t)$ on its boundary $\partial\Omega_s = \Gamma \cup \gamma$; we may write F_Γ and F_γ for the restrictions of F to Γ and γ. We denote by ρ_s, ρ_f the specific mass of the shell and the fluid which we assume to be constants ; g is the gravity, c (= constant > 0) is the sound velocity in the fluid.

We denote by ε_{ij} the components of the strain tensor ε in Ω_s,

$$\varepsilon_{ij} = \varepsilon_{ij}(u) = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \quad i, j = 1, 2, 3 .$$

By the law of linear elasticity, we have at each point

$$\sigma_{ij} = \sum_{k,l=1}^{3} a_{ijkl}\, \varepsilon_{ij} \tag{2.1}$$

where the constants a_{ijkl} are the compliances coefficients. We shall write symbolically (2.1) in the form

$$\sigma = A\varepsilon , \quad \text{or} \quad \sigma(u) = A\varepsilon(u) , \tag{2.2}$$

where A is a linear positive self adjoint invertible operator in the space of symmetric tensors of order 2 on $\mathbb{R}^3$.

The equations

In Ω_s we write the usual equation of conservation of momentum

$$\rho_s \frac{\partial^2 u_i}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} \quad \text{in} \quad \Omega \times (0, T) \tag{2.3}$$

and the continuity of stresses on γ

$$F_i = \sum_{k=1}^{3} \sigma_{ik}\, \nu_k \quad \text{on} \quad \gamma \times (0, T) , \quad i = 1, 2, 3 . \tag{2.4}$$

Other general equations are the coupling conditions on Γ (continuity of normal stresses and normal displacements on Γ) :

$$\rho_f \frac{\partial^2 \varphi}{\partial t^2} \nu_i + F_i = \sum_{k=1}^{3} \sigma_{ik}\, \nu_k \quad \text{on} \quad \Gamma \times (0, T) \tag{2.5}$$

$$\frac{\partial \varphi}{\partial \nu} = u.\nu \quad \text{on} \quad \Gamma \times (0, T) . \tag{2.6}$$

Other equations which depend on the specific problem will be given in each case.

2.2. A problem in hydroelasticity

We consider the motion of a fluid Ω_f in a container Ω_s with a free surface Σ for the fluid, $\partial\Omega_f = \Sigma \cup \Gamma$ (see *fig.* 2.1).

The displacements of the fluid (including that of Σ) are small and given by grad φ. The fluid is incompressible and irrotational and then

(2.7) $$\Delta\varphi = 0 \quad \text{in} \quad \Omega_f \times (0, T)\,.$$

Finally we add the condition on Σ

(2.8) $$g\frac{\partial\varphi}{\partial\nu} - \frac{\partial^2\varphi}{\partial t^2} = 0 \quad \text{on} \quad \Sigma \times (0, T) \quad (^3)\,.$$

Our aim is now to set the problem (2.3)-(2.8) into the framework of Section 1. The spaces and forms are chosen as follows :

(2.9) $$\mathscr{V} = H^1(\Omega_s)^3 \times \{H^1(\Omega_f)/\mathbb{R}\} \times L^2(\Sigma)$$

(2.10) $$\mathscr{H} = L^2(\Omega_s)^3 \times \{H^1(\Omega_f)/\mathbb{R}\} \times H^{-1/2}(\Sigma)\,.$$

For Ω an open set of $\mathbb{R}^3$ we denote by $L^2(\Omega)$ the space of real square

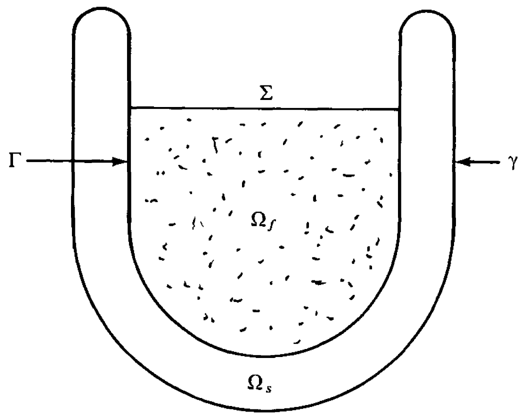

Figure 2.1.

(³) This problem was suggested by Professor R. W. Yeung, University of California, Berkeley.

integrable functions on Ω and by $H^1(\Omega)$ the Sobolev space of order 1

$$H^1(\Omega) = \left\{ u \in L^2(\Omega), \frac{\partial u}{\partial x_1} \in L^2(\Omega), i = 1, 2, 3 \right\}.$$

The space $L^2(\Omega)$ is endowed with the Hilbert scalar product

$$\int_\Omega f(x)\, g(x)\, dx \tag{2.11}$$

and $H^1(\Omega)$ is endowed with the Hilbert scalar product

$$\int_\Omega f(x)\, g(x)\, dx + \int_\Omega \nabla f(x) \,.\, \nabla g(x)\, dx\,. \tag{2.12}$$

The space $L^2(\Sigma)$ is the space of real functions on Σ which are L^2 for the surface measure $d\Gamma$; $H^{1/2}(\partial\Omega_f)$ is the space of traces on $\partial\Omega_f$ of functions in $H^1(\Omega_f)$ (see [7]), $H^{-1/2}(\partial\Omega_f)$ is its dual and $H^{-1/2}(\Sigma)$ is the space of distributions on Σ which are restriction of distributions in $H^{-1/2}(\partial\Omega_f)$, this space being endowed with the usual quotient norm (Σ is open in $\partial\Omega_f$).

The space V is

$$V = \left\{ X = (u, \varphi, y) \in \mathcal{V}, \Delta\varphi = 0 \text{ in } \Omega_f, \frac{\partial\varphi}{\partial\nu} = y \text{ on } \Sigma, \frac{\partial\varphi}{\partial\nu} = u.\nu \text{ on } \Gamma \right\}. \tag{2.13}$$

The forms b, a, L are defined as follows

$$b(X, Y) = \int_{\Omega_s} \rho_s\, uv\, dx + \int_{\Omega_f} \rho_f \nabla\varphi\, \nabla\psi\, dx\,, \quad \forall X, Y \in H \tag{2.14}$$

$$a(X, Y) = \int_\Sigma \rho_f\, gyz\, d\Gamma + \int_{\Omega_s} A\varepsilon(u) \,.\, \varepsilon(v)\, dx\,, \quad \forall X, Y \in V \tag{2.15}$$

$$(L, Y) = \int_{\Gamma\cup\gamma} Fv\, d\Gamma\,, \quad \forall Y = (v, \psi, z) \in V\,. \tag{2.16}$$

Let us show that (1.14) is indeed a weak formulation of the problem. We assume that X is sufficiently regular and satisfies (1.14). This relation is written

$$\forall Y = (v, \psi, z) \in V$$

$$\int_{\Omega_s} \rho_s\, \ddot{u} \,.\, v\, dx + \int_{\Omega_f} \rho_f \nabla\ddot{\varphi} \,.\, \nabla\psi\, dx + \int_\Sigma \rho_f\, gyz\, d\Gamma + \int_{\Omega_s} \sigma(u) \,.\, \varepsilon(v)\, dx = \int_{\Gamma\cup\gamma} Fv\, d\Gamma\,. \tag{2.17}$$

We will repeatedly use the Stokes formula as follows (the repeated indices summation convention is understood) :

$$(2.18)\quad \int_{\Omega_s} \sigma_{ij}\, \varepsilon_{ij}(v)\, dx = \int_{\Omega_s} \sigma_{ij} \frac{\partial v_i}{\partial x_j}\, dx = \int_{\partial\Omega_s} \sigma_{ij}\, v_i\, \nu_j\, d\Gamma - \int_{\Omega_s} \frac{\partial \sigma_{ij}}{\partial x_j} v_i\, dx\,.$$

Also if φ, ψ are two harmonic functions in Ω_f, then

$$(2.19)\qquad 0 = \int_{\Omega_f} \varphi \,.\, \Delta\psi\, dx = -\int_{\partial\Omega_f} \varphi \frac{\partial\psi}{\partial\nu}\, d\Gamma - \int_{\Omega_f} \nabla\varphi \,.\, \nabla\psi\, dx$$

(ν is pointing inward on $\partial\Omega_f$).

We now write (2.17) with $Y = (v, \psi, z)$, $\psi = z = 0$, and v a test function vanishing on $\partial\Omega_s$ ($v \in \mathscr{C}_0^\infty(\Omega)^3$, $\mathscr{C}_0^\infty(\Omega) =$ the space of real $\mathscr{C}^\infty$ functions in Ω with a compact support). We infer from (2.19) that

$$(2.20)\qquad \int_{\Omega_s} \rho_s\, \ddot{u}_i\, v_i\, dx - \int_{\Omega_s} \frac{\partial\sigma_{ij}}{\partial x_j} v_i\, dx = 0\,,\quad \forall v \in \mathscr{C}_0^\infty(\Omega)^3$$

and (2.3) follows. Then using any function $v \in H^1(\Omega_s)^3$ such that

$$(2.21)\qquad v \,.\, \nu = 0 \quad \text{on} \quad \Gamma$$

and $\psi = z = 0$, we obtain from (2.17)-(2.18)

$$\int_{\Gamma\cup\gamma} \sigma_{ij}\, \nu_j\, v_i\, d\Gamma = \int_{\Gamma\cup\gamma} F_i\, v_i\, d\Gamma\,.$$

Consequently, since the trace of v on γ is arbitrary and the tangential component of the trace of v on Γ is arbitrary too, we find

$$F_i = \sigma_{ij}\, \nu_j \quad \text{on} \quad \gamma\,,$$

which is (2.4) and

$$(2.22)\qquad F_\tau = (\sigma \,.\, \nu)_\tau \quad \text{on} \quad \Gamma\,.$$

We have denoted by $\sigma \,.\, \nu$ the vector with components $(\sigma \,.\, \nu)_i = \sigma_{ij}\, \nu_j$ and a_τ, a_ν are the tangential and normal components on $\partial\Omega_s$ or $\partial\Omega_f$ of a vector a :

$$(2.23)\qquad a_\nu = (a \,.\, \nu)\,\nu\,,\qquad a_\tau = a - a_\nu\,.$$

Then we write (2.17) with an arbitrary $Y = (v, \psi, z)$ in V. Using (2.18), (2.19), (2.3), (2.4) and (2.22) we arrive at

$$-\int_{\Sigma\cup\Gamma} \rho_f\, \ddot{\varphi} \frac{\partial\psi}{\partial\nu}\, d\Gamma + \int_\Sigma \rho_f\, gyz\, d\Gamma + \int_\Gamma (\sigma \,.\, \nu)_\nu\, v_\nu\, d\Gamma = \int_\Gamma F_\nu\, v_\nu\, d\Gamma\,.$$

Thus $\left(z = \frac{\partial \psi}{\partial \nu} \text{ on } \Sigma, v_\nu = \frac{\partial \psi}{\partial \nu} \text{ on } \Gamma\right)$:

$$\int_\Sigma \rho_f(gy - \ddot{\varphi}) \frac{\partial \psi}{\partial \nu} d\Gamma + \int ((\sigma \cdot \nu)_\nu - \rho_f \ \ddot{\varphi} - F_\nu) v_\nu d\Gamma = 0 , \qquad \forall Y = (v, \psi, z) . \tag{2.24}$$

Taking $v = 0$ and observing that $\frac{\partial \psi}{\partial \nu}$ is still arbitrary on Σ, we obtain (2.8). Lastly we observe that v_ν is arbitrary on Γ and (2.24) implies then

$$\rho_f \ \ddot{\varphi} + F_\nu = (\sigma \cdot \nu)_\nu \quad \text{on} \quad \Gamma \tag{2.25}$$

which, together with (2.22) provides (2.5). In conclusion the conditions (2.3)-(2.6) and (2.7)-(2.8) are satisfied partly by the fact that $X(t) \in V$ and partly by the arguments above. It is elementary to go back along the computations and to show that a solution of (2.3)-(2.6), (2.7)-(2.8) satisfies formally (1.13) which is thus a weak formulation of the problem.

There remains to show that the assumptions (1.1)-(1.4) and (1.7) are satisfied. It is clear that b is a bilinear continuous form on H (or $\mathscr{H}$). For the coercivity on H, we note that

$$\int_{\Omega_s} \rho_s \, uv \, dx \quad \text{and} \quad \int_{\Omega_f} \rho_f \, \nabla\varphi \, \nabla\psi \, dx$$

are coercive on $L^2(\Omega_s)^3$ and $H^1(\Omega_f)/\mathbb{R}$. The last integral (with the condition $\frac{\partial \varphi}{\partial \nu} = y$ on Σ) provides also the coercivity on $L^2(\Sigma)$. This follows from a trace theorem in [7] : if $\varphi \in H^1(\Omega)$ and $\Delta\varphi \in L^2(\Omega)$ then $\frac{\partial \varphi}{\partial \nu}$ is defined on $\partial\Omega$ and belongs to $H^{-1/2}(\partial\Omega)$ (Ω open set of $\mathbb{R}^n$), and we have :

$$\left\| \frac{\partial \varphi}{\partial \nu} \right\|_{H^{-1/2}(\partial\Omega)} \leqslant c_1(\Omega) \{ |\nabla\varphi|_{L^2(\Omega)^n} + |\Delta\varphi|_{L^2(\Omega)} \} , \tag{2.25}$$

where $c_1 = c_1(\Omega)$ depends only on Ω. Thus for $X \in V$, $X = (u, \varphi, y)$

$$\begin{aligned} \frac{1}{2} \int_{\Omega_f} \rho_f |\nabla\varphi|^2 \, dx &\geqslant \frac{1}{2 \, c_1(\Omega_f)^2} \rho_f \left\| \frac{\partial \varphi}{\partial \nu} \right\|^2_{H^{-1/2}(\partial\Omega_f)} \\ &\geqslant \frac{1}{2 \, c_1(\Omega_f)^2} \rho_f \| Y \|^2_{H^{-1/2}(\Sigma)} . \end{aligned} \tag{2.26}$$

Similarly it is clear that a is a bilinear continuous form on V (or $\mathscr{V}$). For the coercivity of $a + \lambda b$, $\lambda > 0$, we note that, due to Korn's ~~inequality (see for instance [10])~~

$$\int_{\Omega_s} A\varepsilon(u) \cdot \varepsilon(u) \, dx + \lambda \int_{\Omega_s} \rho_s \, u^2 \, dx \geqslant c_1' \| u \|^2_{H^1(\Omega_s)^3} \tag{2.27}$$

for some $c_1' > 0$ depending only on Ω_s, λ, A and ρ_s. It is also clear that the forms

$$\int_{\Omega_f} \rho_f \nabla\varphi \nabla\psi \, dx \quad \text{and} \quad \int_{\Sigma} \rho_f \, gyz \, d\Gamma$$

provide the coercivity on $H^1(\Omega_f)/\mathbb{R}$ and $L^2(\Sigma)$.

Finally we must check the compactness assumption (1.7). If $X_j = (u_j, \varphi_j, y_j)$ is a sequence bounded in V then u_j is bounded in $H^1(\Omega_s)^3$ and by Rellich's theorem, this sequence is relatively compact in $L^2(\Omega_s)^3$. Similarly the sequence y_j is bounded in $L^2(\Sigma)$ and since the injection of $L^2(\Sigma)$ into $H^{-1/2}(\Sigma)$ is compact (*cf.* J. L. Lions and E. Magenes [7]), this sequence is relatively compact in $H^{-1/2}(\Sigma)$. For φ_j we note that $\frac{\partial \varphi_j}{\partial \nu} = u_j \cdot \nu$ on Γ and $= y_j$ on Σ ; hence $\frac{\partial \varphi_j}{\partial \nu}$ is bounded in $L^2(\partial\Omega_f)$ ([4]). Now $\Delta\varphi_j = 0$ and $\frac{\partial \varphi_j}{\partial \nu}$ belongs to $L^2(\partial\Omega_f)$; by the regularity results for the Neumann problem (*cf.* [1]) this implies that φ_j belongs to $H^{3/2}(\Omega_f)$ and is bounded in $H^{3/2}(\Omega_f)/\mathbb{R}$; finally since the imbedding of $H^{3/2}(\Omega)_f$ into $H^1(\Omega)_f$ is compact, the sequence φ_f is relatively compact in $H^1(\Omega_f)/\mathbb{R}$. We have proved that X_j is relatively compact in H.

We now apply the results of Section 1.

The spectral problem

There exists a family of elements $Y_j = (v_j, \psi_j, z_j)$ which is orthonormal in H and orthogonal in V, there exists a sequence of numbers λ_j, $\lambda_j \geq 0$,

$$\lambda_j \to +\infty \quad \text{as} \quad j \to \infty$$

such that

$$\sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k}(v_j) = -\lambda_j \, \rho_s (v_j)_i \quad \text{in} \quad \Omega_s \tag{2.28}$$

$$\sum_{k=1}^{3} \sigma_{ik}(v_j) \, \nu_k = \left| \begin{array}{ll} 0 & \text{on} \quad \gamma \\ -\rho_f \lambda_j \psi_j \nu_i & \text{on} \quad \Gamma \end{array} \right. \tag{2.29}$$

$$\frac{\partial \psi_j}{\partial \nu} = u_j \cdot \nu \quad \text{on} \quad \Gamma \,, \qquad \frac{\partial \psi_j}{\partial \nu} = z_j \quad \text{on} \quad \Sigma \tag{2.30}$$

$$g \frac{\partial \psi_j}{\partial \nu} + \lambda_j \psi_j = 0 \quad \text{on} \quad \Sigma \tag{2.31}$$

$$\Delta \psi_j = 0 \quad \text{in} \quad \Omega_f \,. \tag{2.32}$$

([4]) $u_j \cdot \nu$ is bounded in $H^{1/2}(\partial\Omega_s)$ and thus in $L^2(\Gamma)$.

The evolution problem

Theorem 1.1 gives :

THEOREM 2.1 : *Given* $F \in L^2((0,T) \times \partial\Omega_s)$ *and given* $u_0, u_1, \varphi_0, \varphi_1$, *such that*

$$\begin{aligned}
& u_0 \in H^1(\Omega_s)^3 , \quad u_1 \in L^2(\Omega_s)^3 \\
& \varphi_0, \varphi_1 \in H^1(\Omega_1) , \quad \Delta\varphi_0 = \Delta\varphi_1 = 0 \\
& \frac{\partial \varphi_i}{\partial \nu} = u_i \cdot \nu \quad on \quad \Gamma , \quad \left.\frac{\partial \varphi_i}{\partial \nu}\right|_\Sigma \in L^2(\Sigma) , \qquad i = 1, 2 ,
\end{aligned}$$

there exists a unique function $X = X(t) = (u(t), \varphi(t), y(t))$, *which satisfies* (1.16), *the initial conditions* (1.15) *and is solution to the evolution problem* (2.3)-(2.6), (2.7)-(2.8).

2.3 A problem in elastoacoustics

The fluid completely fills a bounded domain Ω_f which is limited by the shell Ω_s ; the boundary of Ω_f is Γ and the boundary of Ω_s consists of Γ and γ (*cf. fig.* 2.2).

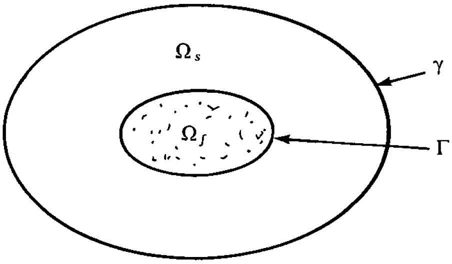

Figure 2.2.

The fluid is now compressible and if $c > 0$ (constant) is the velocity of the sound in the fluid we have

$$\frac{\partial^2 \varphi}{\partial t^2} - c^2 \Delta\varphi = 0 \quad \text{in} \quad \Omega_f \times (0, T) . \tag{2.33}$$

The equations of motion are (2.3)-(2.6) and (2.33), (*cf.* J. Boujot [3]) ; by comparison with the previous problem (Section 2.2), (2.33) replaces (2.7) and (2.8) has disappeared since there is no free surface for the fluid.

We now set this problem into the framework of Section 1. The definition of the spaces and the forms is the following

(2.34) $\mathscr{V} = H^1(\Omega_s)^3 \times W$

$$W = \left\{\varphi \in H^1(\Omega_f)\,,\quad \Delta\varphi = 0 \quad \text{in} \quad \Omega_f\,,\quad \int_{\Omega_f} \varphi\, dx = 0\right\} \quad (^5)$$

(2.35) $$\mathscr{H} = L^2(\Omega_s)^3 \times H^1(\Omega_f)/\mathbb{R}\,.$$

(2.36) $$V = \left\{X = (u, \varphi) \in \mathscr{V}\,, \frac{\partial\varphi}{\partial\nu} = u\,.\,\nu \quad \text{on} \quad \Gamma\right\}$$

(2.37) $$a(X, Y) = \int_{\Omega_s} A\varepsilon(u)\,.\,\varepsilon(v)\,dx + c^2\int_{\Omega_f} \Delta\varphi\,.\,\Delta\psi\,dx$$

$$\forall X, Y \in V\,,\quad X = (u, \varphi)\,,\quad Y = (v, \psi)\,,$$

(2.38) $$b(X, Y) = \int_{\Omega_s} \rho_s\, uv\, dx + \int_{\Omega_f} \rho_f\, \nabla\varphi\, \nabla\psi\, dx$$

$$\forall X, Y \in H\,,\quad X = (u, \varphi)\,,\quad Y = (v, \psi)\,,$$

(2.39) $$(L, Y) = \int_{\Gamma \cup \gamma} Fv\, d\Gamma\,,\quad \forall Y = (v, \psi) \in V\,.$$

It is clear that a and b are bilinear continuous forms respectively on V and H (or $\mathscr{V}$ and $\mathscr{H}$) ; it is clear also that b is coercive on H (or $\mathscr{H}$) and that $a + \lambda b$, $\lambda > 0$, is coercive on V (we proceed as in Section 2.2 using Korn's inequality for the u component). For the compactness assumption (1.7), let $X_j = (u_j, \varphi_j)$ be a bounded sequence in V. Then u_j is bounded in $H^1(\Omega_s)^3$ and relatively compact in $L^2(\Omega_s)^3$; we note also that $u_j\,.\,\nu = \dfrac{\partial\varphi_j}{\partial\nu}$ is bounded in $H^{1/2}(\Gamma)$ ($\Gamma = \partial\Omega_f$), and since $\Delta\varphi_j$ is bounded in $L^2(\Omega_f)$, we conclude by using the regularity results for the Neumann problem [1], that φ_j is bounded in $H^{3/2}(\Omega_f)$ and relatively compact in $H^1(\Omega_f)$.

We now show that (1.14) is a weak formulation of the problem. We

---

(^{5}) This space is a Hilbert space of the natural scalar product

$$\int_{\Omega_f} \nabla\varphi\,.\,\nabla\psi\,dx + \int_{\Omega_f} \Delta\varphi\,.\,\Delta\psi\,dx.$$

The condition $\int_{\Omega_f} \varphi\, dx = 0$ has been added top avoid having to take the quotient by $\mathbb{R}$ (see [4]).

assume that X is sufficiently regular (in x and t) and satisfies (1.14), i.e.

$$\int_{\Omega_s} \rho_s \ddot{u} v \, dx + \int_{\Omega_f} \rho_f \nabla \ddot{\varphi} \,.\, \nabla \psi \, dx + \int_{\Omega_s} A \varepsilon(u) \,.\, \varepsilon(v) \, dx + \tag{2.40}$$

$$+ \int_{\Omega_f} c^2 \Delta \varphi \, \Delta \psi \, dx = \int_{\Gamma \cup \gamma} F v \, d\Gamma \,, \quad \forall Y = (v, \psi) \in V \,.$$

We choose first a test function $Y = (v, \psi)$, with $\psi = 0$, $v \in \mathscr{C}_0^\infty(\Omega)^3$, and we infer that (2.3) is satisfied ; then we choose $Y = (v, \psi)$, $\psi = 0$, $v \in H^1(\Omega_s)^3$, and $v \,.\, \nu = 0$ on Γ ; using (2.18) and (2.3) we conclude that

$$\int_{\Gamma \cup \gamma} \sigma_{ij} \, \nu_j \, v_i \, d\Gamma = \int_{\Gamma \cup \gamma} F_i \, v_i \, d\Gamma \,, \quad \forall v$$

and this implies (2.4) and (2.22) (which is part of (2.5)). Then we write (2.40) with $Y = (v, \psi)$, $v = 0$, $\psi \in \mathscr{C}_0^\infty(\Omega)$:

$$\int_{\Omega_f} \rho_f \nabla \ddot{\varphi} \,.\, \nabla \psi \, dx + \int_{\Omega_f} c^2 \Delta \varphi \,.\, \Delta \psi \, dx = 0 \,, \quad \forall \psi \,;$$

(2.33) follows. Finally we write (2.40) with a general test function $Y = (v, \psi)$. Using (2.18)-(2.19) and the results already proved we find

$$\int_\Gamma (\sigma \,.\, \nu)_\nu \, v_\nu \, d\Gamma - \int_\Gamma \rho_f \, \ddot{\varphi} \frac{\partial \psi}{\partial \nu} \, d\Gamma = \int_\Gamma \mathrm{F}_\nu v_\nu \, d\Gamma \,, \quad \forall (v, \psi) \,.$$

Since $v_\nu = \dfrac{\partial \psi}{\partial \nu}$ on Γ and this arbitrary, we conclude that

$$(\sigma \,.\, \nu)_\nu - \rho_f \, \ddot{\varphi} = F_\nu \quad \text{on} \quad \Gamma$$

which together with (2.22) provides (2.5).

All the conditions (2.3)-(2.6) and (2.33) have been proved. Conversely one can reverse easily the steps of the proof and show that if X is regular and satisfies (2.3)-(2.6) and (2.33) then (1.14) is valid.

The proof of the assumptions is complete, we can apply the general results of Section 1.

The spectral problem

There exists a family of elements $Y_j = (v_j, \psi_j)$ which is orthonormal in H and orthogonal in V, there exists a sequence of numbers λ_j, $\lambda_j \geq 0$, $\lambda_j \to +\infty$ as $j \to \infty$, such that (2.28)-(2.29) holds and

$$\frac{\partial \psi_j}{\partial \nu} = u_j \,.\, \nu \quad \text{on} \quad \Gamma \tag{2.41}$$

$$c^2 \Delta \psi_j + \lambda_j \, \psi_j = 0 \quad \text{in} \quad \Omega_f \,. \tag{2.42}$$

The evolution problem

Theorem 1.1 gives :

THEOREM 2.2 : *Given* $F \in L^2((0,T) \times \partial\Omega_s)$ *and given* u_0, u_1, φ_0, φ_1, *such that*

$$u_0 \in H^1(\Omega_s)^3 \,, \quad u_1 \in L^2(\Omega_s)^3$$

$$\varphi_0, \varphi_1 \in W \,, \quad \frac{\partial \varphi_i}{\partial \nu} = u_i \,.\, \nu \quad on \quad \Gamma \,, \quad i = 0, 1 \,,$$

there exists a unique function $X = X(t) = (u(t), \varphi(t))$ *which satisfies* (1.16), *the initial condition* (1.15) *and is solution to the evolution problem* (2.3) *to* (2.6) *and* (2.33).

2.4 Closed shell imbedded in an unlimited fluid

The shell Ω_s is closed (in the physical sense) and limited by γ in its interior and Γ in the exterior. The unlimited fluid fills the region outside the shell ; we shall denote by Ω_i the bounded open set $\complement \overline{\Omega}_f$ (*cf. fig.* 2.3).

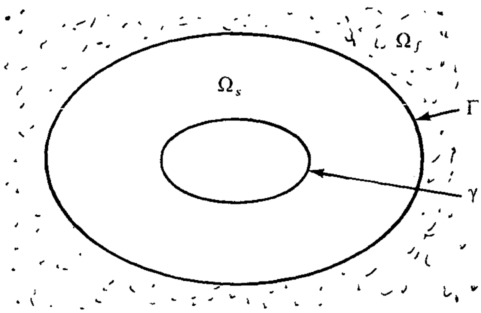

Figure 2.3.

The fluid is incompressible, irrotational and at rest at infinity. The equations of motion (2.3)-(2.6) are completed by

$$\Delta\varphi = 0 \quad \text{in} \quad \Omega_f \tag{2.43}$$

$$\varphi(x) = O\left[\frac{1}{|x|}\right] \quad \text{and} \quad |\nabla\varphi(x)| = O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to \infty \,. \tag{2.44}$$

In order to set this problem in the framework of Section 1 we first recall of some results on harmonic functions and introduce the operator $\mathcal{T}$.

The operator $\mathcal{C}$

We shall consider functions φ which are defined in Ω_f and belong to the space

(2.45) $$\left\{\varphi \in L^2(\Omega_f), \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega_f)\,, \quad i = 1, 2, 3\right\}$$

which is a Hilbert space for the scalar product [7] :

(2.46) $$\int_{\Omega_f} \nabla\varphi \,.\, \nabla\psi \, dx\,.$$

When φ belongs to the space (2.45) and $\Delta\varphi = 0$ then $\left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma}$ makes sense (hereafter we write it as $\left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_f}$) ([6]) it belongs to $H^{-1/2}(\Gamma)$ and (see J. C. Nedelec [9])

(2.47) $$\left\|\left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_f}\right\|_{H^{-1/2}(\Gamma)} \leqslant c_2(\Omega_f)|\nabla\varphi|_{L^2(\Omega_f)^3}$$

where $c_2(\Omega_f)$ depends only on Ω_f. Now we can (and shall) extend such a function φ in Ω_i ; the trace of φ on Γ exists and belongs to $H^{1/2}(\Gamma)$ and we define φ in Ω_i by setting

(2.48) $$\Delta\varphi = 0 \quad \text{in} \quad \Omega_i$$
$$\varphi|_{\Gamma_i} = \varphi|_{\Gamma_f}$$

where Γ_i and Γ_f indicate whether Γ is considered as the boundary of Ω_i or that of Ω_f. Using (2.25) we see that $\left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_i}$ makes sense, belongs to $H^{-1/2}(\Gamma)$ and it is usually different from $\left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_f}$; we shall write

(2.49) $$s = \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_i} - \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_f}.$$

We know (see [2, 4]) that the value of φ on Γ can be recovered from s by an integral

(2.50) $$\varphi(y) = \frac{1}{4\pi}\int_{\Gamma} \frac{s(x)}{|x - y|}\, d\Gamma(x)\,, \quad \forall y \in \Gamma\,.$$

([6]) Hereafter we extend such a function φ to the whole space $\mathbb{R}^3$, and we distinguish the traces of $\frac{\partial \varphi}{\partial \nu}$ on Γ, corresponding to $\Omega_f\left(= \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_f}\right)$ and to $\Omega_i\left(= \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma_i}\right)$.

We write $\mathcal{T}s$ the expression on the right side of (2.50), and we have thus a linear mapping

$$\left(s = \frac{\partial\varphi}{\partial\nu}\bigg|_{\Gamma_i} - \frac{\partial\varphi}{\partial\nu}\bigg|_{\Gamma_f}\right) \to \mathcal{T}s = \varphi|_\Gamma$$

from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. It is easy to see that $\mathcal{T}$ is a linear self-adjoint and coercive operator from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$; indeed if ψ is another function and r is the analogue of $s(\mathcal{T}r = \psi|_\Gamma)$, then by Stokes formula $(\Delta\varphi = \Delta\psi = 0$ in $\Omega_i \cup \Omega_f)$. We have

$$(2.51) \qquad (\mathcal{T}s, r) = \int_\Gamma \left(\frac{\partial\varphi}{\partial\nu}\bigg|_{\Gamma_i} - \frac{\partial\varphi}{\partial\nu}\bigg|_{\Gamma_i}\right)\psi\, d\Gamma = \int_{\Omega_i \cup \Omega_f} \nabla\varphi\, \nabla\psi\, dx$$

$$(2.52) \qquad \begin{aligned}(\mathcal{T}s, s) &= \int_{\Omega_i \cup \Omega_f} |\nabla\varphi|^2\, dx \\ &\geqslant \text{(by (2.25) and (2.47))} \\ &\geqslant \frac{1}{c_2^2}\left\|\frac{\partial\varphi}{\partial\nu}\bigg|_{\Gamma_f}\right\|^2_{H^{-1/2}(\Gamma)} + \frac{1}{c_1^2}\left\|\frac{\partial\varphi}{\partial\nu}\bigg|_{\Gamma_i}\right\|^2_{H^{-1/2}(\Gamma)} \\ &\geqslant c_2^1 \|s\|^2_{H^{-1/2}(\Gamma)}\end{aligned}$$

for some $c_2^1 > 0$ depending only on Ω_f.

The spaces and forms

We now choose the spaces and forms :

$$(2.53) \qquad \mathcal{V} = H^1(\Omega_s)^3 \times H^1(\Omega_i)/\mathbb{R}$$

$$(2.54) \qquad \mathcal{H} = L^2(\Omega_s)^3 \times H^1(\Omega_i)/\mathbb{R}$$

$$(2.55) \qquad V = \left\{X = (u, \varphi) \in \mathcal{V},\ \Delta\varphi = 0 \text{ in } \Omega_f,\ \frac{\partial\varphi}{\partial\nu}\bigg|_{\Gamma_f} = u \cdot \nu \text{ on } \Gamma\right\} \quad (^7)\,.$$

$$(2.56) \qquad a(X, Y) = \int_{\Omega_s} A\varepsilon(u) \cdot \varepsilon(u)\, dx\,, \qquad \forall X, Y \in V\,,\quad X = (u, \varphi)\,,\quad Y = (v, \psi)$$

$$(2.57) \qquad b(X, Y) = \int_{\Omega_s} \rho_s\, uv\, dx + \int_{\Omega_f} \rho_f\, \nabla\varphi\, \nabla\psi\, dx\,, \qquad \forall X, Y \in H\,,\quad X = (u, \varphi)\,,\quad Y = (v, \psi)$$

$$(2.58) \qquad (L, Y) = \int_{\Gamma \cup \gamma} Fv\, d\Gamma\,, \quad \forall Y = (v, \psi) \in V\,.$$

---

(7) The functions φ defined on Ω_i are now extended to harmonic functions on Ω_f in a way totally similar to (2.48).

It is easy to check that a is bilinear continuous on V. For b, the continuity and coercivity on $L^2(\Omega_s)^3$ of the term involving u, v is clear but we have to study the term involving

$$\int_{\Omega_f} \nabla\varphi \, \nabla\psi \, dx \,. \tag{2.59}$$

These results follow from

LEMMA 2.1 : *The form* (2.59) *is bilinear continuous coercive on the space* $W/\mathbb{R}$:

$$W = \{\varphi \in H^1(\Omega_i),\ \Delta\varphi = 0 \ in\ \Omega_i\} \,.$$

Proof: As indicated above the functions in W are extended to Ω_f as harmonic functions, and it suffices to show that

$$\left| \nabla\varphi \,|_{\Omega_f} \right|_{L^2(\Omega_f)^3}$$

defines on $W/\mathbb{R}$ a norm equivalent to the natural norm

$$\left| \nabla\varphi \,|_{\Omega_i} \right|_{L^2(\Omega_i)^3} \,.$$

By the Stokes formula and (2.47)

$$\begin{aligned} \int_{\Omega_f} |\nabla\varphi|^2 \, dx &= -\int_\Gamma \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_f} \varphi \, d\Gamma \\ &\leq \|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)} \left\| \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_f} \right\|_{H^{-1/2}(\Gamma)} \\ &\leq c_2(\Omega_f) \|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)} |\nabla\varphi|_{L^2(\Omega_f)^3} \end{aligned}$$

$$|\nabla\varphi|_{L^2(\Omega_f)^3} \leq c_2(\Omega_f) \|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)} \,. \tag{2.60}$$

Also since

$$\int_{\Omega_f} \Delta\varphi \, dx = \int_\Gamma \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_f} d\Gamma = 0 \,,$$

we can replace $\varphi|_\Gamma$ by $\varphi + C$, $C \in \mathbb{R}$ and we can improve (2.60) as

$$|\nabla\varphi|_{L^2(\Omega_f)^3} \leq c_2(\Omega_f) \|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)/\mathbb{R}} \,. \tag{2.61}$$

By the trace theorem for $H^1(\Omega_i)$ we then have

$$\|\varphi\|_\Gamma\|_{H^{1/2}(\Gamma)/\mathbb{R}} \leq c_3(\Omega_i) |\nabla\varphi|_{L^2(\Omega_i)^3} \tag{2.62}$$

for some $c_3(\Omega_i)$ depending on Ω_i. Hence

$$|\nabla\varphi|_{L^2(\Omega_f)^3} \leqslant c_2\, c_3 |\nabla\varphi|_{L^2(\Omega_i)^3}\,. \tag{2.63}$$

For the opposite inequality we note that

$$\begin{aligned} |\nabla\varphi|_{L^2(\Omega_i)^3} &\leqslant c_4(\Omega_i)\|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)} \\ |\nabla\varphi|_{L^2(\Omega_i)^3} &\leqslant c_4(\Omega_i)\|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)/\mathbb{R}} \end{aligned} \tag{2.64}$$

because of the results on the regularity of the solution of the Dirichlet problem in Ω_i, [1]. Then let $\mathcal{O}$ be the intersection of Ω_f with a ball sufficiently large so that $\bar{\Omega}_i$ is included in this ball. Then Γ is a connected component of $\partial\mathcal{O}$ and by the trace theorem in $H^1(\mathcal{O})$ [7],

$$\begin{aligned} \|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)} &\leqslant c_5(\mathcal{O})|\nabla\varphi|_{L^2(\mathcal{O})^3} \\ \|\varphi|_\Gamma\|_{H^{1/2}(\Gamma)/\mathbb{R}} &\leqslant c_5|\nabla\varphi|_{L^2(\Omega_f)^3}\,. \end{aligned} \tag{2.65}$$

By combining this with (2.64),

$$|\nabla\varphi|_{L^2(\Omega_i)^3} \leqslant c_4\, c_5|\nabla\varphi|_{L^2(\Omega_f)^3} \tag{2.66}$$

and the lemma is proved

Q.E.D.

Finally we have to prove that $a + \lambda b$ is coercive on V, $\forall\lambda > 0$ and that the injection of V into H is compact. We have already observed several times that

$$\lambda \int_{\Omega_s} \rho_s\, uv\, dx + \int_{\Omega_s} A\varepsilon(u)\,.\,\varepsilon(u)\,dx$$

is coercive on $H^1(\Omega_s)^3$, while the coercivity of (2.59) on $H^1(\Omega_i)/\mathbb{R}$ is again provided by Lemma 2.1.

Concerning (1.7), let $X_j = (u_j, \varphi_j)$ be a bounded sequence in V. Then u_j is bounded in $H^1(\Omega_s)^3$, relatively compact in $L^2(\Omega_s)$. Also $\Delta\varphi_j = 0$ in Ω_f, $\left.\dfrac{\partial\varphi_j}{\partial\nu}\right|_{\Gamma_f} = u_j\,.\,\nu$ is bounded in $H^{1/2}(\Gamma)$. By the regularity results for the Neumann problem in Ω_f, [1], φ_j is in $H^2(\mathcal{O})$ for every bounded subset $\mathcal{O}$ of Ω_f, and

$$\|\varphi_j\|_{H^2(\mathcal{O})} \leqslant c_6(\mathcal{O}, \Omega_f)\left\{\left\|\left.\frac{\partial\varphi_j}{\partial\nu}\right|_{\Gamma_f}\right\|_{H^{1/2}(\Gamma)} + |\Delta\varphi_j|_{L^2(\Omega_f)}\right\}.$$

Thus $\varphi_j|_{\mathcal{O}}$ is bounded in $H^2(\mathcal{O})/\mathbb{R}$ and $\varphi_j|_\Gamma$ is bounded in $H^{3/2}(\Gamma)$ and finally

$\varphi_j|_{\Omega_i}$ belongs to $H^2(\Omega_i)$ and is bounded in $H^2(\Omega_i)/\mathbb{R}$. The sequence $\varphi_j|_{\Omega_i}$ is then relatively compact in $H^1(\Omega_i)/\mathbb{R}$.

Equivalence with (1.14)

We now show that (1.14) is equivalent to our present problem, namely (2.3)-(2.6) and (2.43)-(2.44).

Let $X = X(t)$ be a regular solution of (1.14) which we write explicitly

$$(2.67) \quad \int_{\Omega_s} \rho_s \ddot{u} v \, dx + \int_{\Omega_f} \rho_f \nabla \ddot{\varphi} \nabla \psi \, dx + \int_{\Omega_s} A\varepsilon(u) \, . \, \varepsilon(v) \, dx = \\ = \int_{\Gamma \cup \gamma} Fv \, d\Gamma \quad \forall Y = (v, \psi) \in V .$$

As in the previous cases we first choose a test function $Y = (v, \psi)$ with $\psi = 0$ and $v \in \mathscr{C}_0^\infty(\Omega)^3$, and we obtain (2.3) with $\sigma = \sigma(u) = A\varepsilon(u)$. Then we take $v \in H^1(\Omega_s)^3$, $v \, . \, \nu = 0$ on Γ and $\psi = 0$. Using (2.3) and (2.18) we find (2.4) and (2.22). The next step is to write (2.67) with an arbitrary test function $Y = (v, \psi)$ and use (2.19) and the results already proved ; we obtain

$$\int_\Gamma (\sigma \, . \, \nu)_\nu \, v_\nu \, d\Gamma - \int_\Gamma \rho_f \, \ddot{\varphi} \, \frac{\partial \psi}{\partial \nu}\Big|_{\Gamma_f} d\Gamma = \int_\Gamma F_\nu \, v_\nu \, d\Gamma .$$

Since $v_\nu = \dfrac{\partial \psi}{\partial \nu}\Big|_{\Gamma_f}$ and this function is arbitrary, we find

$$(\sigma \, . \, \nu)_\nu - \rho_f \, \ddot{\varphi} = F_\nu \quad \text{on} \quad \Gamma$$

and this (with (2.22)) completes the proof of (2.5). Therefore (2.3)-(2.6) and (2.43)-(2.44) are satisfied. Conversely it is elementary to show that a smooth function $X = X(t)$ satisfying (2.3)-(2.6) and (2.43)-(2.44) is solution of (1.14) ; the equivalence is proved.

We are now able to apply the general results of Section 1. We obtain :

The spectral problem

There exists a sequence of elements $Y_j = (v_j, \psi_j)$ which is orthonormal in H and orthogonal in V, there exists a sequence of numbers λ_j, $\lambda_j \geqslant 0$, $\lambda_j \to \infty$, such that

$$(2.68) \qquad \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k}(v_j) = - \lambda_j \, \rho_s (v_j)_i \quad \text{in} \quad \Omega_s$$

$$(2.69) \qquad \sum_{k=1}^{3} \sigma_{ik}(v_j) \, \nu_k = \left| \begin{array}{ll} 0 & \text{on} \quad \gamma \\ - \rho_f \, \lambda_j \, \psi_j \, \nu_i & \text{on} \quad \Gamma \end{array} \right.$$

$$\frac{\partial \psi_j}{\partial \nu}\bigg|_{\Gamma_f} = u_j \cdot \nu \quad \text{on} \quad \Gamma \tag{2.70}$$

$$\Delta \psi_j = 0 \quad \text{in} \quad \Omega_f \quad (\text{and}\ \Omega_i)\,. \tag{2.71}$$

The evolution problem

Theorem 1.1 gives :

THEOREM 2.3 : *Given* $F \in L^2((0,T) \times \partial\Omega_s)$, *and given* u_0, u_1, φ_0, φ_1 *such that*

$$u_0 \in H^1(\Omega_s)^3\,, \qquad u_1 \in L^2(\Omega_s)^3$$
$$\varphi_0\,, \quad \varphi_1 \in H^1(\Omega_i)$$
$$\Delta\varphi_j = 0\ in\ \Omega_i\,, \quad \frac{\partial \varphi_j}{\partial \nu}\bigg|_{\Gamma_f} = u_j \cdot \nu\ on\ \Gamma\,, \quad j = 0, 1 \quad (^8)$$

there exists a unique function $X = X(t) = (u(t), \varphi(t))$ *which satisfies* (1.16), (1.15) *and is solution to the evolution problem* (2.3) *to* (2.6) *and* (2.43)-(2.44).

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(8) The extension of the φ_j to Ω_f as above is clear.