JACQUELINE BOUJOT
Mathematical formulation of fluid-structure interaction problems


<http://www.numdam.org/item?id=M2AN_1987__21_2_239_0>
MATHEMATICAL FORMULATION OF
FLUID-STRUCTURE INTERACTION PROBLEMS (*)

by Jacqueline BOUJOT (1)

Communicated by R. TEMAM

Abstract. — In this article we study three different types of fluid-structure interaction problems. A general mathematical framework is given and then we show how the general results can be applied to the three specific problems; evolutionary problems, as well as vibration (eigenvalue) problems are considered.

Résumé. — Dans cet article nous étudions trois types différents de problèmes d'interaction fluide-structure. Un cadre mathématique général est donné et nous montrons ensuite comment les résultats généraux peuvent être appliqués aux trois problèmes spécifiques. On considère des problèmes d'évolution et aussi des problèmes de vibration (valeurs propres).

INTRODUCTION

Our object here is to derive the mathematical formulation of three fluid-structure interaction problems from the point of view of existence and uniqueness of solution and that of the existence of eigenmodes. For the three problems the assumptions leading to linear problems are made: the displacements are small and under the considered assumptions the geometry is fixed (even if a free boundary exists), the solid obeys to the law of linear elasticity and the fluid is either compressible and at rest or is incompressible and subjected to an irrotational motion. Under these hypotheses the evolution of the structure is governed by an evolution equation of the second order in time and the vibrations (the eigenmodes) are solutions of a linear eigenvalue problem.

(*) Received in March 1986.
(1) Département de Mathématiques, Université d'Orléans, France. This posthumous article written by R. Temam is based on the results announced in [4] and on the slides of the lectures given by the author at the University of California at Berkeley and at Stanford University during the fall 1984.
A general mathematical framework which applies to the three physical problems is described in § 1. Then § 2 to 4 are devoted to the study of the following situations:

i) **Case I : Hydroelasticity** (§ 2)

This example corresponds for instance to the motion of the fuel of a rocket. The solid is the container and the fluid (the fuel) is limited by the container and a free surface.

ii) **Case II : Elastoacoustics** (§ 3)

In this case a bounded domain is completely filled by the fluid and limited by the solid. The fluid is compressible and at rest and we study the noise generated in the fluid by the vibrations of the vessel.

iii) **Case III : Closed shell imbedded in an unlimited fluid** (§ 4)

In this case the solid is a closed shell with an empty interior and this shell is imbedded in an incompressible unlimited fluid and we study the interactions of this shell with the surrounding fluid. This case leads to an exterior problem.

1. **The mathematical setting**

1.1 The general assumptions
1.2 The spectral problem
1.3 The abstract evolution problem

2. **Applications in mechanics**

2.1 The general framework
2.2 A problem in hydroelasticity
2.3 A problem in elastoacoustics
2.4 A closed shell imbedded in an unlimited fluid

1. **THE MATHEMATICAL SETTING**

1.1. **The general assumptions**

We are given two Hilbert spaces \( \mathcal{V} \) and \( \mathcal{H} \), with \( \mathcal{V} \subset \mathcal{H} \), the injection being continuous. We are also given a closed subspace \( V \) of \( \mathcal{V} \) and we denote by \( H \) the closure of \( V \) in \( \mathcal{H} \). Then \( V \) and \( H \) are both Hilbert spaces when equipped with the scalar product of \( \mathcal{V} \) and \( \mathcal{H} \), \( V \) is included in \( H \) and the imbedding of \( V \) into \( H \) is continuous. We denote by \( (.,.) \) the...
scalar products in $V$ and $H$, and

$$\|X\| = ((X, X))^{1/2}, \quad |X| = (X, X)^{1/2},$$

represent the corresponding norms.

We consider now $a$ and $b$

(1.1) \(a\) is a bilinear symmetric continuous form on $V$
(1.2) \(b\) is a bilinear symmetric continuous form on $H$
(1.3) \(b\) is coercive on $H$, i.e. There exists $\alpha_0 > 0$ such that

$$b(X, X) \geq \alpha_0 |X|^2, \quad \forall X \in H.$$

Due to (1.2) and (1.3), $X \mapsto b(X, X)^{1/2}$ defines a norm on $H$ which is equivalent to the given one. We also assume that

(1.4) for every $\lambda > 0$, $a + \lambda b$ is coercive on $V$, i.e.

there exists $\alpha_1 = \alpha_1(\lambda) > 0$ such that

$$(a + \lambda b)(X, X) \geq \alpha_1 \|X\|^2, \quad \forall X \in V.$$  

It follows from (1.4) that

(1.5) \(a(X, X) \geq 0, \quad \forall X \in V\)

and that, for every $\lambda > 0$,

$$X \mapsto \{a(X, X) + \lambda b(X, X)\}^{1/2}$$

defines a norm on $V$ which is equivalent to the given one.

1.2. The spectral problem

Let $V'$ be the dual of $V$; we note that $H$ can be identified to a subspace of $V'$ so that we have

(1.6) \(V \subset H \subset V'\)

where each space is dense in the following one and the injections are continuous. We will assume furthermore that

(1.7) The injection of $V$ into $H$ is compact.

The bilinear form $b$ allows us to define a linear continuous operator in $H$, $B$, by setting

(1.8) \((BX, Y) = b(X, Y), \quad \forall X, Y \in H;\)
$B$ is self-adjoint and invertible. Similarly we define $A$, linear continuous from $V$ into $V'$ by setting

$$(1.9) \quad (AX, Y) = a(X, Y), \quad \forall X, Y \in V;$$

$A$ is self-adjoint and due to (1.4), for every $\lambda > 0$, $A + \lambda B$ is invertible linear continuous from $V$ into $V'$. Note also that due to (1.7), since $(A + \lambda B)^{-1}$ is continuous from $V'$ into $V$ and hence from $H$ into $V$,

$$(1.10) \quad (A + \lambda B)^{-1} \text{ is a self-adjoint compact operator in } H, \forall \lambda > 0.$$ 

By standard results of spectral theory (see for instance R. Courant and D. Hilbert [5]) there exists a basis in $H$, \{ $Y_j$ \} $j \in \mathbb{N}$ which is orthonormal in $H$ for the scalar product $b(X, Y)$ and orthogonal in $V$ for $a(X, Y)$, and there exists a sequence of numbers $\lambda_j$, such that

$$(1.11) \quad AY_j = \lambda_j BY_j, \quad \forall j = 1, \ldots,$n (1.12) \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty, \quad j \to \infty.$$ 

The property $\lambda_1 \geq 0$ follows obviously from (1.5).

The spectral problem (1.11) will provide the eigenmodes of the vibration problems that we will consider hereafter.

1.3. The abstract évolution problem

Let $T > 0$ be given and let us assume that we are given a family of linear continuous forms on $V$, $L(t) \in V'$, $0 \leq t \leq T$, such that $L \in L^2(0, T; V')$ (¹); as it will appear in the sequel these forms $L$ are related to applied external forces.

In the évolution problem we are looking for a function $X$ from $(0, T)$ into $V$ such that

$$(1.13) \quad B\ddot{X}(t) + AX(t) = L(t), \quad t \in (0, T)$$

(¹) If $W$ is a Banach space, $L^2(0, T; W); 1 \leq p \leq \infty$, is the space of $L^p$ functions from $(0, T)$ into $W$, which is Banach for the norm

$$\left( \int_0^T \| f(t) \|_W^p \, dt \right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty$$

$$\text{Sup ess} \| f(t) \|_W \quad \text{if} \quad p = \infty.$$ 

If $p = 2$ and $W$ is a Hilbert space, we obtain of course a Hilbert space ; $\mathcal{C}([0, T])$ is the space of continuous functions from $[e, T]$ into $W$, which is Banach for the usual norm

$$\max_{t \in [0, T]} \| f(t) \|_W.$$ 

M² AN Modélisation mathématique et Analyse numérique
Mathematical Modelling and Numerical Analysis
or alternatively

\[ \frac{d^2}{dt^2} b \left( X(t), Y \right) + a \left( X(t), Y \right) = (L(t), Y), \quad \forall t \in (0, T), \quad \forall Y \in V \]

where \( \dot{X} = \frac{dX}{dt} \). This evolution equation is completed by the initial data,

\[ (1.15) \quad X(0) = X_0, \quad \dot{X}(0) = X_1. \]

By application of general results on linear evolution equations of the second order in time (see for instance J. L. Lions and E. Magenes [7]), we have the following

**THEOREM 1.1:** Under the above assumptions, for \( L \) given in \( L^2(0, T; V') \), \( X_0, X_1 \) given in \( V \) and \( H \), there exists a unique function \( X \) satisfying (1.13)-(1.15) and

\[ (1.16) \quad X \in \mathcal{C}([0, T]; V'), \quad \dot{X} \in \mathcal{C}([0, T]; H). \]

**Remark 1.1:**

i) Of course we can obtain a more regular solution of (1.13)-(1.15) if we assume more regularity on the data \( L, X_0, X_1 \); for the details see J. M. Guidaglia and R. Teman [6].

ii) The solution \( X \) satisfies also \( \dot{X} \in L^2(0, T; V') \) or \( \dot{X} \in \mathcal{C}([0, T]; V') \) if we assume that \( L \in \mathcal{C}([0, T]; V') \) and we have an energy equality

\[ (1.17) \quad \frac{1}{2} \frac{d}{dt} b \left( X(t), X(t) \right) + a \left( X(t), X(t) \right) = (L(t), X(t)). \]

This equation is obvious if, as mentioned in i), \( X \) is more regular; it is otherwise proved by approximation (cf. [6]).

**Remark 1.2:** As usual the eigenmodes of the problem are obtained by looking for the unforced solutions \( (L(t) = 0) \) of the evolution equation of the form

\[ X(t) = e^{i\omega t} Y \]

hence we find

\[ AY = \omega^2 BY \]

which is exactly (1.11) with \( Y = Y_j, \omega^2 = \lambda_j \).

**Sketch of the proof of Theorem 1.1**

i) The existence of solution is obtained by the Galerkin method based for instance on the eigenvectors \( Y_j \).

vol. 21, n° 2, 1987
As usual, for every $l \in \mathbb{N}$ we define an approximate solution $X_t(t)$ taking its values in the space spanned by $Y_1, \ldots, Y_l$:

$$X_t(t) = \sum_{j=1}^{l} g_{ij}(t) Y_j$$

such that

\begin{align}
(1.18) & \quad b(\ddot{X}_t(t), Y_j) + a(X_t(t), Y_j) = (L(t), Y_j), \quad 1 \leq j \leq l, \\
(1.19) & \quad (X_t(0), Y_j) = (X_0, Y_j), \quad 1 \leq j \leq l, \\
(1.20) & \quad (\dot{X}_t(0), Y_j) = (X_1, Y_j), \quad 1 \leq j \leq l.
\end{align}

The equation (1.18) amounts to a linear evolution system for the $g_{ij}$ with initial data provided by (1.19)-(1.20) and the existence of $X_t$ follows. We obtain then a priori estimates on the $X_t$ by using the analogue of (1.17) which show that the $X_t$ belong to a bounded set of $L^\infty(0, T; V)$ and the $\dot{X}_t$ belong to a bounded set of $L^\infty(0, T; H)$. These estimates and appropriate compactness results allow us to pass to limit $l \to \infty$ and to obtain a solution $X \in L^\infty(0, T; V)$ with $\dot{X} \in L^\infty(0, T; H)$.

ii) For the uniqueness we use the energy equality (1.17) satisfied with $L = 0$ by the difference $X = \ddot{X} - \ddot{X}$ of two possible solutions $\ddot{X}, \ddot{X}$ of (1.13)-(1.15). Finally the continuity properties in (1.16) are proved (see for instance [6, 7]).

2. APPLICATIONS IN MECHANICS

We first describe the general framework for our fluid-structure interaction problems (Section 2.1) and then in Sections 2.2 to 2.4 we describe the applications.

2.1. The general framework

We denote by $\Omega_s$ an open set of $\mathbb{R}^3$ which will represent the shape of a thick shell; another open set $\Omega_f$ of $\mathbb{R}^3$ represents the domain filled by the fluid. It is assumed that $\Omega_s$ and $\Omega_f$ have smooth boundaries $\partial \Omega_s$ and $\partial \Omega_f$ and that $\partial \Omega_s$ and $\partial \Omega_f$ have in common some part $\Gamma(\subset \partial \Omega_s \cup \partial \Omega_f)$. The rest of the boundary of $\Omega_s$ (the free one) is denoted by $\gamma$; whence $\partial \Omega_s = \Gamma \cup \gamma$. The vector $\nu$ with components $\nu_1, \nu_2, \nu_3$ will represent the unit outward normal on $\partial \Omega_s$ and the unit inward normal on $\partial \Omega_f$.

In a linear theory (\textsuperscript{2}) under the assumption of small displacements, the unknown are the following ones:

\footnote{\textsuperscript{2} One can of course consider non linear theories as well; cf. J. Mathieu [8].}

$M^2$ AN Modélisation mathématique et Analyse numérique Mathematical Modelling and Numerical Analysis
— The potential $\varphi = \varphi(x, t)$ of the fluid displacements $(x \in \Omega_f,
 x = (x_1, x_2, x_3), \ t \in (0, T))$, $u = u(x, t)$ with components $u_i(x, t)$, $1 \leq i \leq 3$.

— The Cauchy stress tensors $\sigma = \sigma(x, t)$ with components $\sigma_{ij}(x, t)$, in
$\Omega_s$. The shell is furthermore subjected to surface forces $F = F(t)$ on its
boundary $\partial \Omega_s = \Gamma \cup \gamma$; we may write $F_\Gamma$ and $F_\gamma$ for the restrictions of $F$
to $\Gamma$ and $\gamma$. We denote by $\rho_s$, $\rho_f$ the specific mass of the shell and the fluid
which we assume to be constants; $g$ is the gravity, $c (= \text{constant} > 0)$ is the
sound velocity in the fluid.

We denote by $\varepsilon_{ij}$ the components of the strain tensor $\varepsilon$ in $\Omega_s$,

$$
\varepsilon_{ij} = \varepsilon_{ij}(u) = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right], \ i, j = 1, 2, 3 .
$$

By the law of linear elasticity, we have at each point

$$(2.1) \quad \sigma_{ij} = \sum_{k,l=1}^{3} a_{ijkl} \varepsilon_{ij}$$

where the constants $a_{ijkl}$ are the compliances coefficients. We shall write
symbolically (2.1) in the form

$$(2.2) \quad \sigma = A \varepsilon , \quad \text{or} \quad \sigma(u) = A \varepsilon(u) ,$$

where $A$ is a linear positive self adjoint invertible operator in the space of
symmetric tensors of order 2 on $\mathbb{R}^3$.

The equations

In $\Omega$, we write the usual equation of conservation of momentum

$$(2.3) \quad \rho_s \frac{\partial^2 u_i}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} \quad \text{in} \quad \Omega \times (0, T)$$

and the continuity of stresses on $\gamma$

$$(2.4) \quad F_i = \sum_{k=1}^{3} \sigma_{ik} \nu_k \quad \text{on} \quad \gamma \times (0, T) , \ i = 1, 2, 3 .$$

Other general equations are the coupling conditions on $\Gamma$ (continuity of
normal stresses and normal displacements on $\Gamma$):

$$(2.5) \quad \rho_f \frac{\partial^2 \varphi}{\partial t^2} \nu_i + F_i = \sum_{k=1}^{3} \sigma_{ik} \nu_k \quad \text{on} \quad \Gamma \times (0, T)$$

$$(2.6) \quad \frac{\partial \varphi}{\partial \nu} = u.\nu \quad \text{on} \quad \Gamma \times (0, T) .$$

Other equations which depend on the specific problem will be given in
each case.

vol. 21, n° 2, 1987
2.2. A problem in hydroelasticity

We consider the motion of a fluid \( \Omega_f \) in a container \( \Omega_s \) with a free surface \( \Sigma \) for the fluid, \( \partial \Omega_f = \Sigma \cup \Gamma \) (see fig. 2.1).

The displacements of the fluid (including that of \( \Sigma \)) are small and given by \( \text{grad} \varphi \). The fluid is incompressible and irrotational and then

\[
\Delta \varphi = 0 \quad \text{in} \quad \Omega_f \times (0, T).
\]

Finally we add the condition on \( \Sigma \)

\[
g \frac{\partial \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial t} = 0 \quad \text{on} \quad \Sigma \times (0, T) \quad (3). \]

Our aim is now to set the problem (2.3)-(2.8) into the framework of Section 1. The spaces and forms are chosen as follows:

\[
\mathcal{V} = H^1(\Omega_s)^3 \times \{ H^1(\Omega_f)/\mathbb{R} \} \times L^2(\Sigma)
\]

\[
\mathcal{H} = L^2(\Omega_s)^3 \times \{ H^1(\Omega_f)/\mathbb{R} \} \times H^{-1/2}(\Sigma).
\]

For \( \Omega \) an open set of \( \mathbb{R}^3 \) we denote by \( L^2(\Omega) \) the space of real square

\[
\text{Figure 2.1.}
\]

\( (3) \) This problem was suggested by Professor R. W. Yeung, University of California, Berkeley.

\[M^2 \text{AN Modélisation mathématique et Analyse numérique}
\]

\[\text{Mathematical Modelling and Numerical Analysis}\]
integrable functions on $\Omega$ and by $H^1(\Omega)$ the Sobolev space of order 1

$$H^1(\Omega) = \left\{ u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, 2, 3 \right\}.$$  

The space $L^2(\Omega)$ is endowed with the Hilbert scalar product

$$\int_{\Omega} f(x) g(x) \, dx$$

and $H^1(\Omega)$ is endowed with the Hilbert scalar product

$$\int_{\Omega} f(x) g(x) \, dx + \int_{\Omega} \nabla f(x) \cdot \nabla g(x) \, dx.$$  

The space $L^2(\Sigma)$ is the space of real functions on $\Sigma$ which are $L^2$ for the surface measure $d\Gamma$; $H^{1/2}(\partial \Omega_f)$ is the space of traces on $\partial \Omega_f$ of functions in $H^1(\Omega_f)$ (see [7]), $H^{-1/2}(\partial \Omega_f)$ is its dual and $H^{-1/2}(\Sigma)$ is the space of distributions on $\Sigma$ which are restriction of distributions in $H^{-1/2}(\partial \Omega_f)$, this space being endowed with the usual quotient norm ($\Sigma$ is open in $\partial \Omega_f$).

The space $V$ is

$$V = \left\{ X = (u, \varphi, y) \in \mathcal{V}, \Delta \varphi = 0 \text{ in } \Omega_f, \frac{\partial \varphi}{\partial \nu} = y \text{ on } \Sigma, \frac{\partial \varphi}{\partial \nu} = u \cdot \nu \text{ on } \Gamma \right\}.$$  

The forms $b, a, L$ are defined as follows

(2.14) $b(X, Y) = \int_{\Omega_s} \rho_s \, u \, v \, dx + \int_{\Omega_f} \rho_f \, \nabla \varphi \cdot \nabla \psi \, dx$, $\forall X, Y \in H$

(2.15) $a(X, Y) = \int_{\Sigma} \rho_f \, g y z \, d\Gamma + \int_{\Omega_s} A \varepsilon(u) \cdot \varepsilon(v) \, dx$, $\forall X, Y \in V$

(2.16) $(L, Y) = \int_{\Gamma \cup \gamma} F v \, d\Gamma$, $\forall Y = (v, \psi, z) \in V$

Let us show that (1.14) is indeed a weak formulation of the problem. We assume that $X$ is sufficiently regular and satisfies (1.14). This relation is written

$$\forall Y = (v, \psi, z) \in V$$

(2.17) $\int_{\Omega_s} \rho_s \, \hat{u} \cdot v \, dx + \int_{\Omega_f} \rho_f \, \nabla \hat{\varphi} \cdot \nabla \psi \, dx + \int_{\Sigma} \rho_f \, g y z \, d\Gamma + \int_{\Omega_s} \sigma(u) \cdot \varepsilon(v) \, dx = \int_{\Gamma \cup \gamma} F v \, d\Gamma$. 

vol. 21, n° 2, 1987
We will repeatedly use the Stokes formula as follows (the repeated indices summation convention is understood):

\[
(2.18) \quad \int_{\Omega} \sigma_{ij} \varepsilon_{ij}(v) \, dx = \int_{\Omega} \sigma_{ij} \frac{\partial v_j}{\partial x_i} \, dx = \int_{\partial \Omega} \sigma_{ij} v_i \, d\Gamma - \int_{\partial \Omega} \frac{\partial \sigma_{ij}}{\partial x_j} v_i \, dx.
\]

Also if \( \varphi, \psi \) are two harmonic functions in \( \Omega_f \), then

\[
(2.19) \quad 0 = \int_{\Omega_f} \varphi \cdot \Delta \psi \, dx = - \int_{\partial \Omega_f} \varphi \frac{\partial \psi}{\partial n} \, d\Gamma - \int_{\Omega_f} \nabla \varphi \cdot \nabla \psi \, dx
\]

(\( n \) is pointing inward on \( \partial \Omega_f \)).

We now write \( (2.17) \) with \( Y = (v, \psi, z) \), \( \psi = z = 0 \), and \( v \) a test function vanishing on \( \partial \Omega_s \) (\( v \in C^\infty_0(\Omega)^3 \), \( C^\infty_0(\Omega) \) the space of real \( C^\infty \) functions in \( \Omega \) with a compact support). We infer from \( (2.19) \) that

\[
(2.20) \quad \int_{\Omega} \rho_{ij} u_i v_j \, dx - \int_{\partial \Omega} \frac{\partial \sigma_{ij}}{\partial x_j} v_i \, dx = 0, \quad \forall v \in C^\infty_0(\Omega)^3
\]

and \( (2.3) \) follows. Then using any function \( v \in H^1(\Omega_s)^3 \) such that

\[
(2.21) \quad v \cdot v = 0 \quad \text{on} \quad \Gamma
\]

and \( \psi = z = 0 \), we obtain from \( (2.17)-(2.18) \)

\[
\int_{\Gamma \cup \gamma} \sigma_{ij} v_j v_i \, d\Gamma = \int_{\Gamma \cup \gamma} F_i v_i \, d\Gamma
\]

Consequently, since the trace of \( v \) on \( \gamma \) is arbitrary and the tangential component of the trace of \( v \) on \( \Gamma \) is arbitrary too, we find

\[
F_i = \sigma_{ij} v_j \quad \text{on} \quad \gamma,
\]

which is \( (2.4) \) and

\[
(2.22) \quad F_\tau = (\sigma \cdot v)_\tau \quad \text{on} \quad \Gamma.
\]

We have denoted by \( \sigma \cdot v \) the vector with components \( (\sigma \cdot v)_\gamma = \sigma_{ij} v_j \) and \( a_x, a_v \) are the tangential and normal components on \( \partial \Omega_s \) or \( \partial \Omega_f \) of a vector \( a \):

\[
(2.23) \quad a_v = (a \cdot v) v, \quad a_\tau = a - a_v.
\]

Then we write \( (2.17) \) with an arbitrary \( Y = (v, \psi, z) \) in \( V \). Using \( (2.18), (2.19), (2.3), (2.4) \) and \( (2.22) \) we arrive at

\[
- \int_{\Sigma \cup \Gamma} \rho_f \phi \frac{\partial \psi}{\partial n} \, d\Gamma + \int_{\Sigma} \rho_f gyz \, d\Gamma + \int_{\Gamma} (\sigma \cdot v)_\gamma v_v \, d\Gamma = \int_{\Gamma} F_v v_v \, d\Gamma.
\]
Thus \( z = \frac{\partial \psi}{\partial v} \) on \( \Sigma \), \( v_v = \frac{\partial \psi}{\partial v} \) on \( \Gamma \):

\[
(2.24) \quad \int_{\Sigma} \rho_f (g_y - \Phi) \frac{\partial \psi}{\partial v} \, d\Gamma + \int \left( (\sigma \cdot v)_v - \rho_f (\Phi - F_v) \right) v_v \, d\Gamma = 0, \quad \forall Y = (v, \psi, z).
\]

Taking \( v = 0 \) and observing that \( \frac{\partial \psi}{\partial v} \) is still arbitrary on \( \Sigma \), we obtain (2.8). Lastly we observe that \( v_v \) is arbitrary on \( \Gamma \) and (2.24) implies then

\[
(2.25) \quad \rho_f \Phi + F_v = (\sigma \cdot v)_v \text{ on } \Gamma
\]

which, together with (2.22) provides (2.5). In conclusion the conditions (2.3)-(2.6) and (2.7)-(2.8) are satisfied partly by the fact that \( X(t) \in V \) and partly by the arguments above. It is elementary to go back along the computations and to show that a solution of (2.3)-(2.6), (2.7)-(2.8) satisfies formally (1.13) which is thus a weak formulation of the problem.

There remains to show that the assumptions (1.1)-(1.4) and (1.7) are satisfied. It is clear that \( b \) is a bilinear continuous form on \( H \) (or \( \mathcal{H} \)). For the coercivity on \( H \), we note that

\[
\int_{\Omega_f} \rho_f \nabla \psi \, dx \quad \text{and} \quad \int_{\Omega_f} \rho_f \nabla \psi \, dx
\]

are coercive on \( L^2(\Omega_f)^3 \) and \( H^1(\Omega_f)/\mathbb{R} \). The last integral (with the condition \( \frac{\partial \Phi}{\partial v} = y \) on \( \Sigma \)) provides also the coercivity on \( L^2(\Sigma) \). This follows from a trace theorem in [7]: if \( \varphi \in H^1(\Omega) \) and \( \Delta \varphi \in L^2(\Omega) \) then \( \frac{\partial \varphi}{\partial v} \) is defined on \( \partial \Omega \) and belongs to \( H^{-1/2}(\partial \Omega) \) \((\Omega \text{ open set of } \mathbb{R}^n)\), and we have :

\[
(2.25) \quad \left\| \frac{\partial \varphi}{\partial v} \right\|_{H^{-1/2}(\partial \Omega)} \leq c_1(\Omega) \left\{ \left| \nabla \varphi \right|_{L^2(\Omega)} + \left| \Delta \varphi \right|_{L^2(\Omega)} \right\},
\]

where \( c_1 = c_1(\Omega) \) depends only on \( \Omega \). Thus for \( X \in V, X = (u, \varphi, y) \)

\[
(2.26) \quad \frac{1}{2} \int_{\Omega_f} \rho_f \left| \nabla \varphi \right|^2 \, dx \geq \frac{1}{2} \frac{1}{c_1(\Omega_f)} \rho_f \left\| \frac{\partial \varphi}{\partial v} \right\|^2_{H^{-1/2}(\partial \Omega_f)} \geq \frac{1}{2} \frac{1}{c_1(\Omega_f)} \rho_f \left\| Y \right\|^2_{H^{-1/2}(\Sigma)}.
\]

Similarly it is clear that \( a \) is a bilinear continuous form on \( V \) (or \( \mathcal{Y} \)). For the coercivity of \( a + \lambda b, \lambda > 0 \), we note that, due to Korn’s inequality (see for instance [10]):

\[
(2.27) \quad \int_{\Omega_f} A e(u) \cdot e(u) \, dx + \lambda \int_{\Omega_f} \rho_f u^2 \, dx \geq c_1^2 \left\| u \right\|^2_{H^1(\Omega_f)^3}
\]
for some $c'_1 > 0$ depending only on $\Omega$, $\lambda$, $A$ and $\rho$. It is also clear that the forms

$$
\int_{\Omega} \rho_f \nabla \varphi \nabla \psi \, dx \quad \text{and} \quad \int_{\Sigma} \rho_g \gamma \nu \, d\Gamma
$$

provide the coercivity on $H^1(\Omega_f)/\mathbb{R}$ and $L^2(\Sigma)$.

Finally we must check the compactness assumption (1.7). If $X_j = (u_j, \varphi_j, y_j)$ is a sequence bounded in $V$ then $u_j$ is bounded in $H^1(\Omega)\mathbb{R}^3$ and by Rellich's theorem, this sequence is relatively compact in $L^2(\Omega)\mathbb{R}^3$. Similarly the sequence $y_j$ is bounded in $L^2(\Sigma)$ and since the injection of $L^2(\Sigma)$ into $H^{-1/2}(\Sigma)$ is compact (cf. J. L. Lions and E. Magenes [7]), this sequence is relatively compact in $H^{-1/2}(\Sigma)$. For $\varphi_j$ we note that

$$
\frac{\partial \varphi_j}{\partial v} = u_j \cdot \nu \quad \text{on} \quad \Gamma \quad \text{and} \quad y_j \quad \Sigma; \quad \text{hence} \quad \frac{\partial \varphi_j}{\partial v} \quad \text{is bounded in} \quad L^2(\partial \Omega_f)
$$

(4). Now $\Delta \varphi_j = 0$ and $\frac{\partial \varphi_j}{\partial v}$ belongs to $L^2(\partial \Omega_f)$; by the regularity results for the Neumann problem (cf. [1]) this implies that $\varphi_j$ belongs to $H^{3/2}(\Omega_f)$ and is bounded in $H^{3/2}(\Omega_f)/\mathbb{R}$; finally since the imbedding of $H^{3/2}(\Omega_f)$ into $H^1(\Omega_f)$ is compact, the sequence $\varphi_j$ is relatively compact in $H^1(\Omega_f)/\mathbb{R}$. We have proved that $X_j$ is relatively compact in $H$.

We now apply the results of Section 1.

**The spectral problem**

There exists a family of elements $Y_j = (\psi_j, \zeta_j, \xi_j)$ which is orthonormal in $H$ and orthogonal in $V$, there exists a sequence of numbers $\lambda_j$, $\lambda_j \geq 0$,

$$
\lambda_j \to + \infty \quad \text{as} \quad j \to \infty
$$

such that

$$
\sum_{k=1}^{3} \frac{\partial \sigma_{ik}(\psi_j)}{\partial x_k} (\xi_j) = - \lambda_j \rho_s(\psi_j) \quad \text{in} \quad \Omega,
$$

$$
\sum_{k=1}^{3} \sigma_{ik}(\psi_j) \nu_k = \left\{
\begin{array}{ll}
0 & \text{on} \quad \gamma \\
- \rho_f \lambda_j \psi_j \nu_i & \text{on} \quad \Gamma
\end{array}
\right.
$$

$$
\frac{\partial \psi_j}{\partial v} = u_j \cdot \nu \quad \text{on} \quad \Gamma, \quad \frac{\partial \psi_j}{\partial v} = z_j \quad \text{on} \quad \Sigma
$$

$$
g \frac{\partial \psi_j}{\partial v} + \lambda_j \psi_j = 0 \quad \text{on} \quad \Sigma
$$

$$
\Delta \psi_j = 0 \quad \text{in} \quad \Omega_f.
$$

(4) $u_j \cdot \nu$ is bounded in $H^{1/2}(\partial \Omega_f)$ and thus in $L^2(\Gamma)$.
The evolution problem

Theorem 1.1 gives:

**THEOREM 2.1:** Given $F \in L^2((0, T) \times \partial \Omega_s)$ and given $u_0, u_1, \varphi_0, \varphi_1$, such that

$$
\begin{align*}
&u_0 \in H^1(\Omega_s)^3, \quad u_1 \in L^2(\Omega_s)^3 \\
&\varphi_0, \varphi_1 \in H^1(\Omega_s), \quad \Delta \varphi_0 = \Delta \varphi_1 = 0 \\
&\frac{\partial \varphi_i}{\partial n} = u_i \cdot n \quad \text{on} \quad \Gamma, \quad \frac{\partial \varphi_i}{\partial n} \bigg|_\Sigma \in L^2(\Sigma), \quad i = 1, 2,
\end{align*}
$$

there exists a unique function $X = X(t) = (u(t), \varphi(t), y(t))$, which satisfies (1.16), the initial conditions (1.15) and is solution to the evolution problem (2.3)-(2.6), (2.7)-(2.8).

2.3 A problem in elastoacoustics

The fluid completely fills a bounded domain $\Omega_f$ which is limited by the shell $\Omega_s$; the boundary of $\Omega_f$ is $\Gamma$ and the boundary of $\Omega_s$ consists of $\Gamma$ and $\gamma$ (cf. fig. 2.2).

The fluid is now compressible and if $c > 0$ (constant) is the velocity of the sound in the fluid we have

$$(2.33) \quad \frac{\partial^2 \varphi}{\partial t^2} - c^2 \Delta \varphi = 0 \quad \text{in} \quad \Omega_f \times (0, T).$$

The equations of motion are (2.3)-(2.6) and (2.33), (cf. J. Boujot [3]) ; by comparison with the previous problem (Section 2.2), (2.33) replaces (2.7) and (2.8) has disappeared since there is no free surface for the fluid.

We now set this problem into the framework of Section 1. The definition
of the spaces and the forms is the following

\begin{equation}
\mathcal{Y} = H^1(\Omega_f)^3 \times W
\end{equation}

\begin{equation}
W = \left\{ \varphi \in H^1(\Omega_f), \quad \Delta \varphi = 0 \quad \text{in} \quad \Omega_f, \quad \int_{\Omega_f} \varphi \, dx = 0 \right\}
\end{equation}

\begin{equation}
\mathcal{K} = L^2(\Omega_f)^3 \times H^1(\Omega_f)/\mathbb{R}.
\end{equation}

\begin{equation}
V = \left\{ X = (u, \varphi) \in \mathcal{Y}, \frac{\partial \varphi}{\partial \nu} = u \cdot v \quad \text{on} \quad \Gamma \right\}
\end{equation}

\begin{equation}
a(X, Y) = \int_{\Omega_f} A \varepsilon(u) : \varepsilon(v) \, dx + c^2 \int_{\Omega_f} \Delta \varphi \cdot \Delta \psi \, dx \\
\forall X, Y \in V, \quad X = (u, \varphi), \quad Y = (v, \psi),
\end{equation}

\begin{equation}
b(X, Y) = \int_{\Omega_f} \rho_f \nabla u \nabla v \, dx + \int_{\Omega_f} \rho_f \nabla \varphi \nabla \psi \, dx
\end{equation}

\begin{equation}
(L, Y) = \int_{\Gamma \cup \gamma} F v \, d\Gamma, \quad \forall Y = (v, \psi) \in V.
\end{equation}

It is clear that $a$ and $b$ are bilinear continuous forms respectively on $V$ and $H$ (or $\mathcal{Y}$ and $\mathcal{K}$); it is clear also that $b$ is coercive on $H$ (or $\mathcal{K}$) and that $a + \lambda b$, $\lambda > 0$, is coercive on $V$ (we proceed as in Section 2.2 using Korn’s inequality for the $u$ component). For the compactness assumption (1.7), let $X_j = (u_j, \varphi_j)$ be a bounded sequence in $V$. Then $u_j$ is bounded in $H^1(\Omega_f)^3$ and relatively compact in $L^2(\Omega_f)^3$; we note also that $u_j \cdot v = \frac{\partial \varphi_j}{\partial \nu}$ is bounded in $H^{1/2}(\Gamma)$ ($\Gamma = \partial \Omega_f$), and since $\Delta \varphi_j$ is bounded in $L^2(\Omega_f)$, we conclude by using the regularity results for the Neumann problem [1], that $\varphi_j$ is bounded in $H^{3/2}(\Omega_f)$ and relatively compact in $H^1(\Omega_f)$.

We now show that (1.14) is a weak formulation of the problem. We

\[ (\text{this space is a Hilbert space of the natural scalar product} \]

\[ \int_{\Omega_f} \nabla \varphi \cdot \nabla \psi \, dx + \int_{\Omega_f} \Delta \varphi \cdot \Delta \psi \, dx. \]

The condition $\int_{\Omega_f} \varphi \, dx = 0$ has been added to avoid having to take the quotient by $\mathbb{R}$ (see [4]).
assume that $X$ is sufficiently regular (in $x$ and $t$) and satisfies (1.14), i.e.

$$
(2.40) \quad \int_{\Omega_t} \rho_s \, u \psi \, dx + \int_{\Omega_t} \rho_f \, \nabla \psi \cdot \nabla \psi \, dx + \int_{\Omega_t} A \epsilon (u) \cdot \epsilon (v) \, dx +
$$

$$
+ \int_{\Omega_f} c^2 \Delta \psi \, \Delta \psi \, dx = \int_{\Gamma \cup \gamma} F v \, d\Gamma, \quad \forall \, Y = (v, \psi) \in V.
$$

We choose first a test function $Y = (v, \psi)$, with $\psi = 0, v \in C^\infty_0 (\Omega)^3$, and we infer that (2.3) is satisfied; then we choose $Y = (v, \psi), \psi = 0, v \in H^1(\Omega)^3$, and $v \cdot v = 0$ on $\Gamma$; using (2.18) and (2.3) we conclude that

$$
\int_{\Gamma \cup \gamma} \sigma_{ij} v_j v_i \, d\Gamma = \int_{\Gamma \cup \gamma} F_i v_i \, d\Gamma, \quad \forall \, v
$$

and this implies (2.4) and (2.22) (which is part of (2.5)). Then we write (2.40) with $Y = (v, \psi), v = 0, \psi \in C^\infty_0 (\Omega)$:

$$
\int_{\Omega_t} \rho_f \, \nabla \psi \cdot \nabla \psi \, dx + \int_{\Omega_f} c^2 \Delta \psi \cdot \Delta \psi \, dx = 0, \quad \forall \psi;
$$

(2.33) follows. Finally we write (2.40) with a general test function $Y = (v, \psi)$. Using (2.18)-(2.19) and the results already proved we find

$$
\int_{\Gamma} (\sigma \cdot v)_\nu v_\nu \, d\Gamma - \int_{\Gamma} \rho_f \, \psi \frac{\partial \psi}{\partial v} \, d\Gamma = \int_{\Gamma} F_\nu v_\nu \, d\Gamma, \quad \forall \, (v, \psi).
$$

Since $v_\nu = \frac{\partial \psi}{\partial v}$ on $\Gamma$ and this arbitrary, we conclude that

$$
(\sigma \cdot v)_\nu - \rho_f \, \psi = F_\nu \quad \text{on} \quad \Gamma
$$

which together with (2.22) provides (2.5).

All the conditions (2.3)-(2.6) and (2.33) have been proved. Conversely one can reverse easily the steps of the proof and show that if $X$ is regular and satisfies (2.3)-(2.6) and (2.33) then (1.14) is valid.

The proof of the assumptions is complete, we can apply the general results of Section 1.

The spectral problem

There exists a family of elements $Y_j = (v_j, \psi_j)$ which is orthonormal in $H$ and orthogonal in $V$, there exists a sequence of numbers $\lambda_j, \lambda_j \geq 0, \lambda_j \to + \infty$ as $j \to \infty$, such that (2.28)-(2.29) holds and

$$
(2.41) \quad \frac{\partial \psi_j}{\partial v} = u_j \cdot v \quad \text{on} \quad \Gamma
$$

$$
(2.42) \quad c^2 \Delta \psi_j + \lambda_j \psi_j = 0 \quad \text{in} \quad \Omega_f.
$$
The evolution problem

Theorem 1.1 gives:

\textbf{Theorem 2.2:} Given $F \in L^2((0, T) \times \partial \Omega_s)$ and given $u_0$, $u_1$, $\varphi_0$, $\varphi_1$, such that

$$u_0 \in H^1(\Omega_s)^3, \quad u_1 \in L^2(\Omega_s)^3$$

$$\varphi_0, \varphi_1 \in W, \quad \frac{\partial \varphi_i}{\partial n} = u_i \cdot \nu \quad \text{on} \quad \Gamma, \quad i = 0, 1,$$

there exists a unique function $X = X(t) = (u(t), \varphi(t))$ which satisfies (1.16), the initial condition (1.15) and is solution to the evolution problem (2.3) to (2.6) and (2.33).

2.4 Closed shell imbedded in an unlimited fluid

The shell $\Omega_s$ is closed (in the physical sense) and limited by $\gamma$ in its interior and $\Gamma$ in the exterior. The unlimited fluid fills the region outside the shell; we shall denote by $\Omega_i$ the bounded open set $\mathbb{C} \overline{\Omega_f}$ (cf. fig. 2.3).

![Figure 2.3.](image)

The fluid is incompressible, irrotational and at rest at infinity. The equations of motion (2.3)-(2.6) are completed by

\begin{align}
\Delta \varphi & = 0 \quad \text{in} \quad \Omega_f \\
\varphi(x) & = O \left( \frac{1}{|x|} \right) \quad \text{and} \quad |\nabla \varphi(x)| = O \left( \frac{1}{|x|^2} \right) \quad \text{as} \quad |x| \to \infty.
\end{align}

In order to set this problem in the framework of Section 1 we first recall of some results on harmonic functions and introduce the operator $\mathcal{G}$. 

\textit{M$^2$ AN Modélisation mathématique et Analyse numérique}

Mathematical Modelling and Numerical Analysis
The operator \( \mathcal{G} \)

We shall consider functions \( \varphi \) which are defined in \( \Omega_f \) and belong to the space

\[
\varphi \in L^2(\Omega_f), \quad \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega_f), \quad i = 1, 2, 3
\]

which is a Hilbert space for the scalar product [7]:

\[
\int_{\Omega_f} \nabla \varphi \cdot \nabla \psi \, dx.
\]

When \( \varphi \) belongs to the space (2.45) and \( \Delta \varphi = 0 \) then \( \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma} \) makes sense.

(Hereafter we write it as \( \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_f} \)) (6) it belongs to \( H^{-1/2}(\Gamma) \) and (see J. C. Nedelec [9])

\[
\left\| \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_f} \right\|_{H^{-1/2}(\Gamma)} \leq c_2(\Omega_f) |\nabla \varphi|_{L^2(\Omega_f)}
\]

where \( c_2(\Omega_f) \) depends only on \( \Omega_f \). Now we can (and shall) extend such a function \( \varphi \) in \( \Omega \); the trace of \( \varphi \) on \( \Gamma \) exists and belongs to \( H^{1/2}(\Gamma) \) and we define \( \varphi \) in \( \Omega \) by setting

\[
\Delta \varphi = 0 \quad \text{in} \quad \Omega
\]

\[
\varphi |_{\Gamma_i} = \varphi |_{\Gamma_f}
\]

where \( \Gamma_i \) and \( \Gamma_f \) indicate whether \( \Gamma \) is considered as the boundary of \( \Omega_i \) or that of \( \Omega_f \). Using (2.25) we see that \( \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_i} \) makes sense, belongs to \( H^{-1/2}(\Gamma) \) and it is usually different from \( \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_f} \); we shall write

\[
s = \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_i} - \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_f}
\]

We know (see [2, 4]) that the value of \( \varphi \) on \( \Gamma \) can be recovered from \( s \) by an integral

\[
\varphi(y) = \frac{1}{4 \pi} \int_{\Gamma} \frac{s(x)}{|x - y|} d\Gamma(x), \quad \forall y \in \Gamma.
\]

(6) Hereafter we extend such a function \( \varphi \) to the whole space \( \mathbb{R}^3 \), and we distinguish the traces of \( \frac{\partial \varphi}{\partial v} \) on \( \Gamma \), corresponding to \( \Omega_f \left( = \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_f} \right) \) and to \( \Omega_i \left( = \left. \frac{\partial \varphi}{\partial v} \right|_{\Gamma_i} \right) \).

vol. 21, n° 2, 1987
We write $\mathcal{C}s$ the expression on the right side of (2.50), and we have thus a linear mapping

$$
\left( s = \frac{\partial \varphi}{\partial v} \bigg|_{\Gamma_i} - \frac{\partial \varphi}{\partial v} \bigg|_{\Gamma_f} \right) \rightarrow \mathcal{C}s = \varphi \big|_{\Gamma}
$$

from $H^{-\frac{1}{2}}(\Gamma)$ into $H^{\frac{1}{2}}(\Gamma)$. It is easy to see that $\mathcal{C}$ is a linear self-adjoint and coercive operator from $H^{-\frac{1}{2}}(\Gamma)$ into $H^{\frac{1}{2}}(\Gamma)$; indeed if $\psi$ is another function and $r$ is the analogue of $s'(\mathcal{C}r = \psi |_{\Gamma})$, then by Stokes formula $\left( \Delta \varphi = \Delta \psi = 0 \text{ in } \Omega_i \cup \Omega_f \right)$. We have

(2.51) \hspace{1cm} (\mathcal{C}s, r) = \int_{\Gamma} \left( \frac{\partial \varphi}{\partial v} \bigg|_{\Gamma_i} - \frac{\partial \varphi}{\partial v} \bigg|_{\Gamma_f} \right) \psi \, d\Gamma = \int_{\Omega_i \cup \Omega_f} \nabla \varphi \cdot \nabla \psi \, dx

(2.52) \hspace{1cm} (\mathcal{C}s, s) = \int_{\Omega_i \cup \Omega_f} |\nabla \varphi|^2 \, dx

\geq (\text{by } (2.25) \text{ and } (2.47))

\geq \frac{1}{c_2^2} \left| \frac{\partial \varphi}{\partial v} \right|_{H^{-\frac{1}{2}}(\Gamma)}^2 + \frac{1}{c_2^2} \left| \frac{\partial \varphi}{\partial v} \right|_{H^1(\Gamma)}^2

\geq c_2^2 \| s \|^2_{H^{-\frac{1}{2}}(\Gamma)}

for some $c_2 > 0$ depending only on $\Omega_f$.

The spaces and forms

We now choose the spaces and forms:

(2.53) \hspace{1cm} \mathcal{V} = H^1(\Omega_i)^3 \times H^1(\Omega_i)/\mathbb{R}

(2.54) \hspace{1cm} \mathcal{H} = L^2(\Omega_i)^3 \times H^1(\Omega_i)/\mathbb{R}

(2.55) \hspace{1cm} V = \left\{ X = (u, \varphi) \in \mathcal{V}, \Delta \varphi = 0 \text{ in } \Omega_f, \frac{\partial \varphi}{\partial v} \bigg|_{\Gamma_i} = u \cdot v \text{ on } \Gamma \right\} \quad (\ast) .

(2.56) \hspace{1cm} a(X, Y) = \int_{\Omega_i} A \varepsilon(u) \cdot \varepsilon(u) \, dx ,

\forall X, Y \in V, \quad X = (u, \varphi), \quad Y = (v, \psi)

(2.57) \hspace{1cm} b(X, Y) = \int_{\Omega_i} \rho_s u v \, dx + \int_{\Omega_f} \rho_f \nabla \varphi \cdot \nabla \psi \, dx ,

\forall X, Y \in H, \quad X = (u, \varphi), \quad Y = (v, \psi)

(2.58) \hspace{1cm} (L, Y) = \int_{\Gamma \cup \gamma} F v \, d\Gamma , \quad \forall Y = (v, \psi) \in V .

\(^{(\ast)}\) The functions $\varphi$ defined on $\Omega_i$ are now extended to harmonic functions on $\Omega_f$ in a way totally similar to (2.48).
It is easy to check that $a$ is bilinear continuous on $V$. For $b$, the continuity and coercivity on $L^2(\Omega_i)^3$ of the term involving $u, v$ is clear but we have to study the term involving

\[(2.59) \quad \int_{\Omega_f} \nabla \varphi \nabla \psi \, dx .\]

These results follow from

**Lemma 2.1**: The form (2.59) is bilinear continuous coercive on the space $W/\mathbb{R}$:

\[W = \{ \varphi \in H^1(\Omega_i), \Delta \varphi = 0 \text{ in } \Omega_i \} .\]

**Proof**: As indicated above the functions in $W$ are extended to $\Omega_f$ as harmonic functions, and it suffices to show that

\[|\nabla \varphi|_{\Omega_f}^{L^2(\Omega_f)^3}\]

defines on $W/\mathbb{R}$ a norm equivalent to the natural norm

\[|\nabla \varphi|_{\Omega_i}^{L^2(\Omega_i)^3} .\]

By the Stokes formula and (2.47)

\[(2.60) \quad |\nabla \varphi|_{L^2(\Omega_f)^3} \leq c_2(\Omega_f) \| \varphi \|_{H^{1/2}(\Gamma)} .\]

Also since

\[\int_{\Omega_f} \Delta \varphi \, dx = \int_{\Gamma} \frac{\partial \varphi}{\partial \nu} \bigg|_{\Gamma_f} \, d\Gamma = 0 ,\]

we can replace $\varphi|_{\Gamma}$ by $\varphi + C$, $C \in \mathbb{R}$ and we can improve (2.60) as

\[(2.61) \quad |\nabla \varphi|_{L^2(\Omega_f)} \leq c_2(\Omega_f) \| \varphi \|_{H^{1/2}(\Gamma)} \| /{\mathbb{R}} .\]

By the trace theorem for $H^1(\Omega_i)$ we then have

\[(2.62) \quad \| \varphi \|_{H^{1/2}(\Gamma) / \mathbb{R}} \leq c_3(\Omega_i) |\nabla \varphi|_{L^2(\Omega_i)^3} .\]
for some \( c_3(\Omega_i) \) depending on \( \Omega_i \). Hence

\[
\| \nabla \varphi \|_{L^2(\Omega_i)} \leq c_2 c_3 \| \nabla \varphi \|_{L^2(\Omega_i)^3}.
\]

For the opposite inequality we note that

\[
\| \nabla \varphi \|_{L^2(\Omega_i)} \leq c_4(\Omega_i) \| \varphi \|_{H^{1/2}(\Gamma)}
\]

\[
| \nabla \varphi |_{L^2(\Omega_i)^3} \leq c_4(\Omega_i) \| \varphi \|_{H^{1/2}(\Gamma) / \mathbb{R}}
\]

because of the results on the regularity of the solution of the Dirichlet problem in \( \Omega_i, [1] \). Then let \( \Theta \) be the intersection of \( \Omega_i \) with a ball sufficiently large so that \( \bar{\Omega}_i \) is included in this ball. Then \( \Gamma \) is a connected component of \( \partial \Theta \) and by the trace theorem in \( H^1(\Theta) [7] \),

\[
\| \varphi \|_{H^{1/2}(\Gamma)} \leq c_5(\Theta) | \nabla \varphi |_{L^2(\Theta)^3}
\]

\[
\| \varphi \|_{H^{1/2}(\Gamma) / \mathbb{R}} \leq c_5 | \nabla \varphi |_{L^2(\Theta)^3}.
\]

By combining this with (2.64),

\[
| \nabla \varphi |_{L^2(\Omega_i)^3} \leq c_4 c_5 | \nabla \varphi |_{L^2(\Omega_i)^3}
\]

and the lemma is proved

Q.E.D.

Finally we have to prove that \( a + \lambda b \) is coercive on \( V \), \( \forall \lambda > 0 \) and that the injection of \( V \) into \( H \) is compact. We have already observed several times that

\[
\lambda \int_{\Omega_i} \rho \rho \ dx + \int_{\Omega_i} A \varepsilon (u) \cdot \varepsilon (u) \ dx
\]

is coercive on \( H^1(\Omega_i)^3 \), while the coercivity of (2.59) on \( H^1(\Omega_i) / \mathbb{R} \) is again provided by Lemma 2.1.

Concerning (1.7), let \( X_j = (u_j, \varphi_j) \) be a bounded sequence in \( V \). Then \( u_j \) is bounded in \( H^1(\Omega_i)^3 \), relatively compact in \( L^2(\Omega_i) \). Also \( \Delta \varphi_j = 0 \) in \( \Omega_f, \frac{\partial \varphi_j}{\partial \nu} |_{r_f} = u_j \cdot v \) is bounded in \( H^{1/2}(\Gamma) \). By the regularity results for the Neumann problem in \( \Omega_f, [1] \), \( \varphi_j \) is in \( H^2(\Theta) \) for every bounded subset \( \Theta \) of \( \Omega_f \), and

\[
\| \varphi_j \|_{H^2(\Theta)} \leq c_6(\Theta, \Omega_f) \left\{ \left\| \frac{\partial \varphi_j}{\partial \nu} |_{r_f} \right\|_{H^{1/2}(\Gamma)} + | \Delta \varphi_j |_{L^2(\Omega_f)} \right\}.
\]

Thus \( \varphi_j \big|_{\Theta} \) is bounded in \( H^2(\Theta) / \mathbb{R} \) and \( \varphi_j \big|_{\Gamma} \) is bounded in \( H^{3/2}(\Gamma) \) and finally

M² AN Modélisation mathématique et Analyse numérique
Mathematical Modelling and Numerical Analysis
\( \varphi|_{\Omega_i} \) belongs to \( H^2(\Omega_i) \) and is bounded in \( H^2(\Omega_i)/\mathbb{R} \). The sequence \( \varphi|_{\Omega_i} \) is then relatively compact in \( H^1(\Omega_i)/\mathbb{R} \).

**Equivalence with (1.14)**

We now show that (1.14) is equivalent to our present problem, namely (2.3)-(2.6) and (2.43)-(2.44).

Let \( X = X(t) \) be a regular solution of (1.14) which we write explicitly

\[
\int_{\Omega_i} \rho_s \, u \, v \, dx + \int_{\Omega_f} \rho_f \, \nabla \varphi \, \nabla v \, dx + \int_{\Omega_i} A\varepsilon(u) \cdot \varepsilon(v) \, dx = 0
\]

Equivalence with (1.14)

We now show that (1.14) is equivalent to our present problem, namely (2.3)-(2.6) and (2.43)-(2.44).

Let \( X = X(t) \) be a regular solution of (1.14) which we write explicitly

\[
(2.67) \quad \int_{\Omega_i} \rho_s \, u \, v \, dx + \int_{\Omega_f} \rho_f \, \nabla \varphi \, \nabla v \, dx + \int_{\Omega_i} A\varepsilon(u) \cdot \varepsilon(v) \, dx = 0
\]

As in the previous cases we first choose a test function \( Y = (v, \psi) \) with \( \psi = 0 \) and \( v \in \mathcal{D}'(\Omega_f) \), and we obtain (2.3) with \( \sigma = \sigma(u) = A\varepsilon(u) \). Then we take \( v \in H^1(\Omega_s) \), \( v \cdot v = 0 \) on \( \Gamma \) and \( \psi = 0 \). Using (2.3) and (2.18) we find (2.4) and (2.22). The next step is to write (2.67) with an arbitrary test function \( Y = (v, \psi) \) and use (2.19) and the results already proved; we obtain

\[
\int_{\Gamma} (\sigma \cdot v)_v \, d\Gamma - \int_{\Gamma} \rho_f \, \frac{\partial \psi}{\partial v} \bigg|_{\Gamma_f} \, d\Gamma = \int_{\Gamma} F_v \, v \, d\Gamma.
\]

Since \( v = \frac{\partial \psi}{\partial v} \bigg|_{\Gamma_f} \) and this function is arbitrary, we find

\[
(\sigma \cdot v)_v - \rho_f \, \psi = F_v \quad \text{on} \quad \Gamma
\]

and this (with (2.22)) completes the proof of (2.5). Therefore (2.3)-(2.6) and (2.43)-(2.44) are satisfied. Conversely it is elementary to show that a smooth function \( X = X(t) \) satisfying (2.3)-(2.6) and (2.43)-(2.44) is solution of (1.14); the equivalence is proved.

We are now able to apply the general results of Section 1. We obtain:

**The spectral problem**

There exists a sequence of elements \( Y_j = (v_j, \psi_j) \) which is orthonormal in \( H \) and orthogonal in \( V \), there exists a sequence of numbers \( \lambda_j, \lambda_j \geq 0 \), \( \lambda_j \to \infty \), such that

\[
(2.68) \quad \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} (v_j) = -\lambda_j \rho_s (v_j)_h \quad \text{in} \quad \Omega_s
\]

\[
(2.69) \quad \sum_{k=1}^{3} \sigma_{ik}(v_j) \nu_k = \begin{cases} 0 & \text{on} \quad \gamma \\ -\rho_f \lambda_j \psi_j v_i & \text{on} \quad \Gamma \end{cases}
\]
(2.70) \( \frac{\partial \psi_j}{\partial n} \bigg|_{\Gamma_f} = u_j \cdot v \) on \( \Gamma \)

(2.71) \( \Delta \psi_j = 0 \) in \( \Omega_f \) (and \( \Omega_i \)).

The evolution problem

Theorem 1.1 gives:

**Theorem 2.3:** Given \( F \in L^2((0, T) \times \partial \Omega_s) \), and given \( u_0, u_1, \varphi_0, \varphi_1 \) such that

\[
\begin{align*}
  u_0 &\in H^1(\Omega_s)^3, &  u_1 &\in L^2(\Omega_s)^3 \\
  \varphi_0, & \varphi_1 \in H^1(\Omega_i) \\
  \Delta \varphi_j &= 0 \text{ in } \Omega_i, & \frac{\partial \varphi_j}{\partial n} \bigg|_{\Gamma_f} &= u_j \cdot v \text{ on } \Gamma, & j &= 0, 1
\end{align*}
\]

there exists a unique function \( X = X(t) = (u(t), \varphi(t)) \) which satisfies (1.16), (1.15) and is solution to the evolution problem (2.3) to (2.6) and (2.43)-(2.44).

**REFERENCES**


---

The extension of the \( \varphi_j \) to \( \Omega_f \) as above is clear.