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SUPERCONVERGENCE OF MIXED FINITE ELEMENT METHODS FOR PARABOLIC EQUATIONS (*)

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Abstract. — *An asymptotic expansion of the mixed finite element solution of a linear parabolic problem is used to derive superconvergence results. Optimal order error estimates in Sobolev spaces of negative index are also shown.*

Résumé. — *Un développement asymptotique de la solution par élément fini mixte d'un problème parabolique linéaire est utilisé pour obtenir des résultats de superconvergence. On obtient aussi des estimations d'erreur d'ordre optimal dans les espaces de Sobolev d'exposant négatif.*

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the linear parabolic problem

$$(1.1)\quad \begin{aligned} &(a)\ d\frac{\partial p}{\partial t} - \nabla.(a\,\nabla p + bp) + cp = f\,, && x \in \Omega, t \in J\,,\\ &(b) \qquad\qquad\qquad\qquad p = g\,, && x \in \partial\Omega, t \in J\,,\\ &(c) \qquad\qquad\qquad\qquad p = p_0\,, && x \in \Omega, T = 0\,, \end{aligned}$$

where $J = (0, T)$, and a, b, c are smooth functions of x alone such that a and d are bounded below by a positive constant. Assume that the elliptic part of the operator is invertible.

Denote by $(,)$ the natural inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$, and by $\langle,\rangle$ the one in $L^2(\partial\Omega)$. Let

$$V = H(\text{div}, \Omega) = \{q \in L^2(\Omega)^2 : \text{div}\, q \in L^2(\Omega)\}\ ,$$

with the norm

$$\|q\|_V^2 = \|q\|^2 + \|\text{div}\, q\|^2 = \|q\|^2_{L^2(\Omega)^2} + \|\text{div}\, q\|^2_{L^2(\Omega)}\ ,$$

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and $W = L^2(\Omega)$. Form the parabolic mixed method as follows. Set

$$u = -(a\,\nabla p + bp)\,.$$

Then, if $\alpha = a^{-1}$ and $\beta = a^{-1}b$, integration by parts in the relation

$$(\alpha u + \nabla p + \beta p, v) = 0\,, \qquad v \in V\,,$$

gives

$$(1.2) \qquad (\alpha u, v) - (\operatorname{div} v, p) + (\beta p, v) = -\langle g, v \cdot \nu\rangle\,, \quad v \in V\,,$$

where ν denotes the unit outward normal vector to $\partial\Omega$. The partial differential equation is represented by the weak form

$$(1.3) \qquad \left(d\,\frac{\partial p}{\partial t}, w\right) + (\operatorname{div} u, w) + (cp, w) = (f, w)\,, \quad w \in W\,.$$

In order to define the mixed finite element method for (1.1) let $V_h \times W_h$ denote the Raviart-Thomas-Nedelec space [11, 12, 14] of index $k \geqslant 0$ associated with a quasi-regular partition $\mathfrak{J}_h$ of Ω that is such that

i) if $T \in \mathfrak{J}_h$, T is either a triangle or a rectangle,
ii) if $T \subset \Omega$, T has straight edges,
iii) if T is a boundary triangle or rectangle, the boundary edge can be curved,
iv) all vertex angles are bounded below by a positive constant,
v) diam $(T) = h_T$, $\max\limits_T h_T = h$,
vi) rectangles : ratio of edges bounded.

The definition of $V_h \times W_h$ is as follows. Let $P_k(T)$ denote the restriction of the polynomials of total degree k to the set T and let $P_{k,l}(T)$ denote the restriction of $P_k(R) \otimes P_l(R)$ to T. If T is a triangle, let

$$\begin{aligned} V(T) &= P_k(T)^2 \oplus \operatorname{span}\,(xP_k(T))\,, \\ W(T) &= P_k(T)\,, \end{aligned}$$

where $x = (x_1, x_2)$. Similarly, if T is a rectangle, let

$$\begin{aligned} V(T) &= P_{k+1,k}(T) \times P_{k,k+1}(T) \\ W(T) &= P_{k,k}(T)\,. \end{aligned}$$

Then, set

$$\begin{aligned} V_h &= V(k, \mathfrak{J}_h) = \{v \in V : v|_T \in V(T), T \in \mathfrak{J}_h\}\,, \\ W_h &= W(k, \mathfrak{J}_h) = \{w \in W : w|_T \in W(T), T \in \mathfrak{J}_h\}\,. \end{aligned}$$

Note that $\operatorname{div} V_h = W_h$, and

$$V_h = \left\{ v \in \prod_{T \in \mathfrak{J}_h} V(T) : v|_{T_i} \cdot \nu_i + v|_{T_j} \cdot \nu_j = 0, \text{on } \bar{T}_i \cap \bar{T}_j \right\},$$

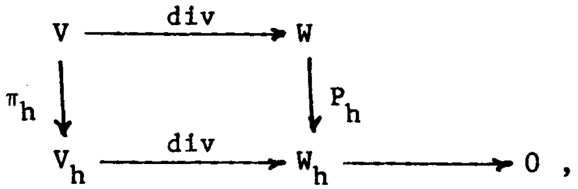
where ν_l is the outer normal to T_l. The above definition of $V_h \times W_h$ coincides with that of Raviart and Thomas [12] for rectangular elements and is the modification due to Nedelec [11] on triangular elements. Next, consider the projection

$$\pi_h \times P_h : V \times W \to V_h \times W_h$$

defined by Raviart and Thomas for polygonal domains [12] and which satisfies

(a) P_h is the $L^2(\Omega)$-projection ;

(b) the following diagram commutes :



i.e., $\operatorname{div} \pi_h = P_h \operatorname{div} : V \xrightarrow{\text{onto}} W_h$

(c) the following approximation properties hold :

$$(1.4) \quad \begin{aligned} &\text{(i)} && \|u - \pi_h u\| \leq Q \|u\|_r h^r, && 1 \leq r \leq k+1, \\ &\text{(ii)} && \|\operatorname{div}(u - \pi_h u)\|_{-s} \leq Q \|\operatorname{div} u\|_r h^{r+s}, && 0 \leq r, s \leq k+1, \\ &\text{(iii)} && \|p - P_h p\|_{-s} \leq Q \|p\|_r h^{r+s}, && 0 \leq r, s \leq k+1. \end{aligned}$$

Arnold-Douglas-Roberts [8] have introduced a modification on the definition of π_h in order to include boundary triangles with curved edges.

The semidiscrete mixed finite element method for (1.1) consists of finding $\{u_h, p_h\} : J \to V_h \times W_h$ such that, for $t \in J$,

$$(1.5) \quad \begin{aligned} &(a) \quad (\alpha u_h, v) - (\operatorname{div} v, p_h) + (\beta p_h, v) = - \langle g, v \cdot \nu \rangle, \quad v \in V_h, \\ &(b) \quad \left(d \frac{\partial p_h}{\partial t}, w \right) + (\operatorname{div} u_h, w) + (c p_h, w) = (f, w), \quad w \in W_h. \end{aligned}$$

It is also necessary to specify the initial value for p_h (as those for u_h then follow from (1.5a)). This specification will be done later.

The main objective of this work is the establishment of superconvergence of the solution of a semidiscrete mixed finite element method to the solution of the linear parabolic problem in $\mathbb{R}^2$. In a single space variable, knot superconvergence has been demonstrated for semidiscrete Galerkin approximation of solutions of linear parabolic problems by Douglas-Dupont-Wheeler [5]. The basic tool they have used consists of an asymptotic expansion of the Galerkin solution by means of a sequence of elliptic projections they have called a quasi-projection. Arnold and Douglas [1] have carried out the results for quasilinear parabolic equations. Using a different approach Thomée [15] has also analysed superconvergence phenomena in Galerkin methods for parabolic problems in $\mathbb{R}^n$. In this work, a quasi-projection for mixed methods for linear parabolic problems is introduced and then used to produce asymptotic expansions to high order of the mixed method solution. Superconvergence is then derived by post-processing.

Note that by using this post-processing procedure the number of parameters involved in the calculation of u_h and p_h is much smaller than it would be in order to obtain the same accuracy by either increasing the index of $V_h \times W_h$ or by reducing h. Thus, the cost of obtaining a given accuracy is much reduced by the employment of the post-processing, which is quite inexpensive in comparison to the evaluation of u_h and p_h.

A brief outline of this paper is as follows. In sections 2, 3 and 4 the quasi-projection for parabolic mixed methods for problems in $\mathbb{R}^2$ with Dirichlet boundary conditions is defined and optimal order estimates in Sobolev spaces of negative index are derived for the approximations to the solution and its associated flow field. In sections 5 and 6, following Bramble and Schatz [2, 3], an averaging operator is introduced and estimates on difference quotients of the error are derived. These estimates then imply the superconvergence error estimates for the post-processed approximations.

2. FORMULATION OF THE MIXED METHOD QUASI-PROJECTION

Let

$$\zeta = p - p_h \quad \text{and} \quad \sigma = u - u_h .$$

Then

$$(2.1) \quad \begin{aligned} &(a)\ (\alpha\sigma, v) - (\operatorname{div} v, \zeta) + (\beta\zeta, v) = 0 , \qquad v \in V_h, \\ &(b)\ \left(d \frac{\partial \zeta}{\partial t}, w\right) + (\operatorname{div} \sigma, w) + (c\zeta, w) = 0 , \quad w \in W_h . \end{aligned}$$

Next, let $\{\tilde{u}_h, \tilde{p}_h\} : J \to V_h \times W_h$ denote the elliptic mixed method projection of the solution $\{u, p\}$ into $V_h \times W_h$: if

$$\eta = p - \tilde{p}_h \quad \text{and} \quad \xi = u - \tilde{u}_h , \tag{2.2}$$

then

$$\text{(2.3)} \quad \begin{aligned} &(a) \quad (\alpha\xi, v) - (\operatorname{div} v, \eta) + (\beta\eta, v) = 0 , \quad v \in V_h , \\ &(b) \quad (\operatorname{div} \xi, w) + (c\eta, w) = 0 , \quad w \in W_h . \end{aligned}$$

Let s be a nonnegative integer and let $H^s(\Omega)$ be the usual Sobolev space ; i.e., the set of all functions in $L^2(\Omega)$ whose distributional derivatives of order not greater than s are also in $L^2(\Omega)$; $H^s(\Omega)$ is normed by

$$\|w\|_s^2 = \sum_{|\alpha| \leq s} \|D^\alpha w\|^2 ,$$

where $\alpha = (\alpha_1, \alpha_2)$, α_i a nonnegative integer,

$$|\alpha| = \alpha_1 + \alpha_2 , \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \, \partial x_2^{\alpha_2}} ,$$

and

$$\|z\|^2 = \|z\|_{L^2(\Omega)}^2 = \int_\Omega z^2 \, dx .$$

Consider also the dual space of $H^s(\Omega)$, denoted by $H^{-s}(\Omega)$, normed by

$$\|\varphi\|_{-s} = \sup \left\{ (\varphi, \psi)/\|\psi\|_s : \psi \in H^s(\Omega), \ \|\psi\|_s \neq 0 \right\} .$$

The object of a quasi-projection is to produce an expansion of ζ and σ that begins with the pair η, ξ and then has terms decreasing in magnitude by a factor of h^2 until the limit of the negative norm estimates for the corresponding elliptic mixed method is reached. Its construction is as follows. First, let

$$\eta_0 = \tilde{p}_h - p_h \quad \text{and} \quad \xi_0 = \tilde{u}_h - u_h .$$

Subtract (2.3) from (2.1)

$$\text{(2.4)} \quad \begin{aligned} &(a) \ (\alpha\xi_0, v) - (\operatorname{div} v, \eta_0) + (\beta\eta_0, v) = 0 , \quad v \in V_h, \\ &(b) \ \left(d \frac{\partial \eta_0}{\partial t}, w \right) + (\operatorname{div} \xi_0, w) + (c\eta_0, w) = - \left(d \frac{\partial \eta}{\partial t}, w \right), \ w \in W_h. \end{aligned}$$

Next, let $\{\xi_1, \eta_1\} : J \to V_h \times W_h$ be defined by

$$(2.5)\qquad \begin{aligned} &(a)\ \ (\alpha\xi_1, v) - (\operatorname{div} v, \eta_1) + (\beta\eta_1, v) = 0, && v \in V_h\,,\\ &(b)\ \ (\operatorname{div} \xi_1, w) + (c\eta_1, w) = \left(d\,\frac{\partial\eta}{\partial t}, w\right), && w \in W_h\,. \end{aligned}$$

Recursively define $\{\xi_j, \eta_j\} : J \to V_h \times W_h$ by

$$(2.6)\qquad \begin{aligned} &(a)\ \ (\alpha\xi_j, v) - (\operatorname{div} v, \eta_j) + (\beta\eta_j, v) = 0\,, && v \in V_h\,,\\ &(b)\ \ (\operatorname{div} \xi_j, w) + (c\eta_j, w) = -\left(d\,\frac{\partial\eta_{j-1}}{\partial t}, w\right), && w \in W_h\,, \end{aligned}$$

for $j = 2, 3, \ldots$.

Let

$$(2.7)\qquad \begin{aligned} (a)\quad & p_0 = \tilde{p}_h\,,\quad p_j = \tilde{p}_h + \eta_1 + \cdots + \eta_j\,,\\ & \theta_0 = p_0 - p_h = \eta_0\,,\quad \theta_j = p_j - p_h = \sum_{i=0}^{j} \eta_i\,,\\ (b)\quad & u_0 = \tilde{u}_h\,,\quad u_j = \tilde{u}_h + \xi_1 + \cdots + \xi_j\,,\\ & \psi_0 = u_0 - u_h = \xi_0\,,\quad \psi_j = u_j - u_h = \sum_{i=0}^{j} \xi_i\,, \end{aligned}$$

so that

$$(2.8)\qquad p - p_h = \eta - \sum_{i=1}^{j} \eta_i + \theta_j\,,\quad u - u_h = \xi - \sum_{i=1}^{j} \xi_i + \psi_j\,.$$

It follows inductively that

$$(2.9)\qquad \begin{aligned} &(a)\ \ (\alpha\psi_j, v) - (\operatorname{div} v, \theta_j) + (\beta\theta_j, v) = 0\,, && v \in V_h\,,\\ &(b)\ \ \left(d\,\frac{\partial\theta_j}{\partial t}, w\right) + (\operatorname{div} \psi_j, w) + (c\theta_j, w) = \left(d\,\frac{\partial\eta_j}{\partial t}, w\right), && w \in W_h\,. \end{aligned}$$

The expansions p_j and u_j are called *quasi-projections* and the terms θ_j and ψ_j are the *residuals*.

The objective now is to carry out negative norm estimates for η, ξ, η_j, and ξ_j. It will be necessary to consider their time-derivatives as well, but the independence of the coefficients of the time variable will make this easy.

The arguments will involve duality and will follow the development by Douglas and Roberts [8] in obtaining global error estimates for the correspondent linear elliptic problem. Their argument has been based on the duality lemma to follow.

LEMMA 2.1 : *Let s be a nonnegative integer and let the index k of $V_h \times W_h$ be at least one. Assume that Ω is $(s+2)$-regular. Let $\tau \in V$, $F \in V'$ and $G \in W'$. If F has the form*

$$F(v) = (F_0, v) + (F_1, \operatorname{div} v), \quad v \in V,$$

and if $z \in W_h$ satisfies

$$(2.10) \quad \begin{aligned} &(a) \ (\alpha\tau, v) - (\operatorname{div} v, z) + (\beta z, v) = F(v), \quad v \in V_h, \\ &(b) \ (\operatorname{div} \tau, w) + (cz, w) = G(w), \qquad w \in W_h, \end{aligned}$$

then

$$\|z\|_{-s} \leqslant Q\{h^{\min(s+1, k+1)}[\|\tau\| + \|F_0\|] + h^{\min(s, k+1)}\|F_1\| + \\ + h^{\min(s+2, k+1)}[\|\operatorname{div} \tau\| + \|G\|] + \|F_0\|_{-s-1} + \|F_1\|_{-s} + \|G\|_{-s-2}\}.$$

The main theorem in Douglas and Roberts [8] of interest here is the following :

THEOREM 2.2 : *Let s be a nonnegative integer and let η and ξ be given by (2.2). Then,*

(*a*) $\|\eta\|_{-s} \leqslant Qh^{r+\min(s, k+1)}\|p\|_{r+(s-k+1)^+}$,
for $2-(s-k+1)^+ \leqslant r \leqslant k+1$;

(*b*) $\|\xi\|_{-s} \leqslant Qh^{r+\min(s, k+1)}\|p\|_{r+1+(s-k)^+}$,
for $1-(s-k)^+ \leqslant r \leqslant k+1$;

(*c*) $\|\operatorname{div} \xi\|_{-s} \leqslant Qh^{r+s}\|p\|_{r+2}$, *for* $0 \leqslant r \leqslant k+1$.

3. A PRIORI ESTIMATES FOR THE QUASI-PROJECTION

Assume that s is a nonnegative integer throughout this section, that the index k is odd and satisfies $k \geqslant 1$, and that Ω is $(s+2)$-regular. Lemma 2.1 implies that,

$$(3.1) \quad \|\eta_j\|_{-s} \leqslant Q\left\{h^{\min(s+1, k+1)}\|\xi_j\| + \\ + h^{\min(s+2, k+1)}\left[\|\operatorname{div} \xi_j\| + \left\|\frac{\partial \eta_{j-1}}{\partial t}\right\|\right] + \left\|\frac{\partial \eta_{j-1}}{\partial t}\right\|_{-s-2}\right\},$$

for $j = 1, 2, \ldots,$ and

$$(3.2)\quad \|\eta_1\|_{-s} \leqslant Q\left\{h^{\min(s+1,k+1)}\|\xi_1\| + h^{\min(s+2,k+1)}\left[\|\operatorname{div}\xi_1\| + \left\|\frac{\partial\eta}{\partial t}\right\|\right] + \left\|\frac{\partial\eta}{\partial t}\right\|_{-s-2}\right\}.$$

If the choice $s = 0$ is made in (3.1) and (3.2), then

$$(3.3)\quad \|\eta_j\| \leqslant Q\left\{h\|\xi_j\| + h^2\left(\|\operatorname{div}\xi_j\| + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|\right) + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-2}\right\}$$

for $j = 2, 3, \ldots,$ and

$$(3.4)\quad \|\eta_1\| \leqslant Q\left\{h\|\xi_1\| + h^2\left(\|\operatorname{div}\xi_1\| + \left\|\frac{\partial\eta}{\partial t}\right\|\right) + \left\|\frac{\partial\eta}{\partial t}\right\|_{-2}\right\}.$$

Now take $w = \operatorname{div}\xi_j$ in (2.7b) and $w = \operatorname{div}\xi_1$ in (2.5b) :

$$(3.5)\quad \|\operatorname{div}\xi_j\| \leqslant Q\begin{cases}\left(\|\eta_j\| + \left\|\dfrac{\partial\eta_{j-1}}{\partial t}\right\|\right), & j = 2, 3, \ldots, \\ \left(\|\eta_1\| + \left\|\dfrac{\partial\eta}{\partial t}\right\|\right), & j = 1.\end{cases}$$

Next, choosing the test function $v = \xi_j$ in (2.6a) or $v = \xi_1$ in (2.5a) leads easily to

$$(3.6)\quad \|\xi_j\|^2 \leqslant Q\{\|\operatorname{div}\xi_j\|\,\|\eta_j\| + \|\eta_j\|^2\}, \quad \text{for} \quad j = 1, 2, \ldots,$$

and it follows from (3.5) that

$$(3.7)\quad \|\xi_j\| \leqslant Q\begin{cases}\left(\|\eta_j\| + \|\eta_j\|^{1/2}\left\|\dfrac{\partial\eta_{j-1}}{\partial t}\right\|^{1/2}\right), & j = 2, 3, \ldots, \\ \left(\|\eta_1\| + \|\eta_1\|^{1/2}\left\|\dfrac{\partial\eta}{\partial t}\right\|^{1/2}\right), & j = 1.\end{cases}$$

Now, it follows from (3.3), (3.5) and (3.7) that

$$(3.8)\quad \|\eta_j\| \leqslant Q\left\{h^2\left\|\frac{\partial\eta_{j-1}}{\partial t}\right\| + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-2}\right\}, \quad j = 2, 3, \ldots$$

Similarly,

$$(3.9)\quad \|\eta_1\| \leqslant Q\left\{h^2\left\|\frac{\partial\eta}{\partial t}\right\| + \left\|\frac{\partial\eta}{\partial t}\right\|_{-2}\right\}.$$

Thus,

$$\text{(3.10)}\quad \|\xi_j\| \leqslant Q \begin{cases} \left(h\left\|\dfrac{\partial\eta_{j-1}}{\partial t}\right\| + h^{-1}\left\|\dfrac{\partial\eta_{j-1}}{\partial t}\right\|_{-2}\right), & j = 2, 3, \dots, \\ \left(h\left\|\dfrac{\partial\eta}{\partial t}\right\| + h^{-1}\left\|\dfrac{\partial\eta}{\partial t}\right\|_{-2}\right), & j = 1 . \end{cases}$$

The bounds given by (3.5) and (3.10) can be applied to (3.1) and (3.2) to obtain the inequalities

$$\text{(3.11)}\quad \begin{aligned} \|\eta_j\|_{-s} \leqslant Q \Big\{ & h^{\min(s, k+1)}\left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-2} + \\ & + h^{\min(s+1, k+1)}\|\eta_j\| + h^{\min(s+2, k+1)}\left\|\frac{\partial\eta_{j-1}}{\partial t}\right\| \\ & + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-s-2}\Big\}, \quad j = 2, 3, \dots, \end{aligned}$$

and

$$\text{(3.12)}\quad \begin{aligned} \|\eta_1\|_{-s} \leqslant Q \Big\{ & h^{\min(s, k+1)}\left\|\frac{\partial\eta}{\partial t}\right\|_{-2} + h^{\min(s+1, k+1)}\|\eta_1\| + \\ & + h^{\min(s+2, k+1)}\left\|\frac{\partial\eta}{\partial t}\right\| + \left\|\frac{\partial\eta}{\partial t}\right\|_{-s-2}\Big\} . \end{aligned}$$

First consider the estimation of η_1. Theorem 2.2 implies that

$$\text{(3.13)}\quad \left\|\frac{\partial\eta}{\partial t}\right\|_{-s} \leqslant Qh^{r+\min(s, k+1)}\left\|\frac{\partial p}{\partial t}\right\|_{r+(s+1-k)^+}, \quad \text{for } 2-(s-k+1)^+ \leqslant r \leqslant k+1 .$$

Thus,

$$\text{(3.14)}\quad \|\eta_1\| \leqslant Qh^{r+2}\left\|\frac{\partial p}{\partial t}\right\|_{r+(3-k)^+}, \quad \text{for } 2 \leqslant r \leqslant k+1 .$$

Next, it follows from (3.12), (3.13), and (3.14) that

$$\text{(3.15)}\quad \|\eta_1\|_{-s} \leqslant Qh^{r+\min(s+2, k+1)}\left\|\frac{\partial p}{\partial t}\right\|_{r+(s+3-k)^+}, \quad \text{for } 2 \leqslant r \leqslant k+1 .$$

Similarly,

$$\text{(3.16)}\quad \left\|\frac{\partial^l\eta_1}{\partial t^l}\right\|_{-s} \leqslant Qh^{r+\min(s+2, k+1)}\left\|\frac{\partial^{l+1} p}{\partial t^{l+1}}\right\|_{r+(s+3-k)^+}, \quad \text{for } 2 \leqslant r \leqslant k+1 ,$$

and l a positive integer. Next, consider the estimation of $\dfrac{\partial^l \eta_j}{\partial t^l}$.

Assume from now on that j is a positive integer such that $2j \leq k+1$.

LEMMA 3.1 :

$$\left\| \frac{\partial^l \eta_j}{\partial t^l} \right\|_{-s} \leq Q h^{r + \min(s+2j, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (s+2j+1-k)^+}, \quad \text{for } 2 \leq r \leq k+1 . \tag{3.17}$$

Proof: The proof will proceed by induction. The case $j = 1$ is just (3.16). So, assume that $j \geq 2$ and (3.17) holds for $j-1$. First, it follows from (3.11) and differentiation with respect to time l times that

$$\left\| \frac{\partial^l \eta_j}{\partial t^l} \right\|_{-s} \leq Q \left\{ h^{\min(s, k+1)} \left\| \frac{\partial^{l+1} \eta_{j-1}}{\partial t^{l+1}} \right\|_{-2} + h^{\min(s+1, k+1)} \left\| \frac{\partial^l \eta_j}{\partial t^l} \right\| + h^{\min(s+2, k+1)} \left\| \frac{\partial^{l+1} \eta_{j-1}}{\partial t^{l+1}} \right\| + \left\| \frac{\partial^{l+1} \eta_{j-1}}{\partial t^{l+1}} \right\|_{-s-2} \right\}. \tag{3.18}$$

If the choice $s = 0$ is made in (3.18), then

$$\left\| \frac{\partial^l \eta_j}{\partial t^l} \right\| \leq Q \left\{ \left\| \frac{\partial^{l+1} \eta_{j-1}}{\partial t^{l+1}} \right\|_{-2} + h^2 \left\| \frac{\partial^{l+1} \eta_{j-1}}{\partial t^{l+1}} \right\| \right\}, \tag{3.19}$$

so that

$$\left\| \frac{\partial^l \eta_j}{\partial t^l} \right\| \leq Q h^{r+2j} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (2j+1-k)^+}, \quad 2 \leq r \leq k+1 . \tag{3.20}$$

Next, (3.18), (3.20), and the induction hypothesis imply that

$$\begin{aligned} \left\| \frac{\partial^l \eta_j}{\partial t^l} \right\|_{-s} \leq Q \Big\{ & h^{r+2j+\min(s, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (2j+1-k)^+} \\ & + h^{r+2j+\min(s+1, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (2j+1-k)^+} \\ & + h^{r+2j-2+\min(s+2, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (2j-1-k)^+} \\ & + h^{r+\min(s+2j, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (s+2j+1-k)^+} \Big\} \\ \leq {} & Q h^{r+\min(s+2j, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (s+2j+1-k)^+}, \end{aligned}$$

and the lemma is proved.

In order to estimate $\|\xi_j\|_{-s}$, let $\gamma \in H^s(\Omega)^2$ for some integer $s \geqslant 1$, and let $\varphi \in H^{s+1}(\Omega)$ satisfy

$$\begin{aligned} -\nabla \cdot (a \nabla\varphi) &= \nabla \cdot \gamma\,, && \text{in } \Omega \\ \varphi &= 0\,, && \text{on } \partial\Omega\,. \end{aligned}$$

Then,

$$\begin{aligned} \gamma &= -a \nabla\varphi + \delta\,, && \text{in } \Omega\,, \\ \operatorname{div} \delta &= 0\,, && \text{in } \Omega\,, \\ \delta \cdot \nu &= \gamma \cdot \nu + a \frac{\partial\varphi}{\partial\nu}\,, && \text{on } \partial\Omega\,. \end{aligned}$$

The function (vector) δ can be represented in the form $\delta = \operatorname{grad} \varepsilon$, where

$$\begin{aligned} \Delta\varepsilon &= 0\,, && \text{in } \Omega\,, \\ \frac{\partial\varepsilon}{\partial\nu} &= \gamma \cdot \nu + a \frac{\partial\varphi}{\partial\nu}\,, && \text{on } \partial\Omega\,. \end{aligned}$$

Consequently,

$$\|\delta\|_1 \leqslant \|\varepsilon\|_2 \leqslant Q\left\{ |\gamma \cdot \nu|_{1/2} + \left|\frac{\partial\varphi}{\partial\nu}\right|_{1/2} \right\} \leqslant Q\|\gamma\|_1\,.$$

Also,

$$\|\delta\|_q \leqslant Q\|\gamma\|_q\,, \quad q \geqslant 1\,.$$

Now, consider

$$(\alpha\xi_j, \gamma) = (\alpha\xi_j, -a \nabla\varphi) + (\alpha\xi_j, \delta)\,.$$

First,

$$\begin{aligned} (\alpha\xi_j, -a\nabla\varphi) &= -(\xi_j, \nabla\varphi) = (\operatorname{div} \xi_j, \varphi) \\ &= -\left(c\eta_j + d\frac{\partial\eta_{j-1}}{\partial t}, \varphi\right) \\ &\quad + \left(c\eta_j + d\frac{\partial\eta_{j-1}}{\partial t} + \operatorname{div} \xi_j, \varphi - P_h\varphi\right)\,. \end{aligned}$$

Thus,

$$(3.21) \qquad |(\alpha\xi_j, -a\nabla\varphi)| \leqslant Q\|\varphi\|_{s+1}\left\{ h^{\min(s+1, k+1)}\left[\|\eta_j\| + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|\right] + \right. \\ \left. + \|\eta_j\|_{-s-1} + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-s-1}\right\}\,.$$

Next, since $\operatorname{div} \delta = 0$ and $(\operatorname{div}(\delta - \pi_h\delta), \eta_j) = 0$,

$$\begin{aligned} (\alpha\xi_j, \delta) &= (\alpha\xi_j, \pi_h\delta) + (\alpha\xi_j, \delta - \pi_h\delta) \\ &= (\beta\eta_j + \alpha\xi_j, \delta - \pi_h\delta) - (\beta\eta_j, \delta)\,. \end{aligned}$$

Hence

$$(3.22)\quad |(\alpha\xi_j, \delta)| \leq Q\,\|\delta\|_s\left\{h^{\min(s,k+1)}[\,\|\xi_j\| + \|\eta_j\|\,] + \|\eta_j\|_{-s}\right\}.$$

It follows now from (3.21) and (3.22) that, for $s \geq 1$,

$$(3.23)\ \|\xi_j\|_{-s} \leq Q\left\{h^{\min(s,k+1)}[\,\|\xi_j\| + \|\eta_j\|\,] + h^{\min(s+1,k+1)}\left\|\frac{\partial\eta_{j-1}}{\partial t}\right\| + \|\eta_j\|_{-s} + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-s-1}\right\}.$$

The estimate for $s = 0$ is given by (3.10) and, together with (3.23), implies that

$$(3.24)\quad \|\xi_j\|_{-s} \leq Q\left\{h^{\min(s,k+1)}\left[\,\|\eta_j\| + h\left\|\frac{\partial\eta_{j-1}}{\partial t}\right\| + h^{-1}\left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-2}\right] + h^{\min(s+1,k+1)}\left\|\frac{\partial\eta_{j-1}}{\partial t}\right\| + \|\eta_j\|_{-s+1} + \left\|\frac{\partial\eta_{j-1}}{\partial t}\right\|_{-s-1}\right\}$$

for $s \geq 1$ and $j = 2, 3, \ldots$. Similarly,

$$(3.25)\quad \|\xi_1\|_{-s} \leq Q\left\{h^{\min(s,k+1)}\left[\,\|\eta_1\| + h\left\|\frac{\partial\eta}{\partial t}\right\| + h^{-1}\left\|\frac{\partial\eta}{\partial t}\right\|_{-2}\right] + h^{\min(s+1,k+1)}\left\|\frac{\partial\eta}{\partial t}\right\| + \|\eta_1\|_{-s+1} + \left\|\frac{\partial\eta}{\partial t}\right\|_{-s-1}\right\}.$$

Next, (3.10), (3.13), and Lemma 3.1 imply the following lemma :

LEMMA 3.2 : *For* $2 \leq r \leq k+1$

$$(3.26)\quad \begin{aligned} &(a)\quad \|\xi_j\| \leq Qh^{r+2j-1}\left\|\frac{\partial^j p}{\partial t^j}\right\|_{r+(2j+1-k)^+}, \\ &(b)\quad \|\xi_j\|_{-s} \leq Qh^{r+\min(s+2j-1,k+1)}\left\|\frac{\partial^j p}{\partial t^j}\right\|_{r+(s+2j-k)^+}, \end{aligned}$$

for $s \geq 1$.

4. BOUNDS FOR THE RESIDUALS

Recall that

$$(4.1)\quad \begin{aligned} &(a)\ (\alpha\psi_j, v) - (\operatorname{div} v, \theta_j) + (\beta\theta_j, v) = 0\,, && v \in V_h\,, \\ &(b)\ \left(d\frac{\partial\theta_j}{\partial t}, w\right) + (\operatorname{div}\psi_j, w) + (c\theta_j, w) = \left(d\frac{\partial\eta_j}{\partial t}, w\right), && w \in W_h\,. \end{aligned}$$

Take $v = \psi_j$ and $w = \theta_j$ in (4.1) and add :

$$\frac{1}{2}\frac{d}{dt}(d\theta_j, \theta_j) + (\alpha\psi_j, \psi_j) + (c\theta_j, \theta_j) = \left(d\frac{\partial \eta_j}{\partial t}, \theta_j\right) - (\beta\theta_j, \psi_j).$$

Thus, integration with respect to time leads to

$$\text{(4.2)} \qquad \|\theta_j\|_{L^\infty(L^2)} + \|\psi_j\|_{L^2(L^2)} \leq Q\left\{\left\|\frac{\partial \eta_j}{\partial t}\right\|_{L^2(L^2)} + \|\theta_j(0)\|_{L^2(\Omega)}\right\},$$

where

$$\|f\|^2_{L^2(X)} = \|f\|^2_{L^2(J;X)} = \int_0^T \|f(t)\|_X^2\, dt,$$

and

$$\|\varphi\|_{L^\infty(X)} = \|\varphi\|_{L^\infty(J;X)} = \operatorname*{ess\,sup}_{0 \leq t \leq T} \|\varphi(t)\|_X$$

for an arbitrary normed space X. Next, differentiate (4.1a) with respect to the time variable and set $v = \psi_j$. Now, set $w = \frac{\partial \theta_j}{\partial t}$ in (4.1b), add the two resulting equations and integrate in time. It follows that

$$\text{(4.3)} \qquad \|\theta_j\|_{L^\infty(L^2)} + \|\psi_j\|_{L^\infty(L^2)} + \left\|\frac{\partial \theta_j}{\partial t}\right\|_{L^2(L^2)} \leq$$
$$\leq Q\left\{\left\|\frac{\partial \eta_j}{\partial t}\right\|_{L^2(L^2)} + \|\theta_j(0)\|_{L^2(\Omega)} + \|\psi_j(0)\|_{L^2(\Omega)}\right\}.$$

Let

$$\text{(4.4)} \qquad 2J = k + 1.$$

Then, Lemma 3.1 implies that

$$\text{(4.5)} \qquad \left\|\frac{\partial \eta_J}{\partial t}\right\|_{L^2(L^2)} \leq Qh^{r+k+1}\left\|\frac{\partial^{J+1} p}{\partial t^{J+1}}\right\|_{L^2(H^{r+2})} \quad \text{for} \quad 2 \leq r \leq k+1.$$

Hence

$$\text{(4.6)} \qquad \|\theta_J\|_{L^\infty(L^2)} + \|\psi_J\|_{L^\infty(L^2)} + \left\|\frac{\partial \theta_J}{\partial t}\right\|_{L^2(L^2)} \leq$$
$$\leq Q\left\{h^{r+k+1}\left\|\frac{\partial^{J+1} p}{\partial t^{J+1}}\right\|_{L^2(H^{r+2})} + \|\theta_J(0)\|_{L^2(\Omega)} + \|\psi_J(0)\|_{L^2(\Omega)}\right\}$$

for $2 \leq r \leq k+1$. Thus, it is desirable that $\theta_J(0)$ and $\psi_J(0)$ be chosen so that they are $0(h^{r+k+1})$. Consider the initialization of u_h and v_h.

Recall that

$$\theta_J = \tilde{p}_h + \eta_1 + \cdots + \eta_J - p_h ,$$
$$\psi_J = \tilde{u}_h + \xi_1 + \cdots + \xi_J - u_h .$$

Thus, it would suffice to take

$$(4.7) \quad \begin{aligned} &(a)\ p_h(0) = \tilde{p}_h(0) + \sum_{i=1}^{J} \eta_i(0) , \quad \text{or} \quad \theta_J(0) = 0 , \\ &(b)\ u_h(0) = \tilde{u}_h(0) + \sum_{i=1}^{J} \xi_i(0) , \quad \text{or} \quad \psi_J(0) = 0 . \end{aligned}$$

These values can be computed using no more than the data f and p_0 and the differential operator. To see that, first note that $\partial^l p(0)/\partial t^l$ can be computed from p_0 and the partial differential equation ; thus $\partial^l \tilde{p}_h(0)/\partial t^l$ and $\partial^l \tilde{u}_h(0)/\partial t^l$ can be evaluated from (2.3). Finally, (2.5) and (2.7) can be used to complete the computation of $u_h(0)$ and $p_h(0)$. Hence, the following has been proved.

LEMMA 4.1 : *If the mixed finite element method is initialized by* (4.7) *and J is defined by* (4.4), *then*

$$(4.8) \quad \|\theta_J\|_{L^\infty(L^2)} + \|\psi_J\|_{L^\infty(L^2)} + \left\| \frac{\partial \theta_J}{\partial t} \right\|_{L^2(L^2)} \leq Q h^{r+k+1} \left\| \frac{\partial^{J+1} p}{\partial t^{J+1}} \right\|_{L^2(H^{r+2})}$$

for $2 \leq r \leq k+1$.

The optimal order negative norm estimates for the error in the mixed method solution can now be obtained by applying the results of this section and Theorem 2.2.

THEOREM 4.2 : *If the mixed method is initialized by* (4.7) *and J is given by* (4.4), *then*

$$(4.10) \quad \|p - p_h\|_{L^\infty(H^{-k-1})} + \|u - u_h\|_{L^\infty(H^{-k-1})} \leq \leq Q h^{2k+2} \sum_{i=0}^{J+1} \left\| \frac{\partial^i p}{\partial t^i} \right\|_{L^2(H^{k+3})} .$$

Proof: Write

$$p - p_h = \eta - \sum_{i=1}^{J} \eta_i + \theta_J ,$$
$$u - u_h = \xi - \sum_{i=1}^{J} \xi_i + \psi_J ,$$

so that,

$$\|p - p_h\|_{L^\infty(H^{-k-1})} \leq \|\eta\|_{L^\infty(H^{-k-1})} + \sum_{i=1}^{J} \|\eta_i\|_{L^\infty(H^{-k-1+2i})} + \|\theta_J\|_{L^\infty(L^2)},$$

$$\|u - u_h\|_{L^\infty(H^{-k-1})} \leq \|\xi\|_{L^\infty(H^{-k-1})} + \sum_{i=1}^{J} \|\xi_i\|_{L^\infty(H^{-k-2+2i})} + \|\psi_J\|_{L^\infty(L^2)}.$$

The theorem now follows from Lemmas 3.1, 3.2, and 4.1.

If k is even and J is now such that $2J = k + 2$, then Theorem 4.2 remains valid [13].

Optimal order estimates for div $(u - u_h)$ have also been derived in [13].

5. CONVERGENCE OF DIFFERENCE QUOTIENTS

Throughout this section and the next let Ω be a rectangle in $\mathbb{R}^2$ and assume that all coefficients of the partial differential equation in (1.1) as well as f and p_0 can be extended periodically to all of $\mathbb{R}^2$ while satisfying the same smoothness conditions as in section 1. Let $\mathfrak{J}_h$ be a uniform grid, and consider the mixed method for the periodic problem associated with (1.1) ; i.e., for any $t \in J$, find $\{u_h, p_h\} \in V_h \times W_h$ satisfying

$$(5.1)\quad \begin{aligned} &(a)\ (\alpha u_h, v) - (\operatorname{div} v, p_h) + (\beta p_h, v) = 0\,, && v \in V_h\,,\\ &(b)\ \left(d\frac{\partial p_h}{\partial t}, w\right) + (\operatorname{div} u_h, w) + (c p_h, w) = (f, w)\,, && w \in W_h\,,\\ &(c)\ \{u_h, p_h\} \text{ periodic of period } \Omega\,. \end{aligned}$$

The projection $\{\tilde{u}_h, \tilde{p}_h\}$ and the terms $\{\xi_i, \eta_i\}$ of the quasi-projection are now to be taken as periodic of period Ω as are the residuals $\{\theta_j, \psi_j\}$.

Next, consider the introduction of *forward difference quotients.* Let $\mu = (\mu_1, \mu_2)$ have integer components and define the translation operator T_h^μ by

$$T_h^\mu v(x) = v(x + \mu h)\,.$$

The forward difference quotient is then defined by

$$\partial_{h,j}\, u = h^{-1}(T_h^{e_j} - I)\, u\,, \quad j = 1, 2\,,$$

where e_j is the unit vector in the direction of x_j, and I is the identity operator. For an arbitrary multi-index μ set

$$\partial_h^\mu = \partial_{h,1}^{\mu_1}\, \partial_{h,2}^{\mu_2}\,.$$

The discrete Leibnitz formula will be used :

$$\partial_h^{\nu}(uv) = \sum_{\mu \leq \nu} \binom{\nu}{\mu} T_h^{\mu} \partial_h^{\nu-\mu} u \, \partial_h^{\mu} v \tag{5.2}$$

where

$$\binom{\nu}{\mu} = \binom{\nu_1}{\mu_1}\binom{\nu_2}{\mu_2} .$$

In this section estimates for difference quotients of the error

$$(a)\ \ p - p_h = \eta - \sum_{i=1}^{j} \eta_i + \theta_j ,$$

$$(b)\ \ u - u_h = \xi - \sum_{i=1}^{j} \xi_i + \psi_j$$

will be carried out. These estimates will then be used in the next section to prove superconvergence. Let ν be a multi-index and assume from now on that $|\nu| = \nu_1 + \nu_2 \leq k + 1$. First, the estimates for $\{\xi, \eta\}$ follow from Douglas-Milner [6] for $s \geq 0$:

$$\| \partial^{\nu} \eta \|_{-s} \leq Q h^{r + \min(s, k+1)} \| p \|_{r + (s+1-k)^+ + |\nu|} \tag{5.3}$$

for $2 - (s - k + 1)^+ \leq r \leq k + 1$;

$$\| \partial^{\nu} \xi \|_{-s} \leq Q h^{r + \min(s, k+1)} \| p \|_{r + 1 + (s-k)^+ + |\nu|} \tag{5.4}$$

for $1 - (s - k)^+ \leq r \leq k + 1$.

Next, since the spaces $V_h \times W_h$ are translation invariant, (2.6) and (5.2) imply that, for $|\nu| > 0$ and $j = 2, 3, \ldots$,

(5.5)

$$(a)\ \ (\alpha\, \partial^{\nu} \xi_j, v) - (\operatorname{div} v, \partial^{\nu} \eta_j) + (\beta\, \partial^{\nu} \eta_j, v) =$$
$$= - \sum_{\mu < \nu} \binom{\nu}{\mu} (T_h^{\mu} \partial_h^{\nu-\mu} \alpha\, \partial_h^{\mu} \xi_j + T_h^{\mu} \partial_h^{\nu-\mu} \beta\, \partial_h^{\mu} \eta_j, v) , \quad v \in V_h ,$$

$$(b)\ \ (\operatorname{div}(\partial^{\nu} \xi_j), w) + (c\, \partial^{\nu} \eta_j, w) =$$
$$= - \left(\partial^{\nu}\left(d \frac{\partial \eta_{j-1}}{\partial t} \right), w \right) - \sum_{\mu < \nu} \binom{\nu}{\mu} (T_h^{\mu} \partial_h^{\nu-\mu} c\, \partial_h^{\mu} \eta_j, w) , \quad w \in W_h .$$

Assume for the time being that $j \geq 2$, and assume from now on that $s \geq 0$. Lemma 2.1 implies that

$$\| \partial^{\nu} \eta_j \|_{-s} \leq Q \left\{ h^{\min(s+1, k+1)} \left[\| \partial^{\nu} \xi_j \| + \sum_{\mu < \nu} (\| \partial^{\mu} \eta_j \| + \| \partial^{\mu} \xi_j \|) \right] \right. \tag{5.6}$$

$$+ h^{\min(s+2,k+1)}\left[\|\operatorname{div}\xi_j\| + \left\|\partial^{\nu}\left(d\frac{\partial\eta_{j-1}}{\partial t}\right)\right\| + \sum_{\mu<\nu}\|\partial^{\mu}\eta_j\|\right]$$
$$+ \sum_{\mu<\nu}(\|\partial^{\mu}\xi_j\|_{-s-1} + \|\partial^{\mu}\eta_j\|_{-s-1})$$
$$+ \left\|\partial^{\nu}\left(d\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|_{-s-2} + \sum_{\mu<\nu}\|\partial^{\mu}\eta_j\|_{-s-2}\Bigg\} .$$

Note that, since d is assumed to be smooth,

$$\left\|\partial^{\nu}\left(d\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|_{-s} \leq Q\sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|_{-s}, \tag{5.7}$$

and then,

$$\|\partial^{\nu}\eta_j\|_{-s} \leq Q\Bigg\{h^{\min(s+1,k+1)}\left[\|\partial^{\nu}\xi_j\| + \sum_{\mu<\nu}(\|\partial^{\mu}\xi_j\| + \|\partial^{\mu}\eta_j\|)\right] \tag{5.8}$$
$$+ h^{\min(s+2,k+1)}\left[\|\operatorname{div}\partial^{\nu}\xi_j\| + \sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|\right]$$
$$+ \sum_{\mu<\nu}(\|\partial^{\mu}\xi_j\|_{-s-1} + \|\partial^{\mu}\eta_j\|_{-s-1}) + \sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|_{-s-2}\Bigg\} .$$

Next, take $w = \operatorname{div}(\partial^{\nu}\xi_j)$ in (5.5*b*) :

$$\|\operatorname{div}(\partial^{\nu}\xi_j)\| \leq Q\left\{\|\partial^{\nu}\eta_j\| + \sum_{\mu<\nu}\|\partial^{\mu}\eta_j\| + \sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|\right\} . \tag{5.9}$$

Now, let $v = \partial^{\nu}\xi_j$ in (5.5*a*), and use (5.9) to obtain

$$\|\partial^{\nu}\xi_j\| \leq Q\Bigg\{\|\partial^{\nu}\eta_j\| + \|\partial^{\nu}\eta_j\|^{1/2}\left(\sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|\right)^{1/2} \tag{5.10}$$
$$+ \sum_{\mu<\nu}(\|\partial^{\mu}\xi_j\| + \|\partial^{\mu}\eta_j\|)\Bigg\} .$$

Estimates (5.9) and (5.10), together with (5.8) imply that

(5.11)

$$\|\partial^{\nu}\eta_j\|_{-s} \leq Q\Bigg\{h^{\min(s+1,k+1)}\left[\|\partial^{\nu}\eta_j\| + \|\partial^{\nu}\eta_j\|^{1/2}\left(\sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|\right)^{1/2}\right.$$
$$\left.+ \sum_{\mu<\nu}(\|\partial^{\mu}\xi_j\| + \|\partial^{\mu}\eta_j\|)\right] + h^{\min(s+2,k+1)}\sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|$$
$$+ \sum_{\mu<\nu}(\|\partial^{\mu}\xi_j\|_{-s-1} + \|\partial^{\mu}\eta_j\|_{-s-1}) + \sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_{j-1}}{\partial t}\right)\right\|_{-s-2}\Bigg\}.$$

Next, an estimate for $\| \partial^\nu \xi_j \|_{-s}$ will be derived. Let $\gamma \in H^s(\Omega)^2$ for some integer $s \geq 1$, and let $\varphi \in H^{s+1}(\Omega)$ be the solution of the periodic problem

$$- \nabla \cdot (a \nabla \varphi) = \nabla \cdot \gamma \quad \text{in} \quad \Omega .$$

Then, as in the estimate for $\| \xi_j \|_{-s}$,

$$\begin{aligned} \gamma &= - a \nabla \varphi + \delta \quad && \text{in} \quad \Omega , \\ \operatorname{div} \delta &= 0 && \text{in} \quad \Omega , \end{aligned}$$

and

$$\| \delta \|_q \leq Q \| \gamma \|_q \quad \text{for} \quad q \geq 1 .$$

Now, consider

$$(\alpha \partial^\nu \xi_j, \gamma) = - (\alpha \partial^\nu \xi_j, a \nabla \varphi) + (\alpha \partial^\nu \xi_j, \delta) .$$

First, let $\chi \in W_h$. Then

$$\begin{aligned} - (\alpha \partial^\nu \xi_j, a \nabla \varphi) &= - (\partial^\nu \xi_j, \nabla \varphi) = (\operatorname{div} (\partial^\nu \xi_j), \varphi) \\ &= (\operatorname{div} (\partial^\nu \xi_j), \varphi - \chi) + (\operatorname{div} (\partial^\nu \xi_j), \chi) , \end{aligned}$$

and it follows from (5.5*b*) that

$$\begin{aligned} - (\alpha \partial^\nu \xi_j, a \nabla \varphi) = &\left(\operatorname{div} (\partial^\nu \xi_j) + c \, \partial^\nu \eta_j + \partial^\nu \left(d \frac{\partial \eta_{j-1}}{\partial t} \right), \varphi - \chi \right) \\ &+ \left(c \, \partial^\nu \eta_j - \partial^\nu \left(d \frac{\partial \eta_{j-1}}{\partial t} \right), \varphi \right) \\ &+ \sum_{\mu < \nu} \binom{\nu}{\mu} (T_h^\mu \partial_h^{\nu - \mu} c \, \partial_h^\mu \eta_j, \varphi - \chi) - \sum_{\mu < \nu} \binom{\nu}{\mu} (T_h^\mu \partial_h^{\nu - \mu} c \, \partial_h^\mu \eta_j, \varphi) . \end{aligned}$$

It follows from the equality above and (5.9) that

$$(5.12) \quad \begin{aligned} | (\alpha \partial^\nu \xi_j, a \nabla \varphi) | \leq Q \| \varphi \|_{s+1} \Big\{ &h^{\min (s+1, k+1)} \Big[\| \partial^\nu \eta_j \| \\ &+ \sum_{\mu < \nu} \| \partial^\mu \eta_j \| + \sum_{\mu \leq \nu} \left\| \partial^\mu \left(\frac{\partial \eta_{j-1}}{\partial t} \right) \right\| \Big] \\ &+ \sum_{\mu \leq \nu} \left(\| \partial^\mu \eta_j \|_{-s-1} + \left\| \partial^\mu \left(\frac{\partial \eta_{j-1}}{\partial t} \right) \right\|_{-s-1} \right) \Big\} . \end{aligned}$$

Next, from (5.5*a*)

$$\begin{aligned} (\alpha \partial^\nu \xi_j, \delta) &= (\alpha \partial^\nu \xi_j, \delta - \pi_h \delta) + (\alpha \partial^\nu \xi_j, \pi_h \delta) \\ &= (\alpha \partial^\nu \xi_j + \beta \partial^\nu \eta_j, \delta - \pi_h \delta) \\ &+ \sum_{\mu < \nu} \binom{\nu}{\mu} (T_h^\mu \partial_h^{\nu - \mu} \alpha \, \partial_h^\mu \xi_j + T_h^\mu \partial_h^{\nu - \mu} \beta \, \partial_h^\mu \eta_j, \delta - \pi_h \delta) \\ &- \sum_{\mu < \nu} \binom{\nu}{\mu} (T_h^\mu \partial_h^{\nu - \mu} \alpha \, \partial_h^\mu \xi_j + T_h^\mu \partial_h^{\nu - \mu} \beta \, \partial_h^\mu \eta_j, \delta) - (\beta \partial^\nu \eta_j, \delta) , \end{aligned}$$

since $(\operatorname{div}(\delta - \pi_h \delta), \partial^\nu \eta_j) = 0 = \operatorname{div} \delta$. Thus,

$$(5.13)\quad |(\alpha\, \partial^\nu \xi_j, \delta)| \leqslant Q \|\delta\|_s \Big\{ h^{\min(s,k+1)} \Big[\|\partial^\nu \xi_j\| + \|\partial^\nu \eta_j\| + \sum_{\mu<\nu} (\|\partial^\mu \xi_j\| + \|\partial^\mu \eta_j\|) \Big] + \sum_{\mu<\nu} \|\partial^\mu \xi_j\|_{-s} + \sum_{\mu \leqslant \nu} \|\partial^\mu \eta_j\|_{-s} \Big\}.$$

The estimate for $\|\partial^\nu \xi_j\|_{-s}$, with $s \geqslant 1$, now follows from (5.12) and (5.13) :

$$(5.14)\quad \begin{aligned} \|\partial^\nu \xi_j\|_{-s} \leqslant Q \Big\{ & h^{\min(s,k+1)} \Big[\|\partial^\nu \xi_j\| + \|\partial^\nu \eta_j\| + \sum_{\mu<\nu} (\|\partial^\mu \xi_j\| + \|\partial^\mu \eta_j\|) \Big] \\ & + h^{\min(s+1,k+1)} \sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta_{j-1}}{\partial t} \Big) \Big\| + \|\partial^\nu \eta_j\|_{-s+1} \\ & + \sum_{\mu<\nu} (\|\partial^\mu \xi_j\|_{-s} + \|\partial^\mu \eta_j\|_{-s}) + \sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta_{j-1}}{\partial t} \Big) \Big\|_{-s-1} \Big\}. \end{aligned}$$

The estimate for $s = 0$ is given by the following inequality that follows from (5.10) :

$$(5.15)\quad \|\partial^\nu \xi_j\| \leqslant Q \Big\{ h^{-1} \|\partial^\nu \eta_j\| + h \sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta_{j-1}}{\partial t} \Big) \Big\| + \sum_{\mu<\nu} (\|\partial^\mu \xi_j\| + \|\partial^\mu \eta_j\|) \Big\}.$$

The estimates for $j = 1$ can be derived in a similar way :

$$(5.16)\quad \begin{aligned} \|\partial^\nu \eta_1\|_{-s} \leqslant Q \Big\{ & h^{\min(s+1,k+1)} \Big[\|\partial^\nu \eta_1\| + \|\partial^\nu \eta_1\|^{1/2} \Big(\sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta}{\partial t} \Big) \Big\| \Big)^{1/2} \\ & + \sum_{\mu<\nu} (\|\partial^\mu \xi_1\| + \|\partial^\mu \eta_1\|) \Big] + h^{\min(s+2,k+1)} \sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta}{\partial t} \Big) \Big\| \\ & + \sum_{\mu<\nu} (\|\partial^\mu \xi_1\|_{-s-1} + \|\partial^\mu \eta_1\|_{-s-1}) + \sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta}{\partial t} \Big) \Big\|_{-s-2} \Big\} \end{aligned}$$

for $s \geqslant 0$;

$$(5.17)\quad \begin{aligned} \|\partial^\nu \xi_1\|_{-s} \leqslant Q \Big\{ & h^{\min(s,k+1)} \Big[\|\partial^\nu \xi_1\| + \|\partial^\nu \eta_1\| + \sum_{\mu<\nu} (\|\partial^\mu \xi_1\| + \|\partial^\mu \eta_1\|) \Big] \\ & + h^{\min(s+1,k+1)} \sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta}{\partial t} \Big) \Big\| + \|\partial^\nu \eta_1\|_{-s+1} \\ & + \sum_{\mu<\nu} (\|\partial^\mu \xi_1\|_{-s} + \|\partial^\mu \eta_1\|_{-s}) + \sum_{\mu \leqslant \nu} \Big\| \partial^\mu \Big(\frac{\partial \eta}{\partial t} \Big) \Big\|_{-s-1} \Big\} \end{aligned}$$

for $s \geqslant 1$; and

(5.18)

$$\|\partial^{\nu}\xi_1\| \leqslant Q\left\{h^{-1}\|\partial^{\nu}\eta_1\| + h\sum_{\mu \leqslant \nu}\left\|\partial^{\mu}\left(\frac{\partial\eta}{\partial t}\right)\right\| + \sum_{\mu < \nu}(\|\partial^{\mu}\eta_1\| + \|\partial^{\mu}\xi_1\|)\right\}.$$

Now, if the choice $s = 0$ is made in (5.16), then

$$\|\partial^{\nu}\eta_1\| \leqslant Q\left\{\sum_{\mu < \nu}(\|\partial^{\mu}\xi_1\| + \|\partial^{\mu}\eta_1\|) + \right. \tag{5.19}$$

$$\left. + h^2\sum_{\mu \leqslant \nu}\left\|\partial^{\mu}\left(\frac{\partial\eta}{\partial t}\right)\right\| + \sum_{\mu \leqslant \nu}\left\|\partial^{\mu}\left(\frac{\partial\eta}{\partial t}\right)\right\|_{-2}\right\},$$

and a simple induction argument leads to the following lemma.

LEMMA 5.1 : *For* $2 \leqslant r \leqslant k+1$, *and* l *a positive integer*

(5.20)

$$(a)\quad \left\|\partial^{\nu}\left(\frac{\partial^l\eta_1}{\partial t^l}\right)\right\| \leqslant Qh^{r+2}\left\|\frac{\partial^{l+1}p}{\partial t^{l+1}}\right\|_{r+(3-k)^+ + |\nu|},$$

$$(b)\quad \left\|\partial^{\nu}\left(\frac{\partial^l\xi_1}{\partial t^l}\right)\right\| \leqslant Qh^{r+1}\left\|\frac{\partial^{l+1}p}{\partial t^{l+1}}\right\|_{r+(3-k)^+ + |\nu|}.$$

LEMMA 5.2 : *If* $s \geqslant 1$, *then for* $2 \leqslant r \leqslant k+1$

(5.21)

$$(a)\quad \left\|\partial^{\nu}\left(\frac{\partial^l\eta_1}{\partial t^l}\right)\right\|_{-s} \leqslant Qh^{r+\min(s+2,k+1)}\left\|\frac{\partial^{l+1}p}{\partial t^{l+1}}\right\|_{r+(s+3-k)^+ + |\nu|},$$

$$(b)\quad \left\|\partial^{\nu}\left(\frac{\partial^l\xi_1}{\partial t^l}\right)\right\|_{-s} \leqslant Qh^{r+\min(s+1,k+1)}\left\|\frac{\partial^{l+1}p}{\partial t^{l+1}}\right\|_{r+(s+2-k)^+ + |\nu|}.$$

Proof: The proof will proceed by induction. The case $\nu = 0$ was treated in (3.17) and (3.26). So, assume that $|\nu| \geqslant 1$ and that (5.21) holds for $|\mu| \leqslant |\nu| - 1$. Then, it follows from (5.16), (5.20), and (5.3) that

$$\|\partial^{\nu}\eta_1\|_{-s} \leqslant Qh^{r+\min(s+2,k+1)}\left\|\frac{\partial p}{\partial t}\right\|_{r+(s+3-k)^+ + |\nu|} \quad \text{for} \quad 2 \leqslant r \leqslant k+1.$$

Next, (5.3), (5.17), (5.21a), and the induction hypothesis imply that

$$\|\partial^{\nu}\xi_1\|_{-s} \leqslant Q\left\{h^{\min(s,k+1)}h^{r+1}\left\|\frac{\partial p}{\partial t}\right\|_{r+(3-k)^+ + |\nu|}\right.$$

$$+ h^{\min(s+1,k+1)}h^{r}\left\|\frac{\partial p}{\partial t}\right\|_{r+|\nu|} + h^{r+\min(s+1,k+1)}\left\|\frac{\partial p}{\partial t}\right\|_{r+(s+2-k)^+ + |\nu|}$$

$$\left. + h^{r+\min(s+1,k+1)}\left\|\frac{\partial p}{\partial t}\right\|_{r+(s+3-k)^+ + |\nu| - 1}\right\}.$$

Thus,

$$(5.22)\qquad \| \partial^\nu \xi_1 \|_{-s} \leq Q h^{r + \min(s+1, k+1)} \left\| \frac{\partial p}{\partial t} \right\|_{r + (s+2-k)^+ + |\nu|}$$

for $2 \leq r \leq k+1$. The independence of the coefficients of the time variable now completes the proof.

LEMMA 5.3 : *For* $2 \leq r \leq k+1$

$$(5.23)\qquad \begin{aligned} &(a)\quad \left\| \partial^\nu \left(\frac{\partial^l \eta_j}{\partial t^l} \right) \right\| \leq Q h^{r+2j} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (2j+1-k)^+ + |\nu|}, \\ &(b)\quad \left\| \partial^\nu \left(\frac{\partial^l \xi_j}{\partial t^l} \right) \right\| \leq Q h^{r+2j-1} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (2j+1-k)^+ + |\nu|} \end{aligned}$$

Proof: Use an induction argument on $|\nu|$ and j. The cases given by $\nu = 0$ for $j = 1, 2, \ldots$, and by $|\nu|$ arbitrary with $j = 1$ have been treated, respectively, in section 3 and Lemma 5.2. So, assume $|\nu| \geq 1$, $j \geq 2$, and that (5.23) holds for $|\mu| \leq |\nu| - 1$ and for $j - 1$. A simple calculation now completes the proof.

LEMMA 5.4 : *If* $s \geq 1$, *then*

(5.24)

$$(a)\quad \left\| \partial^\nu \left(\frac{\partial^l \eta_j}{\partial t^l} \right) \right\|_{-s} \leq Q h^{r + \min(s+2j, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (s+2j+1-k)^+ + |\nu|},$$

$$(b)\quad \left\| \partial^\nu \left(\frac{\partial^l \xi_j}{\partial t^l} \right) \right\|_{-s} \leq Q h^{r + \min(s+2j-1, k+1)} \left\| \frac{\partial^{l+j} p}{\partial t^{l+j}} \right\|_{r + (s+2j-k)^+ + |\nu|},$$

for $2 \leq r \leq k+1$.

Proof: Use an induction argument on $|\nu|$ and j.

Since the spaces $V_h \times W_h$ are translation invariant, it follows from (2.9) and (5.2) that

$$(5.25)\qquad (\alpha\, \partial^\nu \psi_j, v) - (\operatorname{div} v, \partial^\nu \theta_j) + (\beta \partial^\nu \theta_j, v) = = - \sum_{\mu < \nu} \binom{\nu}{\mu} (T_h^\mu\, \partial_h^{\nu-\mu} \alpha\, \partial_h^\mu \psi_j + T_h^\mu\, \partial_h^{\nu-\mu}\, \beta\, \partial_h^\mu \theta_j, v),$$

$v \in V_h$. Also,

$$(5.26)\qquad \left(d\, \partial^\nu \left(\frac{\partial \theta_j}{\partial t} \right), w \right) + (\operatorname{div} \partial^\nu \psi_j, w) + (c\, \partial^\nu \theta_j, w) =$$

$$= - \sum_{\mu < \nu} \binom{\nu}{\mu} \left(T_h^\mu\, \partial_h^{\nu-\mu} d\, \partial_h^\mu \left(\frac{\partial \theta_j}{\partial t} \right) + T_h^\mu\, \partial_h^{\nu-\mu} c\, \partial_h^\mu\, \theta_j, w \right)$$

$$+ \sum_{\mu \leq \nu} \binom{\nu}{\mu} \left(T_h^\mu\, \partial_h^{\nu-\mu} d\, \partial_h^\mu \left(\frac{\partial \eta_j}{\partial t} \right), w \right), \quad w \in W_h .$$

Take $v = \partial^{\nu}\psi_j$ in (5.25), $w = \partial^{\nu}\theta_j$ in (5.26), add, and integrate with respect to time to obtain the inequality

$$(5.27)\quad \|\partial^{\nu}\theta_j\|_{L^\infty(L^2)} + \|\partial^{\nu}\psi_j\|_{L^2(L^2)} \leq$$
$$\leq Q\left\{\sum_{\mu<\nu}\left(\|\partial^{\mu}\theta_j\|_{L^\infty(L^2)} + \|\partial^{\mu}\psi_j\|_{L^2(L^2)} + \left\|\partial^{\mu}\left(\frac{\partial\theta_j}{\partial t}\right)\right\|_{L^2(L^2)}\right) + \sum_{\mu\leq\nu}\left\|\partial^{\mu}\left(\frac{\partial\eta_j}{\partial t}\right)\right\|_{L^2(L^2)} + \|\partial^{\nu}\theta_j(0)\|_{L^2(\Omega)}\right\}.$$

At this stage an extra assumption will be made in order to avoid one of the terms arising on the righ-hand side of (5.27).

Assume $d(x) = d$ to constant.

Note that this assumption does not imply a loss of generality. In fact, if d depends on x, then the change of variables $P = dp$ leads to the differential equation

$$\frac{\partial p}{\partial t} - \nabla\cdot\left(\frac{a}{d}\nabla P + \left(\frac{b}{d} - \frac{a\,\nabla d}{d}\right)P\right) + \frac{c}{d}P = f$$

in which the d coefficient is constant. The mixed method will then furnish approximation to P and to the original flow field u associated with p, since the flow field U associated with P is in fact identical to u :

$$U = -\frac{a}{d}\nabla P - \left(\frac{b}{d} - \frac{a\,\nabla d}{d}\right)P = -a\,\nabla p - bp = u\,.$$

Now, with d constant, inequality (5.27) reduces to

(5.28)

$$\|\partial^{\nu}\theta_j\|_{L^\infty(L^2)} + \|\partial^{\nu}\psi_j\|_{L^2(L^2)} \leq Q\left\{\sum_{\mu<\nu}\left(\|\partial^{\mu}\theta_j\|_{L^\infty(L^2)} + \|\partial^{\mu}\psi_j\|_{L^2(L^2)}\right) + \left\|\partial^{\nu}\left(\frac{\partial\eta_j}{\partial t}\right)\right\|_{L^2(L^2)} + \|\partial^{\nu}\theta_j(0)\|_{L^2(\Omega)}\right\}.$$

Next, estimate (5.24*a*) implies that

$$(5.29)\quad \left\|\partial^{\nu}\left(\frac{\partial\eta_J}{\partial t}\right)\right\|_{L^2(L^2)} \leq Qh^{r+k+1}\left\|\frac{\partial^{J+1}p}{\partial t^{J+1}}\right\|_{L^2(H^{r+2+|\nu|})}$$

for $2 \leq r \leq k+1$, where

$$(5.30)\quad 2J = k+1\,.$$

Hence, the initialization given by (4.7) implies that

(5.31)

$$\| \partial^\nu \theta_J \|_{L^\infty(L^2)} + \| \partial^\nu \psi_J \|_{L^2(L^2)} \leq Q \left\{ \sum_{\mu < \nu} (\| \partial^\mu \theta_J \|_{L^\infty(L^2)} + \| \partial^\mu \psi_J \|_{L^2(L^2)}) + \right.$$
$$\left. + h^{r+k+1} \left\| \frac{\partial^{J+1} p}{\partial t^{J+1}} \right\|_{L^2(H^{r+2+|\nu|})} \right\},$$

for $2 \leq r \leq k+1$.

An easy induction argument leads to the following lemma.

LEMMA 5.5 :

$$(5.32) \quad \| \partial^\nu \theta_J \|_{L^\infty(L^2)} + \| \partial^\nu \psi_J \|_{L^2(L^2)} \leq Q h^{r+k+1} \left\| \frac{\partial^{J+1} p}{\partial t^{J+1}} \right\|_{L^2(H^{r+2+|\nu|})}$$

for $2 \leq r \leq k+1$.

6. THE SUPERCONVERGENCE ESTIMATES

In this section a new approximation $\{u_h^*, p_h^*\}$ to $\{u, p\}$ in $V \times W$ will be introduced and then used to establish superconvergent error estimates. As in Bramble-Schatz [2, 3], $\{u_h^*, p_h^*\}$ is obtained by considering certain « averages » of $\{u_h, p_h\}$ that are formed by the convolution of $\{u_h, p_h\}$ with a kernel K_h, defined as follows.

For t real, let

$$\chi(t) = \begin{cases} 1, & |t| \leq 1/2, \\ 0, & |t| > 1/2, \end{cases}$$

and, for l an integer, set

$$\psi_1^{(l)}(t) = \chi * \chi * \cdots * \chi,$$

convolution $l-1$ times. The function $\psi_1^{(l)}$ is the one-dimensional B-spline basis function of order l.

Next, let c_i, $0 \leq i \leq k$, be determined as the unique solution of the linear system of algebraic equations

$$\sum_{i=0}^{k} c_i \int_{\mathbb{R}} \psi_1^{(k+2)}(y)(y+i)^{2m} \, dy = \delta_{0m}, \quad 0 \leq m \leq k.$$

For $x \in \mathbb{R}^2$, define K_h by

$$K_h(x) = K_{h,k+2}^{2k+2}(x) = \prod_{m=1}^{2} \left(\sum_{i=-k}^{k} h^{-1} c_i' \, \psi_1^{(k+2)}(h^{-1} x_m^{-i}) \right),$$

where the constants c_j' are given by

$$c_0' = c_0$$
$$c_{-i}' = c_i' = c_i/2\ , \quad 1 \leqslant i \leqslant k\ .$$

It is known [2, 3] that

$$\text{(6.1)} \quad \| K_h * w - w \| \leqslant Q \| w \|_r h^r\ , \quad \text{if} \quad w \in H^r(\Omega), 0 \leqslant r \leqslant 2k+2\ ,$$

$$\text{(6.2)} \quad \| D^{\nu}(K_h * w) \|_s \leqslant Q \| \partial^{\nu} w \|_s\ , \quad \text{if} \quad w \in H^s(\Omega), s \in \mathbb{Z}\ ,$$

where $D^{\nu} = \dfrac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2}}$, and ∂^{ν} is the corresponding forward difference with step h, as defined in the previous section. Also,

$$\text{(6.3)} \qquad \| w \| \leqslant Q \sum_{|\nu| \leqslant s} \| D^{\nu} w \|_{-s}\ , \quad 0 \leqslant s \in \mathbb{Z}, w \in L^2(\Omega)\ .$$

The main theorem in this work can now be stated and proved.

THEOREM 6.1 : *Let the index k of $V_h \times W_h$ be odd, and at least one. Assume that d is constant, and that $p \in H^r(\Omega)$ for $2 \leqslant r \leqslant 2k+4$. If the mixed method for the periodic problem corresponding to* (1.1) *is initialized by*

$$p_h(0) = \tilde{p}_h(0) + \sum_{i=1}^{J} \eta_i(0)\ ,$$
$$u_h(0) = \tilde{u}_h(0) + \sum_{i=1}^{J} \xi_i(0)\ ,$$

where

$$2J = k+1\ ,$$

then for h sufficient small

$$\| u - u_h^* \|_{L^2(L^2)} + \| p - p_h^* \|_{L^\infty(L^2)} \leqslant Q h^{2k+2} \sum_{i=0}^{J+1} \left\| \frac{\partial^i p}{\partial t^i} \right\|_{L^2(H^{2k+4})}\ .$$

Proof: Recall that

$$u - u_h = \xi - \sum_{i=1}^{J} \xi_i + \psi_J\ ,$$
$$p - p_h = \eta - \sum_{i=1}^{J} \eta_i + \theta_J\ .$$

Thus, it follows from (6.1), (6.2), and (6.3) that

$$\|u - u_h^*\| \leq K\Big\{\|u - K_h * u\| + \sum_{|\nu| \leq k+1} \|D^\nu(K_h * \xi)\|_{-k-1} + \sum_{i=1}^{J} \sum_{|\nu| \leq k+1} \|D^\nu(K_h * \xi_i)\|_{-k-1} + \sum_{|\nu| \leq k+1} \|D^\nu(K_h * \psi_J)\|_{-k-1}\Big\},$$

and

$$\|u - u_h^*\|_{L^2(L^2)} \leq Kh^{2k+2}\|u\|_{L^2(H^{2k+2})} + K_1 \sum_{|\nu| \leq k+1} \Big\{\|\partial^\nu \xi\|_{L^2(H^{-k-1})} + \sum_{i=1}^{J} \|\partial^\nu \xi_i\|_{L^2(H^{-k-2+2i})} + \|\partial^\nu \psi_J\|_{L^2(H^{-k-1})}\Big\}$$

$$\leq K\Big\{h^{2k+2}\|p\|_{L^2(H^{2k+3})} + \sum_{|\nu| \leq k+1} \Big[\|\partial^\nu \xi\|_{L^2(H^{-k-1})} + \sum_{i=1}^{J} \|\partial^\nu \xi_i\|_{L^2(H^{-k-2+2i})} + \|\partial^\nu \psi_J\|_{L^2(L^2)}\Big]\Big\}.$$

Similarly,

$$\|p - p_h^*\|_{L^\infty(L^2)} \leq K\Big\{h^{2k+2}\|p\|_{L^\infty(H^{2k+2})} + \sum_{|\nu| \leq k+1} \Big[\|\partial^\nu \eta\|_{L^\infty(H^{-k-1})} + \sum_{i=1}^{J} \|\partial^\nu \eta_i\|_{L^\infty(H^{-k-1+2i})} + \|\partial^\nu \theta_J\|_{L^\infty(L^2)}\Big]\Big\}.$$

The estimates derived in the previous section now complete the proof.

When the index k of $V_h \times W_h$ is even, the same results can be proved, if J now is such that $2J = k + 2$ [13].

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