RAIRO

MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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RAIRO – Modélisation mathématique et analyse numérique,
tome 21, n° 3 (1987), p. 361-404.

http://www.numdam.org/item?id=M2AN_1987__21_3_361_0

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ASYMPTOTIC BEHAVIOUR FOR THE SOLUTION OF THE COMPRESSIBLE NAVIER-STOKES EQUATION, WHEN THE COMPRESSIBILITY GOES TO ZERO (*)

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Communicated by C. BARDOS

Résumé. — Nous étudions le comportement asymptotique des solutions (u^λ, p^λ) des équations de Navier-Stokes compressibles lorsque la compressibilité tend vers 0 ($\lambda \rightarrow \infty$) :

$$\begin{cases} \rho^\lambda (u_t^\lambda + (u^\lambda \cdot \nabla) u^\lambda) - \nu \Delta u^\lambda = -\lambda^2 \nabla p^\lambda, \\ p_t^\lambda + (\nabla p^\lambda) \cdot u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0, \\ u^\lambda(x, 0) = u_0(x) + \nabla \Phi_0(x) + \frac{u_1(x)}{\lambda}, \quad \operatorname{div} u_0 = 0, \\ p^\lambda(x, 0) = p_0 + \frac{p_1(x)}{\lambda^2}, \quad p_0 = \text{Cte}, \text{ où } p = A\rho^\gamma \text{ avec } \gamma > 1 \text{ et } A > 0. \end{cases}$$

Nous établissons d'abord l'existence globale en temps des solutions (u^λ, p^λ) , les estimations obtenues étant uniformes en λ .

Lorsque $\Phi_0 = 0$, nous prouvons que u^λ converge fortement vers u^∞ , solution des équations de Navier-Stokes incompressibles suivantes :

$$\begin{cases} \rho_0 (u_t^\infty + (u^\infty \cdot \nabla) u^\infty) - \nu \Delta u^\infty = -\nabla p^\infty, \\ \operatorname{div} u^\infty = 0 \quad \text{et} \quad u^\infty(x, 0) = u_0(x). \end{cases}$$

Lorsque $\Phi_0 \neq 0$, nous mettons en évidence un phénomène de couche initiale. Plus précisément, nous prouvons que $u^\lambda - u^\infty - v^\lambda$ converge fortement vers 0, où v^λ est la solution de l'équation couplée suivante :

$$\begin{cases} \rho_0 v_t^\lambda - \nu \Delta v^\lambda + \lambda \nabla q^\lambda = 0, \\ q_t^\lambda + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0, \\ v^\lambda(x, 0) = \nabla \Phi_0(x), \quad q^\lambda(x, 0) = 0. \end{cases}$$

Abstract. — We study the asymptotic behaviour of the solutions (u^λ, p^λ) of compressible Navier-Stokes' equations when compressibility goes to zero ($\lambda \rightarrow +\infty$) :

(*) Received in March 1986.

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$$\begin{cases} \rho^\lambda(u_t^\lambda + (u^\lambda \cdot \nabla) u^\lambda) - \nu \Delta u^\lambda = -\lambda^2 \nabla p^\lambda, \\ p_t^\lambda + (\nabla p^\lambda) \cdot u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0, \\ u^\lambda(x, 0) = u_0(x) + \nabla \Phi_0(x) + \frac{u_1(x)}{\lambda}, \quad \operatorname{div} u_0 = 0, \\ p^\lambda(x, 0) = p_0 + \frac{p_1(x)}{\lambda^2}, \quad p_0 = \text{Cte}, p = A\rho^\gamma \quad \text{with } \gamma > 1 \text{ and } A > 0. \end{cases}$$

We first establish global existence in time of the solutions (u^λ, p^λ) , the obtained estimates being uniform in λ .

When $\Phi_0 = 0$, we prove that u^λ strongly converges to u^∞ , solution of the following Navier-Stokes' incompressible equations :

$$\begin{cases} \rho_0(u_t^\infty + (u^\infty \cdot \nabla) u^\infty) - \nu \Delta u^\infty = -\nabla p^\infty, \\ \operatorname{div} u^\infty = 0 \quad \text{et} \quad u^\infty(x, 0) = u_0(x). \end{cases}$$

When $\Phi_0 \neq 0$, an initial layer phenomenon arises.

More precisely, we prove that $u^\lambda - u^\infty - v^\lambda$ strongly converges to zero, where v^λ is the solution of the following coupled equation :

$$\begin{cases} \rho_0 v_t^\lambda - \nu \Delta v^\lambda + \lambda \nabla q^\lambda = 0, \\ q_t^\lambda + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0, \\ v^\lambda(x, 0) = \nabla \Phi_0(x), \quad q^\lambda(x, 0) = 0. \end{cases}$$

I. INTRODUCTION

Our aim, in this paper, is to study the solutions of the equations of gases' dynamic :

$$(S) \quad \begin{cases} \rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) - \nu \Delta u = -\nabla p, \quad \nu > 0, \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad x \in \Omega \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x), \end{cases}$$

where the velocity u and the density ρ are unknown, the pression p being a given function of ρ .

Klainerman and Majda in [1] have proved the local existence of a smooth solution (u, ρ) of the system (S) in the case where Ω is the torus T^n of $\mathbb{R}^n$. In [2], they show the local existence of a smooth solution of compressible Euler's equations (when $\nu = 0$) for the whole space $\mathbb{R}^n$.

On the other part, Nishida and Matsumura, in [3], have obtained a global in time result for the system (S) coupled with an evolution equation for the temperature. In their work, they consider the case where $\Omega = \mathbb{R}^3$, where the gas is perfect and polytropic, and they are led to impose to the initial data to be small enough in $H^3(\mathbb{R}^3)$ norm.

As far as we are concerned, we are going to study the compressible system (S) when compressibility goes to 0, for the whole space $\mathbb{R}^n$, in any dimension $n \geq 2$.

Let us consider ρ as a function of p .

A. Lagha, in [4], defines compressibility as the quantity :

$$\varepsilon = \left[\frac{\partial p}{\partial \rho} (\rho_0) \right]^{-1},$$

where ρ_0 represents a first approximation of the gases' density.

She obtains a relation of the shape :

$$\rho = \rho_0 + \varepsilon p,$$

which leads her to study the following perturbed system :

$$(S^\varepsilon) \quad \begin{cases} \rho^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla) u^\varepsilon \right) - \nu \Delta u^\varepsilon = -\nabla p^\varepsilon, & x \in \mathbb{R}^n, \\ \varepsilon \frac{\partial p^\varepsilon}{\partial t} + \varepsilon u^\varepsilon \cdot \nabla p^\varepsilon + \rho^\varepsilon \nabla u^\varepsilon = 0, & t \in \mathbb{R}^+, \\ u^\varepsilon(x, 0) = u_0(x), & p^\varepsilon(x, 0) = p_0(x). \end{cases}$$

Temam uses the same definition of compressibility in [5], but he works in a bounded open set Ω of $\mathbb{R}^n$.

On the other hand, Majda, in [6], takes a more physical definition of compressibility by considering the state equation of a perfect gas :

$$p = A\rho^\gamma, \quad \gamma > 1.$$

From the initial system :

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \\ \rho \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla p = 0, \\ \rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x), \end{cases}$$

he is led to consider the following perturbed system :

$$\begin{cases} \frac{\partial \tilde{\rho}}{\partial t'} + \operatorname{div}(\tilde{\rho} \tilde{u}) = 0, \\ \tilde{\rho} \left(\frac{\partial \tilde{u}}{\partial t'} + (\tilde{u} \cdot \nabla) \tilde{u} \right) + \lambda^2 \nabla p(\tilde{\rho}) = 0, \\ \tilde{\rho}(x, 0) = \frac{\rho_0(x)}{\rho_m}, \quad \tilde{u}(x, 0) = \frac{u_0(x)}{|u_m|}, \end{cases}$$

where

$$\tilde{\rho} = \frac{\rho}{\rho_m}, \quad \tilde{u} = \frac{u}{|u_m|}, \quad t' = |u_m| t,$$

$$\rho_m = \max \rho_0(x) \quad \text{and} \quad |u_m| = \max |u_0(x)|.$$

The compressibility is there given by $1/\lambda^2$, with

$$\lambda^2 = \left[\frac{\partial p}{\partial \rho}(\rho_m) / |u_m|^2 \right] (\gamma A)^{-1}.$$

Majda proves, for « small enough » initial data, the existence of a smooth solution for the system (S^λ) , when λ is sufficiently large.

We have choosed to use this last definition of compressibility, while keeping the viscosity term : $-\nu \Delta u$.

This led us to consider a perturbed system, between those studied by A. Lagha and Majda, of the shape :

$$(S^\lambda) \quad \begin{cases} \rho^\lambda \left(\frac{\partial u^\lambda}{\partial t} + (u^\lambda \cdot \nabla) u^\lambda \right) - \nu \Delta u^\lambda = -\lambda^2 \nabla p^\lambda, \\ \frac{\partial p^\lambda}{\partial t} + (\nabla p^\lambda) \cdot u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0, \\ u^\lambda(x, 0) = u_0(x) + \frac{u_1(x)}{\lambda}, \quad p^\lambda(x, 0) = p_0 + \frac{p_1(x)}{\lambda^2}, \quad p_0 = \text{Cte}. \end{cases}$$

The shape of $u^\lambda(x, 0) = u_0^\lambda(x)$ and $p^\lambda(x, 0) = p_0^\lambda(x)$ issues from a formal asymptotic development (see [6]).

In the paragraph II, we have followed Lagha's way of proceeding which was taking its inspiration from Nishida and Matsumura's technics.

We introduce

$$E^\lambda(t) = |u^\lambda(t)|_{H^s}^2 + |\lambda(p^\lambda - p_0)|_{H^s}^2 \quad \text{where} \quad s > \left[\frac{n}{2} \right] + 1,$$

and we prove that, for sufficiently large λ and for « small enough » initial data, there exists some constant K_0 , independent of λ , so that :

$$\forall t \in \mathbb{R}^+, \quad E^\lambda(t) + \int_0^t |\nabla u^\lambda(\tau)|_{H^s}^2 d\tau + \int_0^t |\lambda \nabla(p^\lambda - p_0)|_{H^{s-1}}^2 d\tau \leq K_0.$$

This result permits to conclude, in any dimension $n \geq 2$, that there exists a unic smooth global solution of the system (S^λ) , for small enough initial data :

$$u^\lambda \in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-2}),$$

$$(p^\lambda - p_0) \in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-1}), \quad \text{where} \quad s > \left[\frac{n}{2} \right] + 1.$$

In the following part of our work, we study the asymptotic behaviour of the solutions (u^λ, p^λ) of the system (S^λ) when the compressibility goes to zero, so when λ goes to infinity.

In paragraph III, we add the classical following hypothesis :

$$\operatorname{div} u_0 = 0 ,$$

and we study the convergence of the solutions (u^λ, p^λ) to the solution (u^∞, p^∞) of the incompressible Navier-Stokes equations :

$$(S^\infty) \quad \begin{cases} \rho_0 \left(\frac{\partial u^\infty}{\partial t} + (u^\infty \cdot \nabla) u^\infty \right) - \nu \Delta u^\infty = - \nabla p^\infty , \\ \operatorname{div} u^\infty = 0 , \quad u^\infty(x, 0) = u_0(x) . \end{cases}$$

We first obtain supplementary estimates concerning the time derivatives, independent of λ sufficiently large :

$$\forall t \in \mathbb{R}^+ , \quad |u_t^\lambda|_{H^{s-2}}^2 + |\lambda(p^\lambda - p_0)_t|_{H^{s-2}}^2 + \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 d\tau \leq M(t) ,$$

where $s > \left[\frac{n}{2} \right] + 1$ and $M(t) \in L_{\text{loc}}^\infty(\mathbb{R}^+, \mathbb{R}^+)$.

This leads us to state the following weak convergence result, obtained by Klainerman and Majda in the case of the torus of $\mathbb{R}^n$ and by A. Lagha in $\mathbb{R}^2$:

If $\Omega = \mathbb{R}^n$, with $n \geq 2$, then

$$\begin{aligned} u^\lambda &\rightarrow u^\infty \quad \text{in } C_{\text{loc}}(0, \infty, H_{\text{loc}}^{s-1}) \quad \text{strongly} , \\ \lambda^2 \nabla p^\lambda &\rightarrow \nabla p^\infty \quad \text{in } L_{\text{loc}}^\infty(0, \infty, H^{s-2}) \quad \text{weak star (w.s.)} , \\ \rho^\lambda &\rightarrow \rho_0 \quad \text{in } C_B(0, \infty, W^{\infty, s-2}) \quad \text{strongly} . \end{aligned}$$

However, Klainerman and Majda, in [2], prove the strong convergence of the solutions (u^λ, p^λ) of compressible Euler's equations :

$$\begin{cases} \rho^\lambda \left(\frac{\partial u^\lambda}{\partial t} + (u^\lambda \cdot \nabla) u^\lambda \right) = - \lambda^2 \nabla p^\lambda , \\ \frac{\partial p^\lambda}{\partial t} + (\nabla p^\lambda) \cdot u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0 , \\ u^\lambda(x, 0) = u_0(x) + \frac{u_1(x)}{\lambda} , \quad p^\lambda(x, 0) = p_0 + \frac{p_1(x)}{\lambda^2} \\ p_0 = \text{Cte} , \quad \operatorname{div} u_0 = 0 , \end{cases}$$

to the solution (u^∞, p^∞) of incompressible Euler's equations :

$$\begin{cases} \rho_0 \left(\frac{\partial u^\infty}{\partial t} + (u^\infty \cdot \nabla) u^\infty \right) = -\nabla p^\infty, \\ \operatorname{div} u^\infty = 0, \quad u^\infty(x, 0) = u_0(x), \end{cases}$$

by imposing supplementary conditions to $|p^\infty|_{L^2}$ and $|p_t^\infty|_{L^2}$.

(It is, of course, a convergence on a finite time interval.)

In paragraph IV, we take our inspiration from that technic. We impose to the solution (u^∞, p^∞) of the system (S^∞) to verify the following hypothesis :

$$(H) \quad |p^\infty|_{L^2} + |p_t^\infty|_{L^2} \leq N(t), \quad \text{where } N \in L^\infty_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+).$$

Then, when the initial data $(u_0^\lambda, p_0^\lambda - p_0)$ are in $H^{s+2}(\mathbb{R}^n)$, we prove that there exists a locally bounded function $M(t)$ so that :

$$\forall t \in \mathbb{R}^+, \quad \forall \lambda \geq \lambda_0,$$

$$\lambda^2 |u^\lambda - u^\infty|_{H^s}^2 + |\lambda^2(p^\lambda - p_0) - p^\infty|_{H^s}^2 + \lambda^2 \int_0^t |\nabla(u^\lambda - u^\infty)|_{H^s}^2 d\tau \leq M(t).$$

In paragraph V, we have studied what happens with the convergence of (u^λ, p^λ) to (u^∞, p^∞) when we cut out the fundamental hypothesis : $\operatorname{div} u_0 = 0$. So we consider the initial data with the following more general shape :

$$\begin{aligned} u_0^\lambda(x) &= u_0(x) + \nabla \Phi_0(x) + \frac{u_1(x)}{\lambda}, \quad \text{with } \operatorname{div} u_0 = 0, \\ p_0^\lambda(x) &= p_0 + \frac{p_1(x)}{\lambda^2}, \quad p_0 = \text{Cte}. \end{aligned}$$

In fact, an initial layer phenomenon appears.

A fitting corrector term is given by the solution (v^λ, q^λ) of the following system (C^λ) :

$$(C^\lambda) \quad \begin{cases} \rho_0 \frac{\partial v^\lambda}{\partial t} - \nu \Delta v^\lambda = -\lambda \nabla q^\lambda, \\ \frac{\partial q^\lambda}{\partial t} + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0, \\ v^\lambda(x, 0) = \nabla \Phi_0(x), \quad q^\lambda(x, 0) = 0. \end{cases}$$

We prove, in appendix, that if Φ_0 is choosen regular enough, then v^λ verifies the following inequalities :

$$|v^\lambda(., t)|_{L^\infty} \leq \frac{C}{1 + \lambda t} \quad \text{if } n \geq 3 ,$$

$$|v^\lambda(., t)|_{L^\infty} \leq \frac{C}{\sqrt{1 + \lambda t}} \quad \text{if } n = 2 .$$

We obtain the following result :

If the solution (u^∞, p^∞) of the system (S^∞) satisfies to the hypothesis (H) and if the initial data are regular enough (we'll precise these assumptions later), there exists some locally bounded function $M(t)$ so that, for sufficiently large λ , we have :

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leq \frac{M(t)}{\lambda} (\text{Log } (1 + \lambda t) + 1) \quad \text{if } n \geq 3 ,$$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leq \frac{M(t)}{\sqrt{\lambda}} \quad \text{if } n = 2 .$$

We then end by a remark concerning an initial layer's phenomenon in the compressible Euler's equations.

Notations :

— $|\cdot|_{L^p}$ (or $|\cdot|_p$), $|\cdot|_{H^s}$ and $|\cdot|_{W^{k,p}}$ will design respectively the norms $L^p(\mathbb{R}^n)$, $H^s(\mathbb{R}^n)$ and $W^{k,p}(\mathbb{R}^n)$.

— We'll call « C » different numerical constants and « K » different quantities only depending on initial data.

— Finally, $M(t)$ or $N(t)$ will design different increasing functions of $L_{\text{loc}}^\infty(\mathbb{R}^+, \mathbb{R}^+)$.

II. INDEPENDENT OF λ ESTIMATES. GLOBAL EXISTENCE

A. Independent of λ estimates

Let us consider the system (S^λ) :

$$(2.1) \quad \rho^\lambda(u_t^\lambda + (u^\lambda \cdot \nabla) u^\lambda) - v \Delta u^\lambda = -\lambda^2 \nabla p^\lambda, \quad x \in \mathbb{R}^n ,$$

$$(2.2) \quad p_t^\lambda + \nabla p^\lambda \cdot u^\lambda + \gamma p^\lambda \text{div } u^\lambda = 0, \quad t \in \mathbb{R}^+,$$

$$(2.3) \quad u^\lambda(x, 0) = u_0^\lambda(x), \quad p^\lambda(x, 0) = p_0 + \frac{p_1(x)}{\lambda^2}, \quad p_0 = \text{Cte} ,$$

where $u_0^\lambda \in H^s$, $p_0 > 0$, $p_1 \in H^s$, s being an integer verifying $s > s_0 = \left[\frac{n}{2} \right] + 1$, and where $p = A\rho^\psi$, $\gamma > 1$.

Let us note that equation (2.2) may be written :

$$(2.4) \quad \rho_t^\lambda + \operatorname{div} (\rho^\lambda u^\lambda) = 0.$$

We are going to assume « a priori » that (u^λ, p^λ) satisfies the following $H(K, T)$ hypothesis :

There exists $T > 0$ and $K > 0$ so that (u^λ, p^λ) is a solution of (S^λ) on the intervall $[0, T]$, verifying :

$$\begin{aligned} u^\lambda &\in C([0, T], H^s) \cap C^1([0, T], H^{s-2}), \\ p^\lambda &\in C([0, T], H^s) \cap C^1([0, T], H^{s-1}) \text{ and} \\ \forall t \in [0, T], \quad E^\lambda(t) &\leq K, \end{aligned}$$

where $E^\lambda(t)$ is defined by the relation :

$$E^\lambda(t) = |u^\lambda(t)|_{H^s}^2 + |\lambda(p^\lambda - p_0)|_{H^s}^2.$$

We are going to prove that, in these conditions, there exists some constant $C_0(K)$, independent of T and λ , and there exists $\lambda_0 > 0$, so that :

$$\forall t \in [0, T], \quad \forall \lambda \geq \lambda_0,$$

$$E^\lambda(t) + \int_0^t |\nabla u^\lambda|_{H^s}^2 d\tau + \int_0^t |\lambda \nabla (p^\lambda - p_0)|_{H^{s-1}}^2 d\tau \leq C_0(K) \cdot E_0^\lambda$$

(where $E_0^\lambda = E^\lambda(0)$).

First, let us make some preliminary remarks which will appreciably simplify the proof.

Let us note

$$\begin{aligned} \tilde{p}^\lambda(x, t) &= \lambda(p^\lambda(x, t) - p_0) \quad \text{and} \\ \tilde{\rho}^\lambda(x, t) &= \lambda(\rho^\lambda(x, t) - \rho_0) \quad \text{where} \quad p_0 = A\rho_0^\gamma. \end{aligned}$$

LEMMA 1 : *Under hypothesis $H(K, T)$, and if $\lambda \geq \lambda_1$, then there exists four strictly positive constants p_1, p_2, ρ_1, ρ_2 , so that :*

$$\begin{aligned} \forall x \in \mathbb{R}^n, \quad \forall t \in [0, T], \quad 0 < p_1 \leq p^\lambda \leq p_2 \\ \text{and} \quad 0 < \rho_1 \leq \rho^\lambda \leq \rho_2. \end{aligned}$$

In fact,

$$\begin{aligned} |p^\lambda - p_0|_\infty &\leq |p^\lambda - p_0|_{H^s} \quad \left(\text{since } s > s_0 > \frac{n}{2} \right) \\ &\leq \frac{|\tilde{p}^\lambda|}{\lambda} H^s \leq \frac{K}{\lambda}. \end{aligned}$$

We have just to choose $\lambda_1 = \frac{2K}{p_0}$, which gives $p_1 = \frac{p_0}{2}$, $p_2 = \frac{3p_0}{2}$.

Moreover, if $h(\rho) = A\rho^\gamma = p^\lambda$, then $0 < h^{-1}\left(\frac{p_0}{2}\right) \leq \rho^\lambda \leq h^{-1}\left(\frac{3p_0}{2}\right)$.

LEMMA 2: *There exists two constants C_1 and C_2 and $\lambda_2 = \lambda_2(K)$, so that, if $\lambda \geq \lambda_2 \geq \lambda_1$, we get:*

$$\forall p \in [2, +\infty], \quad C_1 |\tilde{p}^\lambda|_p \leq |\tilde{p}^\lambda|_p \leq C_2 |\tilde{p}^\lambda|_p$$

and

$$C_1 |D\tilde{p}^\lambda|_p \leq |D\tilde{p}^\lambda|_p \leq C_2 |D\tilde{p}^\lambda|_p.$$

Let us note $k = h^{-1}$. Then there exists $p_\theta \in [p_0, p^\lambda]$, so that:

$$\tilde{p}^\lambda = \lambda [k(p^\lambda) - k(p_0)] = \lambda (p^\lambda - p_0) \cdot k'(p_0) + \frac{\lambda}{2} (p^\lambda - p_0)^2 \cdot k''(p_\theta).$$

Then,

$$|\tilde{p}^\lambda - k'(p_0) \tilde{p}^\lambda|_p \leq \frac{1}{2\lambda} |\tilde{p}^\lambda|_p |\tilde{p}^\lambda|_\infty |k''(p_\theta)|_\infty \leq \frac{C}{\lambda} |\tilde{p}^\lambda|_p.$$

So, for large enough λ , $|\tilde{p}^\lambda|_p$ and $|D\tilde{p}^\lambda|_p$ are comparable.

Moreover, $D\tilde{p}^\lambda = k'(p^\lambda) \cdot D\tilde{p}^\lambda$; k and all its derivatives being locally bounded on $\mathbb{R}_+^*$, we may conclude with lemma 1.

LEMMA 3:

(i) $D^s \tilde{p}^\lambda$ may be written:

$$D^s \tilde{p}^\lambda = k'(p^\lambda) \cdot D^s \tilde{p}^\lambda + \frac{\chi}{\lambda} \quad \text{where} \quad |\chi|_{L^2} \leq C |\nabla \tilde{p}^\lambda|_{H^{s-1}}.$$

In particular, $|D\tilde{p}^\lambda|_{H^{s-1}}$ and $|D\tilde{p}^\lambda|_{H^{s-1}}$ are comparable as soon as λ is sufficiently large, $\lambda \geq \lambda_3 \geq \lambda_2$.

(ii) $\left| D^{s-1} \left(\frac{1}{\rho^\lambda} \right) \right|_{L^2} \leq \frac{C}{\lambda}$, as soon as λ is large enough.

Proof:

(i)

$$\begin{aligned} D^s \tilde{p}^\lambda &= k'(p^\lambda) D^s (\tilde{p}^\lambda) + \\ &+ \underbrace{\sum_{p=2}^s \sum_{\substack{i_1 + \dots + i_s = p \\ i_1 + 2i_2 + \dots + (s-1)i_{s-1} = s}} C_{i_p, p} (D\tilde{p}^\lambda)^{i_1} \dots (D^{s-1} \tilde{p}^\lambda)^{i_{s-1}} \frac{k^{(p)}(p^\lambda)}{\lambda^{p-1}}}_{\chi/\lambda}. \end{aligned}$$

If $\lambda \geq \max(K^2, 1)$, we deduce from hypothesis $H(K, T)$ that $|\chi|_{L^2} \leq C |\nabla \tilde{p}^\lambda|_{H^{s-1}}$. Then, since $0 < k'(p_2) \leq k'(p^\lambda) \leq k'(p_1)$, we get that $|D\tilde{p}^\lambda|_{H^{s-1}}$ and $|D\tilde{p}^\lambda|_{H^{s-1}}$ are comparable.

(ii) We just have to note that if $\phi(x) = x^{-1}$, then :

$$D^{s-1} \left(\frac{1}{\rho^\lambda} \right) = \sum_{p=1}^{s-1} \sum_{\substack{i_1 + \dots + i_{s-1} = p \\ i_1 + \dots + (s-1)i_{s-1} = s-1}} C_{i_p, p} (D\tilde{p}^\lambda)^{i_1} \dots (D^{s-1} \tilde{p}^\lambda)^{i_{s-1}} \frac{\phi^{(p)}(\rho^\lambda)}{\lambda^p}.$$

We end with the assumption $H(K, T)$.

LEMMA 4 : If u, v and w are smooth functions,

$$\int (v \cdot \nabla) u \cdot w \, dx = - \int (v \cdot \nabla) w \cdot u \, dx - \int (u \cdot w) \operatorname{div} v \, dx.$$

In particular,

$$\int (v \cdot \nabla) u \cdot u \, dx = - \frac{1}{2} \int |u|^2 \operatorname{div} v \, dx.$$

LEMMA 5 [7] : Let f and g be two smooth functions

$$(2.5) \quad |D^k(fg) - fD^k g|_p \leq C |Df|_r |D^{k-1} g|_{r'} + C |D^k f|_s |g|_{s'};$$

$$(2.6) \quad |D^k(fg)|_p \leq C |f|_r |D^k g|_{r'} + C |D^k f|_s |g|_{s'},$$

where $k > 0, p \in [1, +\infty]$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{r'} = \frac{1}{s} + \frac{1}{s'}$.

We are now able to establish the desired « a priori » estimates.

First step : L^2 -Norms of u^λ and p^λ .

Multiplying (2.1) by u^λ , and (2.4) by $\frac{|u^\lambda|^2}{2}$, we get :

$$\begin{aligned} \frac{|u^\lambda|^2}{2} \rho_t^\lambda + \frac{|u^\lambda|^2}{2} \operatorname{div} (\rho^\lambda u^\lambda) + \frac{\rho^\lambda}{2} |u^\lambda|_t^2 + \\ + (\rho^\lambda u^\lambda \nabla) u^\lambda \cdot u^\lambda - \nu \Delta u^\lambda \cdot u^\lambda = -\lambda \nabla \tilde{p}^\lambda \cdot u^\lambda. \end{aligned}$$

Then, integrating on $\mathbb{R}^n$:

$$\begin{aligned} \frac{\partial}{\partial t} \int \frac{\rho^\lambda |u^\lambda|^2}{2} + \nu \int |\nabla u^\lambda|^2 + \int (\rho^\lambda u^\lambda \nabla) u^\lambda \cdot u^\lambda + \\ + \int \frac{|u^\lambda|^2}{2} \operatorname{div} (\rho^\lambda u^\lambda) = \lambda \int \tilde{p}^\lambda \operatorname{div} u^\lambda. \end{aligned}$$

We deduce from lemma 4 that :

$$(2.7) \quad \frac{\partial}{\partial t} \int \frac{\rho^\lambda |u^\lambda|^2}{2} dx + \nu \int |\nabla u^\lambda|^2 dx = \lambda \int (\tilde{p}^\lambda \operatorname{div} u^\lambda) dx .$$

Let us introduce

$$W(\rho^\lambda) = \int_{\rho_0}^{\rho^\lambda} \frac{\lambda \tilde{p}^\lambda(s)}{s^2} ds .$$

Multiplying (2.4) by $\frac{\partial}{\partial \rho} (\rho^\lambda W)$, we get :

$$\begin{aligned} \frac{\partial}{\partial t} \int \rho^\lambda W dx + \int \operatorname{div} (\rho^\lambda u^\lambda) W dx + \\ + \int (\rho^\lambda)^2 \operatorname{div} u^\lambda \frac{\partial W}{\partial \rho} dx + \int \rho^\lambda u^\lambda \cdot \nabla \rho^\lambda \frac{\partial W}{\partial \rho} dx = 0 . \end{aligned}$$

Now,

$$\int \operatorname{div} (\rho^\lambda u^\lambda) W dx = - \int \rho^\lambda u^\lambda \cdot \nabla W dx = - \int \rho^\lambda u^\lambda \cdot \nabla \rho^\lambda \frac{\partial W}{\partial \rho} dx ,$$

$$\text{and} \quad \int (\rho^\lambda)^2 \operatorname{div} u^\lambda \frac{\partial W}{\partial \rho} dx = \lambda \int \tilde{p}^\lambda \operatorname{div} u^\lambda dx ,$$

what gives us :

$$(2.8) \quad \frac{\partial}{\partial t} \int \rho^\lambda W dx + \lambda \int \tilde{p}^\lambda \operatorname{div} u^\lambda dx = 0 .$$

We then can deduce from (2.7) and (2.8) that :

$$\frac{\partial}{\partial t} \left[\int \rho^\lambda W dx + \int \rho^\lambda \frac{|u^\lambda|^2}{2} dx \right] + \nu \int |\nabla u^\lambda|^2 dx = 0 ,$$

and thanks to lemma 1 :

$$\int \rho^\lambda W dx + \frac{\rho_1}{2} |u^\lambda(t)|_2^2 + \nu \int_0^t |\nabla u^\lambda|_2^2 d\tau \leq \int |\rho^\lambda W(0)| dx + \frac{\rho_2}{2} |u_0^\lambda|_2^2 .$$

So we have to estimate $\int \rho^\lambda W dx$.

(i) *Minoration* : Let us consider

$$\Phi(\rho^\lambda) = \rho^\lambda \int_{\rho_0}^{\rho^\lambda} \frac{\lambda \tilde{p}^\lambda(s)}{s^2} ds.$$

The shape of Φ gives immediatly :

$$\Phi(\rho_0) = \Phi'(\rho_0) = 0 \quad \text{and} \quad \Phi''(\rho_0) = A\lambda^2 \gamma \rho_0^{\gamma-2}$$

and

$$\Phi'''(\rho) = A\gamma(\gamma-2)\lambda^2 \rho^{\gamma-3}.$$

$$\begin{aligned} \text{So} \quad \Phi(\rho^\lambda) &= \frac{A}{2} (\rho^\lambda - \rho_0)^2 \lambda^2 \gamma \rho_0^{\gamma-2} + \frac{A}{6} \gamma(\gamma-2) \lambda^2 (\rho^\lambda - \rho_0)^3 \rho_0^{\gamma-3} \\ &= (\tilde{\rho}^\lambda)^2 \left[\frac{A}{2} \gamma \rho_0^{\gamma-2} + \frac{A}{6\lambda} \gamma(\gamma-2) \tilde{\rho}^\lambda \cdot \rho_0^{\gamma-3} \right], \end{aligned}$$

with $\rho_\theta = \rho_0 + \theta(\rho^\lambda - \rho_0)$, $\theta \in [0, 1]$.

Now,

$$\left| \frac{A}{6\lambda} \gamma(\gamma-2) \tilde{\rho}^\lambda \cdot \rho_0^{\gamma-3} \right|_\infty \leq \frac{A}{6\lambda} \gamma(\gamma-2) K \rho_0^{\gamma-3} \leq \frac{CK}{\lambda}.$$

Since $C = \frac{A}{2} \gamma \rho_0^{\gamma-2} > 0$, we get that : for λ large enough, $\lambda \geq \lambda_4(K) \geq \lambda_3$, we have :

$$\int \Phi(\rho^\lambda) dx \geq \frac{C}{2} |\tilde{\rho}^\lambda|_2^2 \geq \frac{C}{2} C_1 |\tilde{\rho}^\lambda|_2^2.$$

(ii) *Majoration* :

Since $\tilde{p}^\lambda(s) = \lambda A (s^\gamma - \rho_0^\gamma)$, then $\sup_{[\rho_0, \rho^\lambda]} |\tilde{p}^\lambda(s)| = |\tilde{p}^\lambda(\rho^\lambda)|$.

Then

$$\begin{aligned} \int \left| \rho^\lambda \int_{\rho_0}^{\rho^\lambda} \frac{\lambda \tilde{p}^\lambda(s)}{s^2} ds \right| dx &\leq \int \rho^\lambda \lambda |\tilde{p}^\lambda(\rho^\lambda)| \left| \int_{\rho_0}^{\rho^\lambda} \frac{ds}{s^2} \right| dx \\ &\leq \int \rho^\lambda \lambda |\tilde{p}^\lambda| \frac{|\rho^\lambda - \rho_0|}{\rho^\lambda \rho_0} dx \leq \frac{1}{\rho_0} \int |\tilde{\rho}^\lambda| |\tilde{p}^\lambda| dx. \end{aligned}$$

So, thanks to lemma 2, we get that :

$$\int |\rho^\lambda W| dx \leq \frac{C_2}{\rho_0} |\tilde{\rho}^\lambda|_2^2 = C |\tilde{\rho}^\lambda|_2^2.$$

Finally, we conclude that :

$$(2.9) \quad \left\{ \begin{array}{l} \text{Under hypothesis } H(K, T), \text{ there exists } \lambda_4 = \lambda_4(K), \text{ and} \\ \text{some constant } C, \text{ independent of } T, \lambda \text{ and } K, \text{ so that,} \\ \forall t \in [0, T], \quad \forall \lambda \geq \lambda_4, \quad \text{we have :} \\ |u^\lambda|_2^2 + |\tilde{p}^\lambda|_2^2 + |\tilde{\rho}^\lambda|_2^2 + \nu \int_0^t |\nabla u^\lambda|_2^2 d\tau \leq C \cdot E_0^\lambda, \\ \text{where } E_0^\lambda = |u_0^\lambda|_{H^s}^2 + |\tilde{p}_0^\lambda|_{H^s}^2 \text{ and } \tilde{p}_0^\lambda(x) = \lambda(p^\lambda(x, 0) - p_0). \end{array} \right.$$

2nd Step : Estimate of $\int_0^t |D\tilde{p}^\lambda|_2^2 d\tau$.

Multiplying equation (2.1) by $-\frac{\nabla \tilde{p}^\lambda}{\lambda \rho^\lambda}$, and integrating in time and on $\mathbb{R}^n$, we get :

$$\begin{aligned} \int_0^t \int \frac{\nabla \tilde{p}^\lambda \cdot \nabla \tilde{p}^\lambda}{\rho^\lambda} dx d\tau &= - \int_0^t \int u_t^\lambda \frac{\nabla \tilde{p}^\lambda}{\lambda} dx d\tau - \int_0^t \int \frac{(u^\lambda \cdot \nabla) u^\lambda \cdot \nabla \tilde{p}^\lambda}{\lambda} dx d\tau \\ &\quad + \nu \int_0^t \int \frac{\Delta u^\lambda \cdot \nabla \tilde{p}^\lambda}{\lambda \rho^\lambda} dx d\tau. \end{aligned}$$

Now

$$\begin{aligned} \int_0^t \int u_t^\lambda \frac{\nabla \tilde{p}^\lambda}{\lambda} dx d\tau &= \left[\int u^\lambda \frac{\nabla \tilde{p}^\lambda}{\lambda} dx \right]_0^t - \int_0^t \int u^\lambda \cdot \nabla \rho_t^\lambda dx d\tau \\ &= \left[\int u^\lambda \frac{\nabla \tilde{p}^\lambda}{\lambda} dx \right]_0^t + \int_0^t \int \operatorname{div} u^\lambda \cdot \operatorname{div} (\rho^\lambda u^\lambda) dx d\tau. \end{aligned}$$

Finally,

$$\begin{aligned} (a) &= \int_0^t \int \frac{\nabla \tilde{p}^\lambda \cdot \nabla \tilde{p}^\lambda}{\rho^\lambda} dx d\tau \\ &= \left[\int u^\lambda \frac{\nabla \tilde{p}^\lambda}{\lambda} dx \right]_0^t + \int_0^t \int \operatorname{div} u^\lambda \cdot \operatorname{div} (\rho^\lambda u^\lambda) dx d\tau \\ &\quad + \nu \int_0^t \int \frac{\Delta u^\lambda \cdot \nabla \tilde{p}^\lambda}{\lambda \rho^\lambda} dx d\tau - \int_0^t \int \frac{(u^\lambda \cdot \nabla) u^\lambda \cdot \nabla \tilde{p}^\lambda}{\lambda} dx d\tau \\ &= (b) + (c) + (d) + (e). \end{aligned}$$

(i) We get from lemma 1 :

$$(a) = \int_0^t \int \frac{(\nabla \tilde{p}^\lambda)^2}{\rho^\lambda} k'(p^\lambda) dx d\tau \geq \frac{k'(p_2)}{\rho_1} \int_0^t |\nabla \tilde{p}^\lambda|_2^2 d\tau,$$

$$\begin{aligned}
 \text{(ii)} \quad |(b)| &\leq |u^\lambda(t)|_2^2 + \frac{1}{\lambda^2} |D\tilde{p}^\lambda|_2^2 + |u^\lambda(0)|_2^2 + \frac{1}{\lambda^2} |D\tilde{p}^\lambda(0)|_2^2 \\
 &\leq C \cdot E_0^\lambda + \frac{1}{\lambda^2} |D^s \tilde{p}^\lambda(t)|_2^2 \quad \text{as soon as } \lambda \geq \sup(\lambda_4, 1).
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad |(c) + (e)| &\leq 2 \int_0^t |u^\lambda|_\infty |Du^\lambda|_2 \frac{|D\tilde{p}^\lambda|_2}{\lambda} d\tau \\
 &\leq \frac{4K}{\lambda} \int_0^t |Du^\lambda|_2^2 d\tau + \frac{1}{\lambda} \int_0^t |D\tilde{p}^\lambda|_2^2 d\tau \quad (|u|_\infty \leq \sqrt{K}) \\
 &\leq \frac{4KC}{\lambda\nu} E_0^\lambda + \frac{1}{\lambda} \int_0^t |D\tilde{p}^\lambda|_2^2 d\tau \quad (\text{by (2.9)})
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad |(d)| &\leq \frac{\nu^2}{\rho_1^2 \lambda} \int_0^t |\nabla \tilde{p}^\lambda|_2^2 d\tau + \frac{1}{\lambda} \int_0^t |D^2 u^\lambda|_2^2 d\tau \\
 &\leq \frac{\nu^2}{\rho_1^2 \lambda} \int_0^t |\nabla \tilde{p}^\lambda|_2^2 d\tau + \frac{C}{\lambda\nu} E_0^\lambda + \frac{1}{\lambda} \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau \quad (\text{by (2.9)}).
 \end{aligned}$$

We deduce from all above that :

$$\begin{aligned}
 \int_0^t |\nabla \tilde{p}^\lambda|_2^2 d\tau &\leq C \left(1 + \frac{K}{\lambda}\right) E_0^\lambda + \frac{1}{\lambda^2} |D^s \tilde{p}^\lambda|_2^2 + \\
 &\quad + \frac{1}{\lambda} \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau + \frac{C}{\lambda} \int_0^t |\nabla \tilde{p}^\lambda|_2^2 d\tau.
 \end{aligned}$$

We conclude from that :

$$(2.10) \quad \left\{ \begin{array}{l} \text{Under hypothesis } H(K, T), \text{ there exists} \\ \lambda_5 = \lambda_5(K) \geq \max(\lambda_4, 1, K) \\ \text{and some constant } C, \text{ independent of } \lambda, T, \text{ and } K \text{ so that,} \\ \forall t \in [0, T], \quad \forall \lambda \geq \lambda_5, \\ \int_0^t |\nabla \tilde{p}^\lambda(\tau)|_2^2 d\tau \leq C E_0^\lambda + \frac{1}{\lambda^2} |D^s \tilde{p}^\lambda|_2^2 + \frac{1}{\lambda} \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau. \end{array} \right.$$

The norm $|u|_{H^s}$ being equivalent to the norm $(|u|_2^2 + |D^s u|_2^2)$, we go straightly to the :

3rd Step : L^2 -Norm of the derivatives of order s .

Deriving equations (2.1) and (2.2) s times yields to :

$$(2.11) \quad \partial^s(\rho^\lambda u_t^\lambda) + \partial^s(\rho^\lambda(u^\lambda \cdot \nabla) u^\lambda) - \nu \Delta \partial^s u^\lambda = -\lambda \nabla \partial^s \tilde{p}^\lambda,$$

$$(2.12) \quad \partial^s \tilde{p}_t^\lambda + \partial^s(\nabla \tilde{p}^\lambda \cdot u^\lambda) + \gamma \partial^s(\tilde{p}^\lambda \operatorname{div} u^\lambda) + \lambda \gamma p_0 \partial^s \operatorname{div} u^\lambda = 0.$$

The operation

$$\int \left[(2.11) \cdot \gamma p_0 \partial^s u^\lambda + (2.12) \cdot \partial^s \tilde{p}^\lambda + (2.4) \gamma p_0 \frac{(\partial^s u^\lambda)^2}{2} \right] dx$$

leads to the following equality :

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{\gamma p_0}{2} |\sqrt{\rho^\lambda} \partial^s u^\lambda|_2^2 + \frac{1}{2} |\partial^s \tilde{p}^\lambda|_2^2 \right] + \nu \gamma p_0 |\nabla \partial^s u^\lambda|_2^2 = \\ & = -\gamma p_0 \int [\partial^s(\rho^\lambda u_t^\lambda) - \rho^\lambda \partial^s u_t^\lambda] \cdot \partial^s u^\lambda dx - \\ & \quad - \gamma p_0 \int \partial^s((\rho^\lambda u^\lambda \cdot \nabla) u^\lambda) \cdot \partial^s u^\lambda dx \\ & \quad - \gamma p_0 \int \operatorname{div}(\rho^\lambda u^\lambda) \frac{(\partial^s u^\lambda)^2}{2} dx - \int (\nabla(\partial^s \tilde{p}^\lambda) \cdot u^\lambda) \partial^s \tilde{p}^\lambda dx \\ & \quad - \int [\partial^s(\nabla \tilde{p}^\lambda \cdot u^\lambda) - (\partial^s \nabla \tilde{p}^\lambda) u^\lambda] \partial^s \tilde{p}^\lambda dx - \gamma \int \partial^s(\tilde{p}^\lambda \operatorname{div} u^\lambda) \partial^s \tilde{p}^\lambda dx \\ & = (a) + (b) + (c) + (d) + (e) + (f). \end{aligned}$$

(i) Let us estimate (a). Thanks to (2.5), we may write :

$$\begin{aligned} |(a)| & \leq C |D^s u^\lambda|_2 [|D \rho^\lambda|_\infty |D^{s-1} u_t^\lambda|_2 + |D^s \rho^\lambda|_2 |u_t^\lambda|_\infty] \\ & \leq C |D^s u^\lambda|_2 \frac{|D \tilde{p}^\lambda|_\infty}{\lambda} |D^{s-1} u_t^\lambda|_2 + C |D^s u^\lambda|_2 |D^s \tilde{p}^\lambda|_2 \frac{1}{\lambda} |u_t^\lambda|_\infty. \end{aligned}$$

Now (2.1) gives us :

$$u_t^\lambda = -\lambda \frac{\nabla \tilde{p}^\lambda}{\rho^\lambda} + \nu \frac{\Delta u^\lambda}{\rho^\lambda} - (u^\lambda \cdot \nabla) u^\lambda.$$

Thus

$$\frac{1}{\lambda} |u_t^\lambda|_\infty \leq \frac{1}{\rho_1} |D \tilde{p}^\lambda|_\infty + \frac{\nu}{\rho_1 \lambda} |\Delta u^\lambda|_\infty + \frac{\sqrt{K}}{\lambda} |Du^\lambda|_\infty.$$

We now use an inequality due to Gagliardo and Nirenberg [9].

So, with hypothesis $H(K, T)$, we can get that :

$$\begin{aligned} |D^s u^\lambda|_2 |D^s \tilde{p}^\lambda|_2 \frac{1}{\lambda} |u_t^\lambda|_\infty & \leq \\ & \leq C \sqrt{K} \left[|D \tilde{p}^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + |Du^\lambda|_2^2 + \frac{|D^{s+1} u^\lambda|_2^2}{\lambda} \right], \end{aligned}$$

as soon as $\lambda \geq \lambda_5$.

On the other hand,

$$\begin{aligned} |D^{s-1} u_t^\lambda|_2 &= \left| D^{s-1} \frac{\rho^\lambda u_t^\lambda}{\rho^\lambda} \right|_2 \leq \\ &\leq C |\rho^\lambda u_t^\lambda|_\infty \left| D^{s-1} \frac{1}{\rho^\lambda} \right|_2 + C |D^{s-1}(\rho^\lambda u_t^\lambda)|_2 \cdot \left| \frac{1}{\rho^\lambda} \right|_\infty. \end{aligned}$$

We know that (lemma 3 (ii)), as soon as λ is large enough,

$$\left| D^{s-1} \frac{1}{\rho^\lambda} \right|_2 \leq \frac{C}{\lambda}.$$

Moreover, using assertion (2.5) of lemma 5 and hypothesis $H(K, T)$, we get :

$$\begin{aligned} |D^{s-1}(\rho^\lambda u_t^\lambda)|_2 &\leq \lambda |D^s \tilde{p}^\lambda|_2 + \nu |D^{s+1} u^\lambda|_2 + |D^{s-1}(\rho^\lambda u^\lambda \cdot \nabla) u^\lambda|_2 \\ &\leq C \lambda |D^s \tilde{p}^\lambda|_2 + C \sqrt{K} |Du^\lambda|_2 + C \sqrt{K} |D^{s+1} u^\lambda|_2. \end{aligned}$$

So, when λ is large enough, $\lambda \geq \lambda_6(K) \geq \lambda_5$, we have :

$$\begin{aligned} |D^s u^\lambda|_2 |D \tilde{p}^\lambda|_\infty \frac{1}{\lambda} |D^{s-1} u_t^\lambda|_2 &\leq \\ &\leq C (1 + K^{3/2}) \left[|D \tilde{p}^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + |Du^\lambda|_2^2 + \frac{|D^{s+1} u^\lambda|_2^2}{\lambda} \right], \end{aligned}$$

and (a) verifies the same inequality.

(ii) Thanks to lemma 5 (2.6), lemma 3, and hypothesis $H(K, T)$, we deduce the following estimate for (b) + (c) : ($\beta \geq 1$)

$$\begin{aligned} |(b) + (c)| &\leq C (1 + K^\beta) C(\alpha) |Du^\lambda|_2^2 + \\ &+ \alpha |D^{s+1} u^\lambda|_2^2 + \frac{K}{\lambda} (|Du^\lambda|_2^2 + |D^{s+1} u^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2). \end{aligned}$$

(We also need the inequality :

$$\begin{aligned} |D^{s+1} u|_2 |D^s u|_2 &\leq C |Du|_2^{1-a} |D^{s+1} u|_2^{a+1} \leq \\ &\leq C(\alpha) |Du|_2^2 + \alpha |D^{s+1} u|_2^2. \end{aligned}$$

(iii) For (d), we just have to write :

$$\left| \int (\nabla \partial^s \tilde{p}^\lambda \cdot u^\lambda) \partial^s \tilde{p}^\lambda dx \right| = \left| - \int \operatorname{div} u^\lambda \frac{(\partial^s \tilde{p}^\lambda)^2}{2} dx \right| \leq C \sqrt{K} |D^s \tilde{p}^\lambda|_2^2.$$

(iv) Thanks again to lemma 3, to assertions (2.5) and (2.6) of lemma 5 and to $H(K, T)$, we finally estimate (e) and (f) in the following way :

$$|(e) + (f)| \leq (1 + K) C(\alpha) |D^s \tilde{p}^\lambda|_2^2 + \alpha |D^{s+1} u^\lambda|_2^2 + (1 + K) C(\alpha) |Du^\lambda|_2^2.$$

So, taking into account these estimates and lemma 1, we find, integrating on $[0, T]$, that there exists $\beta > 1$ and $\lambda_6 = \lambda_6(K)$ so that :

$$\forall t \in [0, T], \forall \lambda \geq \lambda_6,$$

$$\begin{aligned} |D^s u^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau &\leq \\ &\leq C E_0^\lambda + C(\alpha)(1 + K^\beta) \int_0^t |Du^\lambda|_2^2 d\tau + C(1 + K^{3/2}) \int_0^t |D\tilde{p}^\lambda|_2^2 d\tau \\ &+ C(\alpha)(1 + K^{3/2}) \int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau + \left(\alpha + \frac{KC}{\lambda}\right) \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau. \end{aligned}$$

Then, using results (2.9) and (2.10), choosing $\alpha = 1/4$, and $\lambda_7 = \max(\lambda_6, 4KC)$, we obtain the following result :

$$(2.11) \quad \left| \begin{array}{l} \text{Under hypothesis } H(K, T), \text{ there exists} \\ \lambda_7 = \lambda_7(K) \geq \lambda_6 \geq \dots \geq \lambda_1, \\ \beta > 1, \text{ and some constant } C, \text{ independent of } \lambda, K \text{ and } T, \text{ so that:} \\ \forall t \in [0, T], \forall \lambda \geq \lambda_7, \\ |D^s u^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau \leq \\ \leq C(1 + K^\beta) \left(E_0^\lambda + \int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau \right). \end{array} \right.$$

We now have to estimate $\int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau$, which is the aim of the :

4th Step : Estimate of $\int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau$.

First, let us note that if we call $v^\lambda = \rho^\lambda u^\lambda$, equation (2.1) becomes :

$$(2.12) \quad v_t^\lambda + (v^\lambda \cdot \nabla) u^\lambda + u^\lambda \operatorname{div} v^\lambda - \nu \Delta u^\lambda = -\lambda \nabla \tilde{p}^\lambda.$$

Deriving $(s-1)$ times in x this equation, multiplying by $-\frac{\nabla \partial^{s-1} \tilde{p}^\lambda}{\lambda}$, and integrating on $\mathbb{R}^n \times [0, T]$, we obtain :

$$\begin{aligned}
\int_0^t \int \nabla \partial^{s-1} \tilde{p}^\lambda \cdot \nabla \partial^{s-1} \tilde{p}^\lambda dx d\tau = & \\
& - \int_0^t \int \partial^{s-1} v_t^\lambda \cdot \frac{\nabla \partial^{s-1} \tilde{p}^\lambda}{\lambda} dx d\tau + \nu \int_0^t \int \Delta \partial^{s-1} u^\lambda \cdot \frac{\nabla \partial^{s-1} \tilde{p}^\lambda}{\lambda} dx d\tau \\
& - \int_0^t \int \partial^{s-1} ((v^\lambda \cdot \nabla) u^\lambda) \frac{\nabla \partial^{s-1} \tilde{p}^\lambda}{\lambda} dx d\tau \\
& - \int_0^t \int \partial^{s-1} (u^\lambda \operatorname{div} v^\lambda) \frac{\nabla \partial^{s-1} \tilde{p}^\lambda}{\lambda} dx d\tau = (a) + (b) + (c) + (d).
\end{aligned}$$

(i) From lemma 3, we easily deduce that :

$$\int \nabla \partial^{s-1} \tilde{p}^\lambda \cdot \nabla \partial^{s-1} \tilde{p}^\lambda dx \geq k'(p_2) |\nabla \partial^{s-1} \tilde{p}^\lambda|_2^2 - \frac{C}{\lambda} |\nabla \tilde{p}^\lambda|_{H^{s-1}} |\nabla \partial^{s-1} \tilde{p}^\lambda|_2.$$

It follows that there exists $\lambda_8 = \lambda_8(K)$ so that, for any $\lambda \geq \lambda_8$, we have :

$$\begin{aligned}
\int_0^t \int \nabla \partial^{s-1} \tilde{p}^\lambda \cdot \nabla \partial^{s-1} \tilde{p}^\lambda dx d\tau & \geq \\
& \geq \frac{k'(p_2)}{2} \int_0^t |\nabla \partial^{s-1} \tilde{p}^\lambda|_2^2 d\tau - \frac{C}{\lambda} \int_0^t |D \tilde{p}^\lambda|_2^2 d\tau.
\end{aligned}$$

(ii) *Estimate of (a).*

$$(a) = - \left[\int \partial^{s-1} v^\lambda \frac{\nabla \partial^{s-1} \tilde{p}^\lambda}{\lambda} dx \right]_0^t + \int_0^t \int \partial^{s-1} v^\lambda \cdot \nabla \partial^{s-1} \rho_t^\lambda dx d\tau.$$

Now, $\rho_t^\lambda = -\operatorname{div} (\rho^\lambda u^\lambda) = -\operatorname{div} v^\lambda$. Then,

$$(a) = - \left[\int \partial^{s-1} v^\lambda \frac{\nabla \partial^{s-1} \tilde{p}^\lambda}{\lambda} dx \right]_0^t + \int_0^t |\operatorname{div} \partial^{s-1} v^\lambda|_2^2 d\tau.$$

So,

$$|(a)| \leq \frac{C}{\lambda} E_0^\lambda + |D^{s-1} v^\lambda|_2 \frac{|D^s \tilde{p}^\lambda|_2}{\lambda} + \int_0^t |D^s v^\lambda|_2^2 d\tau.$$

On the other hand, thanks to lemma 5 and to hypothesis $H(K, T)$, we obtain :

$$(2.13) \quad \left| \begin{aligned} |D^k v^\lambda|_2 & \leq C |D^k u^\lambda|_2 + \frac{\sqrt{K}}{\lambda} |D^k \tilde{p}^\lambda|_2 \\ & \leq C |D^k u^\lambda|_2 + C \frac{\sqrt{K}}{\lambda} (|D^k \tilde{p}^\lambda|_2 + |D \tilde{p}^\lambda|_2); \\ |v^\lambda|_\infty & \leq CK; \quad |D v^\lambda|_\infty \leq CK. \end{aligned} \right.$$

What gives finally :

$$| (a) | \leq C E_0^\lambda + \frac{C}{\lambda} |D^s u^\lambda|_2^2 + \frac{C}{\lambda} |D^s \tilde{p}^\lambda|_2^2 + C(\alpha) \int_0^t |Du^\lambda|_2^2 d\tau \\ + \alpha \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau + \frac{K}{\lambda^2} \int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau + \frac{K}{\lambda^2} \int_0^t |D\tilde{p}^\lambda|_2^2 d\tau .$$

(iii) It follows from lemma 5, (2.9) and (2.13) that :

$$|c + d| \leq C \int_0^t \frac{|D^s \tilde{p}^\lambda|_2}{\lambda} [K |Du^\lambda|_2 + \\ + K |D^{s+1} u^\lambda|_2 + \frac{K}{\lambda} |D\tilde{p}^\lambda|_2 + \frac{K}{\lambda} |D^s \tilde{p}^\lambda|_2] d\tau ,$$

and consequently,

$$|c + d| \leq \frac{K^2 C(\alpha)}{\lambda^2} \int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau + \\ + \frac{K^2 C(\alpha)}{\lambda^2} \int_0^t |D\tilde{p}^\lambda|_2^2 d\tau + C E_0^\lambda + \alpha \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau .$$

(iv) At last, we get easily :

$$|b| \leq \alpha \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau + \frac{C(\alpha)}{\lambda^2} \int_0^t (|D^s \tilde{p}^\lambda|_2^2 + |D\tilde{p}^\lambda|_2^2) d\tau .$$

We deduce from the estimates above the following result :

$$(2.14) \quad \left\{ \begin{array}{l} \forall t \in [0, T], \quad \forall \lambda \geq \lambda_8(K) \geq \lambda_7, \\ \int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau \leq C E_0^\lambda + \frac{C}{\lambda} |D^s u^\lambda|_2^2 \\ + \frac{C}{\lambda} |D^s \tilde{p}^\lambda|_2^2 + 3\alpha \int_0^t |D^{s+1} u^\lambda|_2^2 d\tau \\ + \frac{(1+K^2)}{\lambda^2} C(\alpha) \int_0^t |D\tilde{p}^\lambda|_2^2 d\tau + \frac{(1+K^2)}{\lambda^2} C(\alpha) \int_0^t |D^s \tilde{p}^\lambda|_2^2 d\tau . \end{array} \right.$$

Choosing α small enough and putting together the results (2.9), (2.10), (2.13) and (2.14), we can conclude.

Namely :

PROPOSITION (2.15) : *Under hypothesis $H(K, T)$, there exists some constants $N \in \mathbb{N}^*$ and $C \geq 1$, independent of λ , K and T , and $\lambda_9 = \lambda_9(K)$, independent of T , so that :*

$$\forall t \in [0, T], \quad \forall \lambda \geq \lambda_9 ,$$

$$|u^\lambda(t)|_{H^s}^2 + |\tilde{p}^\lambda(t)|_{H^s}^2 + \int_0^t |\nabla u^\lambda|_{H^s}^2 d\tau + \int_0^t |\nabla \tilde{p}^\lambda|_{H^{s-1}}^2 d\tau \leq C(1+K)^N \cdot E_0^\lambda$$

$$\text{and } |\tilde{p}^\lambda(t)|_{H^s}^2 + \int_0^t |\nabla \tilde{p}^\lambda|_{H^{s-1}}^2 d\tau \leq C(1+K)^N \cdot E_0^\lambda.$$

COROLLARY : Under the same assumptions, the following estimate is verified :

$$|u_t^\lambda|_{H^{s-2}}^2 + |\tilde{p}_t^\lambda|_{H^{s-1}}^2 \leq C\lambda(1+K)^M E_0^\lambda \quad (\text{for some } M \in \mathbb{N}^*).$$

(It is a consequence of (2.15)).

B. Global existence

We first have to see that there really exists K and T verifying hypothesis $H(K, T)$.

Taking our inspiration from Nishida and Matsumura's technic in [3], we get the following local existence's result :

PROPOSITION (2.16) : Let $(u_0^\lambda, p_1) \in (H^s(\mathbb{R}^n))^2$, and $p_0^\lambda(x) = p_0 + \frac{p_1(x)}{\lambda^2}$.

Let $E_0^\lambda = |u_0^\lambda|_{H^s}^2 + |\lambda(p_0^\lambda(x) - p_0)|_{H^s}^2$, where $s > \left[\frac{n}{2}\right] + 1$.

Then, for large enough λ , $\lambda \geq \lambda_{10}$, there exists a unic solution of the system (S^λ) on some interval $[0, T^\lambda(E_0^\lambda)]$, verifying :

(i) $T^\lambda(E_0)$ is an decreasing function of E_0 ;

(ii) The solution (u^λ, p^λ) satisfies :

$$\forall t \in [0, T^\lambda(E_0^\lambda)], E^\lambda(t) = |u^\lambda(t)|_{H^s}^2 + |\lambda(p^\lambda(t) - p_0)|_{H^s}^2 \leq \phi(E_0^\lambda) \cdot E_0^\lambda,$$

where ϕ is an increasing function, independent of $\lambda \geq \lambda_{10}$, so that $\phi \geq 1$.

Now, we are going to put together proposition (2.15) and the above result to prove the global existence as soon as λ is large enough.

Let us introduce K_0 realizing the maximum of the function $\Psi(K)$:

$$\Psi(K) = \frac{K}{C(1+K)^N \cdot \phi[C(1+K)^N]}.$$

Let us note $\lambda_0 = \max(\lambda_9(K_0), \lambda_{10})$.

Choosing E_0^λ so that $E_0^\lambda \leq \Psi(K_0) < 1$, we get :

$$\phi(E_0^\lambda) E_0^\lambda \leq \phi(1) E_0^\lambda \leq \phi(1) \leq \phi[C(1 + K_0)^N] \leq \frac{K_0}{C(1 + K_0)^N} \leq K_0.$$

Let us note $T_0^\lambda = T^\lambda(C(1 + K_0)^N E_0^\lambda) \leq T^\lambda(E_0^\lambda)$.

Thus, we deduce that hypothesis $H(K_0, T_0^\lambda)$ is verified as soon as $\lambda \geq \lambda_0$.

It yields, from (2.15), that :

$$\forall t \in [0, T_0^\lambda], \quad \forall \lambda \geq \lambda_0, \quad E^\lambda(t) \leq C(1 + K_0)^N \cdot E_0^\lambda.$$

In particular, $E^\lambda(T_0^\lambda) \leq C(1 + K_0)^N \cdot E_0^\lambda$.

Now, let us apply the result (2.16), taking T_0^λ as initial instant. Since $E^\lambda(T_0^\lambda) \leq C(1 + K_0)^N \cdot E_0^\lambda$, then $T_0^\lambda \leq T^\lambda(E^\lambda(T_0^\lambda))$.

So, it follows that :

$$\forall t \in [T_0^\lambda, 2 T_0^\lambda], \quad \forall \lambda \geq \lambda_0, \quad E^\lambda(t) \leq \phi(E^\lambda(T_0^\lambda)) \cdot E^\lambda(T_0^\lambda).$$

Now, by construction :

$$\begin{aligned} \phi(E^\lambda(T_0^\lambda)) \cdot E^\lambda(T_0^\lambda) &\leq \phi(C(1 + K_0)^N \cdot E_0^\lambda) \cdot C(1 + K_0)^N \cdot E_0^\lambda \\ &\leq \phi(C(1 + K_0)^N) \cdot C(1 + K_0)^N \cdot \Psi(K_0) \leq K_0. \end{aligned}$$

So, $\forall t \in [0, 2 T_0^\lambda], \quad \forall \lambda \geq \lambda_0, \quad E^\lambda(t) \leq K_0$.

Iterating the process, we get the global existence.

Namely :

THEOREM 1 : *There exists $\lambda_0 > 0$ and $K_0 > 0$ so that : If $E_0^\lambda \leq K_0$ and $\lambda \geq \lambda_0$, then the system (S^λ) admits a unic global solution (u^λ, p^λ) verifying :*

$$\begin{aligned} u^\lambda &\in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-2}), \\ (p^\lambda - p_0) &\in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-1}), \end{aligned}$$

and

$$\forall t \geq 0, \quad \forall \lambda \geq \lambda_0,$$

$$\begin{aligned} |u^\lambda|_{H^s}^2 + |\lambda(p^\lambda - p_0)|_{H^s}^2 + \int_0^\infty |\nabla u^\lambda|_{H^s}^2 d\tau + \\ + \int_0^\infty |\lambda \nabla(p^\lambda - p_0)|_{H^{s-1}}^2 d\tau \leq K_0. \end{aligned}$$

Moreover, $|\partial_t p^\lambda|_{H^{s-1}}$ and $|\partial_t p^\lambda|_{H^{s-1}}$ are bounded, independently of $\lambda \geq \lambda_0$.

We are now going to establish some independent of λ estimates on derivatives in time, in order to obtain some convergence's results. This leads us to consider an initial data u_0^λ of the shape :

$$u_0^\lambda(x) = u_0(x) + \frac{1}{\lambda} u_1(x), \quad \text{where} \quad \operatorname{div} u_0(x) = 0.$$

III. A WEAK CONVERGENCE'S RESULT

Hence, we consider the system (S^λ) :

$$(2.1) \quad \rho^\lambda u_t^\lambda + \rho^\lambda (u^\lambda \cdot \nabla) u^\lambda - \nu \Delta u^\lambda = -\lambda \nabla \tilde{p}^\lambda,$$

$$(2.2) \quad \tilde{p}_t^\lambda + u^\lambda \cdot \nabla \tilde{p}^\lambda + \gamma \tilde{p}^\lambda \cdot \operatorname{div} u^\lambda + \lambda \gamma p_0 \operatorname{div} u^\lambda = 0,$$

$$(2.3) \quad u^\lambda(x, 0) = u_0(x) + \frac{1}{\lambda} u_1(x), \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x),$$

with the supplementary condition :

$$(3.1) \quad \operatorname{div} u_0(x) = 0.$$

The operation $\partial_t(2.1) \times \gamma p_0 u_t^\lambda + \partial_t(2.2) \times \tilde{p}_t^\lambda$ gives, after integration on $\mathbb{R}^n$ and thanks to lemma 4 :

$$\begin{aligned} \frac{d}{dt} \left[\frac{\gamma}{2} p_0 \left| \sqrt{\rho^\lambda} u_t^\lambda \right|_2^2 + \frac{1}{2} \left| \tilde{p}_t^\lambda \right|_2^2 \right] + \nu \gamma p_0 \left| \nabla u_t^\lambda \right|_2^2 + \gamma p_0 \int \rho_t^\lambda (u^\lambda \cdot \nabla) u^\lambda \cdot u_t^\lambda dx \\ + \gamma p_0 \int \rho^\lambda (u_t^\lambda \cdot \nabla) u^\lambda \cdot u_t^\lambda dx + \gamma p_0 \int \rho_t^\lambda \left| u_t^\lambda \right|^2 dx + \int u_t^\lambda \cdot \nabla \tilde{p}^\lambda \tilde{p}_t^\lambda dx \\ + \left(\gamma - \frac{1}{2} \right) \int \left| \tilde{p}_t^\lambda \right|^2 \operatorname{div} u^\lambda dx + \gamma \int \tilde{p}^\lambda \tilde{p}_t^\lambda \operatorname{div} u_t^\lambda dx = 0. \end{aligned}$$

We deduce from that, thanks to lemmas 1 and 3, and to the results of theorem 1, the following inequality :

$$\begin{aligned} \left| u_t^\lambda \right|_2^2 + \left| \tilde{p}_t^\lambda \right|_2^2 + \int_0^t \left| \nabla u_t^\lambda \right|_2^2 d\tau \leq \\ \leq C \left[\left| u_t^\lambda(0) \right|_2^2 + \left| \tilde{p}_t^\lambda(0) \right|_2^2 + \int_0^t \left(\left| u_t^\lambda \right|_2^2 + \left| \tilde{p}_t^\lambda \right|_2^2 \right) d\tau \right]. \end{aligned}$$

This part of the reasoning clearly shows the necessity to introduce the assumption (3.1). As a matter of fact, it permits to obtain that, under the hypothesis of theorem 1 :

$$\left| u_t^\lambda(\cdot, 0) \right|_2 \leq \left| u_0^\lambda \cdot \nabla u_0^\lambda \right|_2 + \frac{\nu}{\rho_1} \left| \Delta u_0^\lambda \right|_2 + \frac{\nu}{\rho_1} \left| \nabla p_1 \right|_2 \leq C$$

and

$$|\tilde{p}_t^\lambda(\cdot, 0)|_2 \leq \left| \gamma \left(p_0 + \frac{p_1(\cdot)}{\lambda} \right) \operatorname{div} u_1 \right|_2 + \left| \frac{\nabla p_1}{\lambda} u_0^\lambda \right|_2 \leq C.$$

So, for λ large enough, we have the following result :

$$(3.2) \quad \forall t \geq 0, \quad |u_t^\lambda|_2^2 + |\tilde{p}_t^\lambda|_2^2 + \int_0^t |\nabla u_t^\lambda|_2^2 d\tau \leq C e^{Ct}.$$

Using of the same methods for the derivatives of order $(s-2)$ in x , we get the equality :

$$\begin{aligned} & \frac{d}{dt} \left[\frac{\gamma p_0}{2} |\sqrt{\rho^\lambda} D^{s-2} u_t^\lambda|_2^2 + \frac{1}{2} |D^{s-2} \tilde{p}_t^\lambda|_2^2 \right] + \gamma p_0 |D^{s-1} u_t^\lambda|_2^2 = \\ &= -\gamma p_0 \int [D^{s-2}(\rho^\lambda u_{tt}^\lambda) - \rho^\lambda D^{s-2} u_{tt}^\lambda] \cdot D^{s-2} u_t^\lambda dx \\ &+ \gamma p_0 \int \frac{1}{2} \rho_t^\lambda (D^{s-2} u_t^\lambda)^2 dx \\ &- \gamma p_0 \int D^{s-2}(\rho_t^\lambda u_t^\lambda) D^{s-2} u_t^\lambda dx - \gamma p_0 \int D^{s-2}(\rho_t^\lambda u^\lambda \cdot \nabla u^\lambda) D^{s-2} u_t^\lambda dx \\ &- \gamma p_0 \int D^{s-2}(\rho^\lambda u_t^\lambda \cdot \nabla u^\lambda) D^{s-2} u_t^\lambda dx \\ &- \gamma p_0 \int D^{s-2}(\rho^\lambda u^\lambda \cdot \nabla u_t^\lambda) D^{s-2} u_t^\lambda dx \\ &+ \int D^{s-2}(\nabla \tilde{p}_t^\lambda u_t^\lambda) D^{s-2} \tilde{p}_t^\lambda dx \\ &+ \frac{1}{2} \int \operatorname{div} u^\lambda (D^{s-2} \tilde{p}_t^\lambda)^2 dx + \gamma \int D^{s-2}(\tilde{p}^\lambda \operatorname{div} u_t^\lambda) D^{s-2} \tilde{p}_t^\lambda dx \\ &+ \int (D^{s-2}(\nabla \tilde{p}_t^\lambda u^\lambda) - u^\lambda D^{s-2} \nabla \tilde{p}_t^\lambda) D^{s-2} \tilde{p}_t^\lambda dx \\ &- \gamma \int D^{s-2}(\tilde{p}_t^\lambda \operatorname{div} u^\lambda) D^{s-2} \tilde{p}_t^\lambda dx. \end{aligned}$$

Except the first term of the right member, all the (numerous !) terms of this equality can be estimated by the technics developped all along the preceding paragraph (lemma 5 and estimates of theorem 1).

Let us study this particular term a little more attentively.

Let us write :

$$\begin{aligned} \int_0^t \int [D^{s-2}(\rho^\lambda u_t^\lambda) - \rho^\lambda D^{s-2} u_t^\lambda] D^{s-2} u_t^\lambda dx d\tau &\leq \\ &\leq \int_0^t [|D\rho^\lambda|_\infty |D^{s-3} u_t^\lambda|_2 + |D^{s-2} \rho^\lambda|_r |u_t^\lambda|_{r'}] |D^{s-2} u_t^\lambda|_2 d\tau. \end{aligned}$$

Taking $(r, r') = (\infty, 2)$ when $n = 2$ or 3 , and $(r, r') = \left(\frac{2n}{n-2}, \frac{n}{2}\right)$ when $n \geq 4$, we get :

$$|D^{s-2} \rho^\lambda|_r \leq \frac{1}{\lambda} |\tilde{\rho}^\lambda|_{H^s} \leq \frac{K_0}{\lambda} \quad \text{and} \quad |u_t^\lambda|_{r'} \leq |u_t^\lambda|_{H^{s-3}}.$$

So, we just have to estimate $\int_0^t \frac{1}{\lambda^2} |u_t^\lambda|_{H^{s-3}}^2 d\tau$.

Let us note $\chi = |u_t^\lambda|_{H^{s-2}}^2 + |\tilde{p}_t^\lambda|_{H^{s-2}}^2$, and let us derive in time the equation (2.1).

Proceeding by the now classical method, and using lemma 4, lemma 5 and the results of theorem 1, we get :

$$\int_0^t \frac{1}{\lambda^2} (|u_t^\lambda|_{H^{s-3}}^2) d\tau \leq C \int_0^t \chi(\tau) d\tau + \frac{C}{\lambda^2} \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 d\tau.$$

Which yields, for λ large enough, to the following Gronwald's inequality :

$$\chi(t) + \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 d\tau \leq C\chi(0) + C \int_0^t \chi(\tau) d\tau.$$

We then can state the obtained result in the :

PROPOSITION : *If $u_0^\lambda(x) = u_0(x) + \frac{u_1(x)}{\lambda} \in H^s$, with $\operatorname{div} u_0 = 0$, If $p_0^\lambda(x) = p_0 + \frac{p_1(x)}{\lambda^2}$, with $p_1 \in H^s$ and $s > \left[\frac{n}{2}\right] + 1$, then, under the assumptions of theorem 1, the solutions (u^λ, p^λ) of (S^λ) verify, as soon as λ is large enough, in addition to the already obtained estimates :*

$$(3.3) \quad |u_t^\lambda|_{H^{s-2}}^2 + |\tilde{p}_t^\lambda|_{H^{s-2}}^2 + \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 d\tau \leq M(t).$$

In particular,

$$(3.4) \quad \left\{ \begin{array}{l} |p_t^\lambda|_{H^{s-2}}^2 + |\rho_t^\lambda|_{H^{s-2}}^2 \leq \frac{1}{\lambda^2} M(t), \\ |\nabla \tilde{p}^\lambda|_{H^{s-2}}^2 + |\operatorname{div}(\rho^\lambda u^\lambda)|_{H^{s-2}}^2 + |\operatorname{div} u^\lambda|_{H^{s-2}}^2 \leq \frac{1}{\lambda^2} M(t), \end{array} \right.$$

where $M(t) \in L_{\text{loc}}^\infty(\mathbb{R}_+, \mathbb{R}_+)$.

Now, we have got all that is necessary to prove that the sequence (u^λ, p^λ) weakly converges (in a sense that will be precised), to the solution (u^∞, p^∞) of the viscous incompressible fluid's equation :

$$(S^\infty) \quad \left\{ \begin{array}{l} \rho_0(u_t^\infty + (u^\infty \cdot \nabla) u^\infty) - \nu \Delta u^\infty = -\nabla p^\infty, \\ \operatorname{div} u^\infty = 0, \quad u^\infty(x, 0) = u_0(x). \end{array} \right.$$

Remark : We'll now write « u^λ » for any subsequence of u^λ . In fact, this notation is justified : the unicity of the solutions (u^λ, p^λ) and (u^∞, p^∞) shows, a posteriori, that this is really the sequence (u^λ, p^λ) that converges and not any subsequence.

From the estimates of theorem 1 and from (3.3), we deduce that there exists u^∞ verifying :

$$u^\infty \in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-2}),$$

so that :

$$(3.5) \quad \left\{ \begin{array}{l} u^\lambda \rightarrow u^\infty \quad \text{in } L^\infty(0, \infty, H^s) \text{ w.s.}, \\ u_t^\lambda \rightarrow u_t^\infty \quad \text{in } L_{\text{loc}}^\infty(0, \infty, H^{s-2}) \text{ w.s.}, \end{array} \right.$$

and,

$$(3.6) \quad \left\{ \begin{array}{l} \nabla u^\lambda \rightarrow \nabla u^\infty \quad \text{in } L^2(0, \infty, H^s) \text{ w.s.}, \\ \nabla u_t^\lambda \rightarrow \nabla u_t^\infty \quad \text{in } L_{\text{loc}}^2(0, \infty, H^{s-2}) \text{ w.s.} \end{array} \right.$$

Moreover, from the inequality :

$$|\lambda(\rho^\lambda - \rho_0)|_{H^s} \leq CK_0,$$

we deduce :

$$(3.7) \quad \rho^\lambda \rightarrow \rho_0 \quad \text{in } C_B(0, \infty, W^{\infty, s-2}) \text{ strongly}.$$

Then, $\rho^\lambda u_t^\lambda \rightarrow \rho_0 u_t^\infty$ in $L_{\text{loc}}^\infty(0, \infty, H^{s-2})$ w.s. .

From (3.5), we get that :

$$u^\lambda \rightarrow u^\infty \text{ in } L_{\text{loc}}^\infty(0, \infty, H_{\text{loc}}^{s-1}) \text{ strongly and almost everywhere .}$$

These last points lead to the following result :

$$\rho^\lambda(u^\lambda \cdot \nabla) u^\lambda \rightarrow \rho_0(u^\infty \cdot \nabla) u^\infty \text{ in } D'(0, \infty, H^{s-1}) .$$

Let us now consider ϕ in $D(0, T, H^{s-2})$, so that $\text{div } \phi = 0$. Then,

$$(\rho^\lambda(u_t^\lambda + (u^\lambda \cdot \nabla) u^\lambda) - \nu \Delta u^\lambda, \phi) = 0 .$$

Making λ go to $+\infty$, we deduce from the above results that :

$$(\forall \phi \in D(0, T, H^{s-2})) ,$$

$$(\text{div } \phi = 0 \Rightarrow (\rho_0 u_t^\infty + \rho_0(u^\infty \cdot \nabla) u^\infty - \nu \Delta u^\infty, \phi) = 0) .$$

So, we have shown that there exists some function p^∞ verifying :

$$\rho_0 u_t^\infty + \rho_0(u^\infty \cdot \nabla) u^\infty - \nu \Delta u^\infty = -\nabla p^\infty .$$

By construction, it is clear that :

$$\nabla p^\infty \in C(0, \infty, H^{s-2}) ,$$

$$\text{and } \lambda \nabla \tilde{p}^\lambda \rightarrow \nabla p^\infty \text{ in } L_{\text{loc}}^\infty(0, \infty, H^{s-2}) \text{ w.s. .}$$

We can gather all these results in the following theorem :

THEOREM 2 : *Let us consider initial data of the shape :*

$$u_0^\lambda(x) = u_0(x) + \frac{1}{\lambda} u_1(x) , \quad p_0^\lambda(x) = p_0 + \frac{1}{\lambda^2} p_1(x) ,$$

$$\text{div } u_0 = 0 , \quad p_0 = \text{Cte} ;$$

$$(u_0, u_1, p_1) \in [H^s(\mathbb{R}^n)]^3 , \text{ with } s > \left[\frac{n}{2} \right] + 1 , \text{ and } \|u_0\|_{H^s}^2 < K_0 .$$

Then, the sequence (u^λ, p^λ) converges to (u^∞, p^∞) , solution of the system (S^∞) , in the following sense :

$$u^\lambda \rightarrow u^\infty \text{ in } C_{\text{loc}}(0, \infty, H_{\text{loc}}^{s-1}(\mathbb{R}^n)) \text{ strongly ,}$$

$$\lambda \nabla \tilde{p}^\lambda \rightarrow \nabla p^\infty \text{ in } L_{\text{loc}}^\infty(0, \infty, H^{s-2}(\mathbb{R}^n)) \text{ w.s. .}$$

In addition, $u^\infty \in C_B(0, \infty, H^s) \cap C^1(0, \infty, H^{s-2})$ and

$$\nabla u^\infty \in L^2(0, \infty, H^s) .$$

Remark : We have shown a double stability for the system (S^λ) :

- On one hand, stability of the estimates towards λ large enough.
- On the other hand, stability of the limit (u^∞, p^∞) towards the initial data (u_1, p_1) smooth enough.

In particular, to obtain the results we need concerning the derivatives in time of u^∞ and p^∞ , we can choose $u_1 = p_1 = 0$.

In this case, taking u_0 smooth enough and deriving once more in time the equations (2.1) and (2.2), we just have to proceed as usual to get uniform in λ estimates on u_{tt}^λ and $\tilde{p}_{tt}^\lambda$.

Which, passing to the limit, allows to enounce the following properties :

PROPOSITION : *Let us suppose that $|u_0|_{H^{s+k}}^2 < K_0$ ($k \geq 1$). Then :*

$$|u_{tt}^\infty|_{H^{s+k-4}}^2 + \int_0^t |\nabla u_{tt}^\infty|_{H^{s+k-4}}^2 d\tau + \int_0^t |\nabla \tilde{p}_{tt}^\infty|_{H^{s+k-3}}^2 d\tau \leq M(t).$$

Such a result naturally raises the following question :

« Could we get a best convergence by adding new fitting assumptions ? ».

IV. STRONG CONVERGENCE

Like it often happens, to establish strong convergence's results, we have to give more regularity to the initial data.

Moreover, we have an estimate of $|\nabla p^\infty|_{H^k}$ and $|\nabla p_t^\infty|_{H^{k-2}}$, but we don't know anything about $|p^\infty|_2$ and $|p_t^\infty|_2$.

So, like Klainerman and Majda [2], we are going to impose to $|p^\infty|_2$ and $|p_t^\infty|_2$ to be locally bounded.

We then get the following result :

THEOREM 3 : *Let us consider the system (S^λ) with initial data :*

$$u^\lambda(x, 0) = u_0(x) + \frac{1}{\lambda} u_1(x), \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x),$$

$$\operatorname{div} u_0 = 0, \quad p_0 > 0,$$

$$(u_0, u_1, p_1) \in [H^{s+2}(\mathbb{R}^n)]^3, \quad \text{with } s > \left[\frac{n}{2} \right] + 1, \quad \text{and } |u_0|_{H^{s+2}}^2 < K_0.$$

Let us suppose, in addition, that the following assumption (H) is true :

$$(H) \quad |p^\infty(t)|_2 + |p_t^\infty(t)|_2 \leq M(t), \quad \text{where } M(t) \in L_{\text{loc}}^\infty(\mathbb{R}_+, \mathbb{R}_+).$$

Then, there exists $\lambda_0 \geq 0$, so that :

$$\forall t \geq 0, \quad \forall \lambda \geq \lambda_0, \quad \lambda^2 |u^\lambda - u^\infty|_{H^s}^2 + |\lambda^2(p^\lambda - p_0) - p^\infty|_{H^s}^2 + \\ + \lambda^2 \int_0^t |\nabla(u^\lambda - u^\infty)|_{H^s}^2 d\tau \leq M(t).$$

Remark : The assumption $|u_0|_{H^{s+2}}^2 \leq K_0$ is necessary to assure global existence of (u^λ, p^λ) and (u^∞, p^∞) , as soon as λ is large enough (see theorem 1). Before going on, let us sum up the results that we have already got, in the case where the initial data are in H^{s+k} , with $k \in \mathbb{N}^*$:

$$(4.1) \quad |u^\lambda|_{H^{s+k}}^2 + |\tilde{p}^\lambda|_{H^{s+k}}^2 + \int_0^\infty |\nabla u^\lambda|_{H^{s+k}}^2 d\tau + \int_0^\infty |\nabla \tilde{p}^\lambda|_{H^{s+k-1}}^2 d\tau \leq K_0;$$

$$(4.2) \quad |u_t^\lambda|_{H^{s+k-2}}^2 + |\tilde{p}_t^\lambda|_{H^{s+k-2}}^2 + \int_0^t |\nabla u_t^\lambda|_{H^{s+k-2}}^2 d\tau \leq M(t) \quad (t \geq 0);$$

$$(4.3) \quad |\nabla \tilde{p}^\lambda|_{H^{s+k-2}}^2 + |\operatorname{div} u^\lambda|_{H^{s+k-2}}^2 \leq \frac{M(t)}{\lambda^2};$$

$$(4.4) \quad |\tilde{p}|_{H^{s+k}}^2 \leq CK_0, \quad |\tilde{p}_t|_{H^{s+k-2}}^2 \leq M(t);$$

$$(4.5) \quad |p^\lambda - p_0|_{W^{\infty, s+k-2}}^2 + |\rho^\lambda - \rho_0|_{W^{\infty, s+k-2}}^2 \leq \frac{K_0}{\lambda^2};$$

$$(4.6) \quad |u^\infty|_{H^{s+k}}^2 + \int_0^\infty |\nabla u^\infty|_{H^{s+k}}^2 d\tau \leq K_0;$$

$$(4.7) \quad |u_t^\infty|_{H^{s+k-2}}^2 + |\nabla p^\infty|_{H^{s+k-2}}^2 + \int_0^t |\nabla u_t^\infty|_{H^{s+k-2}}^2 d\tau \leq M(t) \quad (t \geq 0);$$

$$(4.8) \quad \int_0^t |\nabla p_t^\infty|_{H^{s+k-3}}^2 d\tau \leq M(t).$$

Having got all these important results, we are now going to use the usual technics to prove the result of the theorem.

Let us note

$$\hat{u} = \lambda(u^\lambda - u^\infty) \quad \text{and} \quad \hat{p} = \lambda^2(p^\lambda - p_0) - p^\infty.$$

(N.B. : It follows from hypothesis (H) that $\hat{p} \in L^2$ and $\hat{p}_t \in L^2$.)

Then the couple $(\hat{u}, \hat{p})$ is a solution of the following system :

$$(4.9) \quad \rho_0 \hat{u}_t + \tilde{\rho}^\lambda u_t^\lambda + \tilde{\rho}^\lambda (u^\lambda \cdot \nabla) u^\lambda + \rho_0 (u^\lambda \cdot \nabla) \hat{u} + \\ + \rho_0 (\hat{u} \cdot \nabla) u^\infty - \nu \Delta \hat{u} = -\lambda \nabla \hat{p},$$

$$(4.10) \quad \hat{\rho}_t + \lambda \nabla \tilde{p}^\lambda \cdot u^\lambda + \gamma \tilde{p}^\lambda \operatorname{div} \hat{u} + \lambda \gamma p_0 \operatorname{div} \hat{u} = -p_t^\infty, \quad (\operatorname{div} u^\infty = 0)$$

$$(4.11) \quad \hat{u}(x, 0) = u_1(x), \quad \hat{p}(x, 0) = p_1(x) - p^\infty(x, 0).$$

1st Step : L^2 -Norms of $\hat{u}$ and $\hat{p}$.

Multiplying equation (4.9) by $\gamma p_0 \hat{u}$ and equation (4.10) by $\hat{p}$, and integrating on $\mathbb{R}^n$, we get :

$$\begin{aligned} & \frac{d}{dt} \left[\frac{\gamma p_0 \rho_0}{2} |\hat{u}|_2^2 + \frac{1}{2} |\hat{p}|_2^2 \right] + \nu \gamma p_0 |\nabla \hat{u}|_2^2 = \\ & - \gamma p_0 \int \tilde{p}^\lambda (u_t^\lambda + u^\lambda \nabla u^\lambda) \hat{u} \, dx - \gamma p_0 \rho_0 \int (u^\lambda \nabla) \hat{u} \cdot \hat{u} \, dx \\ & - \gamma p_0 \rho_0 \int (\hat{u} \nabla u^\infty) \hat{u} \, dx \\ & - \int u^\lambda (\lambda \nabla \tilde{p}^\lambda) \hat{p} \, dx - \int \gamma \tilde{p}^\lambda \operatorname{div} \hat{u} \hat{p} \, dx - \int p_t^\infty \hat{p} \, dx. \end{aligned}$$

Thanks to estimates (4.1) to (4.7), the right member can be majored by :

$$M(t) + |\hat{u}|_2^2 + \frac{\nu \gamma p_0}{2} |\nabla \hat{u}|_2^2 + |\hat{p}|_2^2 + |p_t^\infty|_2^2.$$

Using the supplementary condition on p_t^∞ , it yields :

$$\forall t \geq 0, \quad |\hat{u}(t)|_2^2 + |\hat{p}(t)|_2^2 + \int_0^t |\nabla \hat{u}(\tau)|_2^2 \, d\tau \leq M(t).$$

2nd Step : L^2 -Norms of $D^s u$ and $D^s p$.

Let us derive s times the equations (4.9) and (4.10), multiply the first obtained equation by $\gamma p_0 \partial^s \hat{u}$, the second by $\partial^s \hat{p}$, and integrate on $\mathbb{R}^n \times [0, t]$. Using the results (4.1) to (4.8) (for $k = 2$), and the usual technics to estimate the obtained terms, we get :

$$\begin{aligned} & |D^s \hat{u}|_2^2 + |D^s \hat{p}(t)|_2^2 + \int_0^t |\nabla D^s \hat{u}|_2^2 \, d\tau \leq M(t) + C |\hat{u}(0)|_{H^s}^2 + |\hat{p}(0)|_{H^s}^2 + \\ & + C \int_0^t (|D^s \hat{u}(\tau)|_2^2 + |D^s \hat{p}(\tau)|_2^2) \, d\tau + C \int_0^t |\nabla D^{s-1} p_t^\infty(\tau)|_2^2 \, d\tau. \end{aligned}$$

$$\text{So,} \quad \forall t \geq 0, \quad |\hat{u}(t)|_{H^s}^2 + |\hat{p}(t)|_{H^s}^2 + \int_0^t |\nabla \hat{u}(\tau)|_{H^s}^2 \, d\tau \leq M(t).$$

Remark : We can get « good » principle parts by scaling non linear terms.

V. AN INITIAL LAYER PHENOMENON WHEN $\operatorname{div} u_0 \neq 0$

Hence we consider the solution (u^λ, p^λ) of the system (S^λ) :

$$\begin{cases} \rho^\lambda (u_t^\lambda + (u^\lambda \cdot \nabla) u^\lambda) - \nu \Delta u^\lambda = -\lambda \nabla \tilde{p}^\lambda, \\ \tilde{p}_t^\lambda + u^\lambda \cdot \nabla \tilde{p}^\lambda + \gamma \tilde{p}^\lambda \operatorname{div} u^\lambda + \lambda \gamma p_0 \operatorname{div} u^\lambda = 0, \\ u^\lambda(x, 0) = u_0(x) + \frac{1}{\lambda} u_1(x), \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x), \end{cases}$$

with now $\operatorname{div} u_0 \neq 0$.

Let us write :

$$(5.1) \quad u_0 = v_0 + \nabla \phi_0, \quad \text{with} \quad \operatorname{div} v_0 = 0.$$

Since the solution (u^∞, p^∞) of the system (S^∞) verifies the condition : $\operatorname{Div} u^\infty = 0$, it clearly appears an initial layer's phenomenon.

A fitting corrector term is provided by the solution (v^λ, q^λ) of the linear following system :

$$(C^\lambda) \quad \begin{cases} (5.2) & \rho_0 v_t^\lambda - \nu \Delta v^\lambda = -\lambda \nabla q^\lambda, \\ (5.3) & q_t^\lambda + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0, \\ (5.4) & v^\lambda(x, 0) = \nabla \phi_0(x), \quad q^\lambda(x, 0) = 0. \end{cases}$$

We'll establish, in an appendix, the following result :

PROPOSITION (5.5) : *If $\phi_0 \in W^{s+n+4}(\mathbb{R}^n)$, then v^λ verifies the following $L^\infty - L^1$ estimate :*

$$\begin{aligned} |v^\lambda|_{W^{s,\infty}} &\leq \frac{C}{(1+\lambda t)} |\phi_0|_{W^{1,s+n+4}} \quad \text{if } n \geq 3, \\ |v^\lambda|_{W^{s,\infty}} &\leq \frac{C}{\sqrt{1+\lambda t}} |\phi_0|_{W^{1,s+6}} \quad \text{if } n = 2. \end{aligned}$$

Let us consider the solution (u^∞, p^∞) of the system (S^∞) :

$$(S^\infty) \quad \begin{cases} \rho_0 (u_t^\infty + (u^\infty \cdot \nabla) u^\infty) - \nu \Delta u^\infty = -\nabla p^\infty, \\ \operatorname{div} u^\infty = 0, \quad u^\infty(x, 0) = v_0(x). \end{cases}$$

Like in paragraph 4, we'll impose, in the whole part left, to p^∞ to verify :

$$(H) \quad |p^\infty|_2^2 + |p_t^\infty|_2^2 \leq M(t), \quad \text{where} \quad M(t) \in L_{\text{loc}}^\infty(\mathbb{R}^+, \mathbb{R}^+).$$

We then prove the :

THEOREM 4 : *Let us consider the system (S^λ) with the initial data :*

$$u^\lambda(x, 0) = v_0(x) + \nabla \phi_0(x) + \frac{1}{\lambda} u_1(x), \quad \text{with } \operatorname{div} v_0(x) = 0,$$

$$p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x), \quad p_0 > 0,$$

$$(v_0, u_1, p_1) \in [H^{s+2}(\mathbb{R}^n)]^3 \quad \text{and} \quad \phi_0 \in W^{1, s+n+5} \subset H^{s+3} \left(s > \left[\frac{n}{2} \right] + 1 \right),$$

$$\text{and } |v_0 + \nabla \phi_0|_{H^{s+2}}^2 < K_0.$$

Let us suppose, in addition, that hypothesis (H) is verified.

Then, there exists $\lambda_0 \geq 0$, so that :

$$\forall t > 0, \quad \forall \lambda \geq \lambda_0,$$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leq M(t) \frac{(1 + \operatorname{Log}(1 + \lambda t))}{\lambda} \\ \text{if } n \geq 3,$$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leq \frac{1}{\sqrt{\lambda}} M(t) \\ \text{if } n = 2.$$

$$\text{Proof: Let us note } w = u^\lambda - u^\infty - v^\lambda \text{ and } b = \tilde{p}^\lambda - \frac{1}{\lambda} p^\infty - q^\lambda.$$

Considering the equations satisfied by (u^λ, p^λ) , (u^∞, p^∞) and (v^λ, q^λ) , we find that (w, b) is a solution of the following system :

$$(5.6) \quad \rho^\lambda w_t + \rho_0 w \nabla u^\infty + \rho_0 u^\lambda \nabla w - v \Delta w + \frac{\tilde{\rho}^\lambda}{\lambda} v_t^\lambda + \\ + \frac{\tilde{\rho}^\lambda}{\lambda} (u_t^\infty + u^\lambda \nabla u^\lambda) + \rho_0 (v^\lambda \nabla u^\infty + u^\lambda \nabla v^\lambda) = -\lambda \nabla b,$$

$$(5.7) \quad b_t + u^\lambda \nabla b + \gamma \tilde{p}^\lambda \operatorname{div} w + \lambda \gamma p_0 \operatorname{div} w + \\ + \left(\frac{p_t^\infty}{\lambda} + \frac{u^\lambda \nabla p^\infty}{\lambda} + \frac{v u^\lambda \Delta v^\lambda}{\lambda} \right) + \gamma \tilde{p}^\lambda \operatorname{div} v^\lambda - \rho_0 v_t^\lambda \frac{u^\lambda}{\lambda} = 0,$$

$$(5.8) \quad w(x, 0) = \frac{1}{\lambda} u_1(x), \quad b(x, 0) = \frac{1}{\lambda} (p_1(x) - p^\infty(x, 0)).$$

Let us note :

$$(5.9) \quad h(x, t) = w_t + \frac{\lambda \nabla b}{\rho^\lambda} \quad \text{and} \quad k(x, t) = b_t + \lambda \gamma p_0 \operatorname{div} w + \frac{1}{\lambda} p_t^\infty.$$

Thanks to estimates (4.1), (4.5), (4.6) and (4.7), we deduce from the smoothness of the initial data ($k = 2$), that :

$$(5.10) \quad \forall t \geq 0, \quad |h(t)|_{H^s} + |k(t)|_{H^s} \leq M(t).$$

Let us also note that equations (5.6) and (5.7) can be written as follows :

$$(5.11) \quad \rho^\lambda w_t + \rho_0(w \nabla) u^\infty + \rho_0(u^\lambda \nabla) w - \nu \Delta w + \frac{\tilde{p}^\lambda}{\lambda} v_t^\lambda + f^\lambda = -\lambda \nabla b,$$

$$(5.12) \quad \left(1 - \frac{\tilde{p}^\lambda}{\lambda p_0}\right) b_t + u^\lambda \cdot \nabla b + \lambda \gamma p_0 \operatorname{div} w - \rho_0 v_t^\lambda \frac{u^\lambda}{\lambda} + g^\lambda = 0,$$

where $f^\lambda = \frac{\tilde{p}^\lambda}{\lambda} (u_t^\infty + (u^\lambda \nabla) u^\lambda) + \rho_0((v^\lambda \nabla) u^\infty + (u^\lambda \nabla) v^\lambda),$

and $g^\lambda = \frac{1}{\lambda} \left(p_t^\infty + u^\lambda \nabla p^\infty + \nu u^\lambda \Delta v^\lambda + \frac{\tilde{p}^\lambda}{p_0} k - \frac{\tilde{p}^\lambda}{\lambda p_0} p_t^\infty \right) + \gamma \tilde{p}^\lambda \operatorname{div} v^\lambda.$

Let $a^\lambda(t)$ be the quantity :

$$a^\lambda(t) = \int_0^t (|f^\lambda(\tau)|_{H^s} + |g^\lambda(\tau)|_{H^s} + |v^\lambda(\tau)|_{W^{\infty, s+1}}) d\tau.$$

We are going to need the following lemma :

LEMMA (5.13) :

$$a^\lambda(t) \leq \frac{M(t)}{\sqrt{\lambda}} \quad \text{if } n = 2 \quad \text{and}$$

$$a^\lambda(t) \leq \frac{M(t)}{\lambda} (1 + \operatorname{Log}(1 + \lambda t)) \quad \text{if } n \geq 3.$$

It is immediatly deduced from proposition (5.5) and from the assumptions of theorem 4.

1st Step : Estimate of w and b in L^2 -norm.

Let us multiply equation (5.11) by $\gamma p_0 w$ and equation (5.12) by b . The only true difficulty lays in the terms :

$$\frac{\tilde{p}^\lambda}{\lambda} v_t^\lambda w \quad \text{and} \quad \frac{u^\lambda}{\lambda} v_t^\lambda b,$$

because we just know that $\frac{v_t^\lambda}{\lambda}$ is bounded.

To avoid this difficulty, we just have to integrate by part, using (5.9). So, we obtain :

$$\begin{aligned}
& \frac{d}{dt} \left[\int \left(\gamma p_0 \rho^\lambda \frac{w^2}{2} + \left(1 - \frac{\tilde{p}^\lambda}{\lambda p_0} \right) \frac{b^2}{2} + \gamma p_0 \frac{\tilde{p}^\lambda}{\lambda} v^\lambda w - \rho_0 \frac{v^\lambda u^\lambda b}{\lambda} \right) dx \right] \\
& + v \gamma p_0 = |\nabla w|_2^2 = \int \left(\gamma p_0 \rho^\lambda \frac{w^2}{2} - p_t^\lambda \frac{b^2}{2 p_0} - \gamma p_0 p_0 (w \nabla) u^\infty \cdot w \right. \\
& \left. + \frac{\gamma}{2} \rho_0 p_0 \operatorname{div} u^\lambda |w|^2 + \operatorname{div} u^\lambda \frac{|b|^2}{2} \right) dx \\
& + \int \left(\gamma p_0 \frac{\tilde{p}_t^\lambda}{\lambda} v^\lambda w - \gamma p_0 f^\lambda w - \gamma p_0 p_0 \nabla (u^\lambda v^\lambda) w \right. \\
& \left. + \gamma p_0 \operatorname{div} \left(\frac{\tilde{p}^\lambda v^\lambda}{\rho^\lambda} \right) b - \rho_0 \frac{v^\lambda}{\lambda} u_t^\lambda b - g^\lambda b \right) dx \\
& + \int \left(\gamma p_0 \tilde{p}^\lambda h \frac{v^\lambda}{\lambda} - \rho_0 u^\lambda \frac{v^\lambda}{\lambda} k + \rho_0 u^\lambda v^\lambda \frac{p_t^\infty}{\lambda^2} \right) dx \\
& = I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

$$\text{Let } \chi_0^2(t) = \sup_{[0, t]} (|w(\tau)|^2 + |b(\tau)|^2).$$

From the results of theorem 1 (§ II), we easily deduce the following estimates

$$\begin{aligned}
\int_0^t |I_1(\tau)| d\tau & \leq K \int_0^t \chi_0^2 d\tau, \\
\int_0^t |I_2(\tau)| d\tau & \leq K \chi_0 a^\lambda(t), \\
\int_0^t |I_3(\tau)| d\tau & \leq K \frac{a^\lambda(t)}{\lambda}.
\end{aligned}$$

Let us also note

$$I_4(t) = \int \left(\gamma p_0 \frac{\tilde{p}^\lambda}{\lambda} v^\lambda w - \rho_0 \frac{v^\lambda u^\lambda b}{\lambda} \right) dx.$$

Then I_4 verifies :

$$|I_4(t)| \leq \frac{K}{\lambda} \chi_0.$$

Now, thanks to hypothesis (H) and (5.8), we deduce that : $\chi_0(0) \leq \frac{K}{\lambda}$.

Thus, we get the following inequality :

$$\chi_0^2(t) \leq \frac{K}{\lambda^2} + \frac{K}{\lambda} \chi_0 + \frac{K}{\lambda} a^\lambda(t) + K \chi_0 a^\lambda(t) + K \int_0^t \chi_0^2(\tau) d\tau,$$

so

$$(5.14) \quad \chi_0^2(t) \leq K \left(|a^\lambda(t)|^2 + \int_0^t \chi_0^2(\tau) d\tau \right).$$

2nd Step : Estimate of $D^s w$ and $D^s b$ in L^2 -norm

We'll use the technics developped in paragraph II (pp. 16-18), the difficulty raised in the first step being solved by integrating by parts again. (We shall use in particular the inequalities (2.5) and (2.6)).

The operation

$$\int D^s(5.6) \gamma p_0 D^s w dx + \int D^s(5.7) \cdot D^s b dx$$

hence gives :

$$\begin{aligned} & \frac{d}{dt} \left[\int \gamma p_0 \rho^\lambda \frac{(D^s w)^2}{2} + \left(1 - \frac{\tilde{p}^\lambda}{\lambda p_0} \right) \frac{(D^s b)^2}{2} + \gamma p_0 \frac{D^s(\tilde{\rho}^\lambda v^\lambda)}{\lambda} D^s w \right. \\ & \quad \left. - \rho_0 \frac{D^s(u^\lambda v^\lambda)}{\lambda} D^s b \right] + \nu \rho_0 p_0 |\nabla D^s w|_2^2 = \gamma p_0 p_0 \int \left(\frac{\rho_t}{\rho_0} \frac{(D^s w)^2}{2} \right. \\ & \quad \left. - D^s(w \cdot \nabla u^\infty) D^s w \right. \\ & \quad \left. + \operatorname{div} u^\lambda \frac{(D^s w)^2}{2} - [D^s(u^\lambda \cdot \nabla w) - u^\lambda D^s \nabla w] \cdot D^s w dx \right. \\ & \quad \left. + \int \left(\operatorname{div} u^\lambda \frac{(D^s b)^2}{2} - \frac{p_t^\lambda}{p_0} \frac{(D^s b)^2}{2} \right. \right. \\ & \quad \left. \left. - [D^s(u^\lambda \cdot \nabla b) - u^\lambda (D^s \nabla b)] \cdot D^s b \right) dx \right. \\ & \quad \left. + \gamma p_0 \int \left(D^s \left(\tilde{\rho}_t^\lambda \frac{v^\lambda}{\lambda} \right) D^s w + D^{s+1}(\tilde{\rho}^\lambda v^\lambda) D^{s-1} \left(\frac{\nabla b}{\rho^\lambda} \right) \right. \right. \\ & \quad \left. \left. - D^s f^\lambda \cdot D^s w - \rho_0 D^s \nabla(u^\lambda v^\lambda) D^s w \right) dx \right. \\ & \quad \left. - \int \left(\rho_0 D^s \left(u_t^\lambda \frac{v^\lambda}{\lambda} \right) D^s b + D^s g^\lambda \cdot D^s b \right) dx \right. \\ & \quad \left. + \int \left(\gamma p_0 D^s(\tilde{\rho}^\lambda v^\lambda) \frac{D^s h}{\lambda} - \rho_0 D^s(u^\lambda v^\lambda) \frac{D^s k}{\lambda} + \rho_0 D^s(u^\lambda v^\lambda) \frac{D^s(p_t^\infty)}{\lambda^2} \right) dx \right. \\ & \quad \left. + \int \left(\gamma p_0 [D^s(\rho^\lambda w_t) - \rho^\lambda (D^s w_t)] \cdot D^s w \right. \right. \\ & \quad \left. \left. + \left[D^s \left(1 - \frac{\tilde{\rho}^\lambda}{\lambda p_0} \right) b_t - \left(1 - \frac{\tilde{p}^\lambda}{\lambda p_0} \right) D^s b_t \right] \cdot D^s b \right) dx \right. \\ & = I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Let
$$\chi_s^2(t) = \sup_{[0, t]} (|D^s w(\tau)|_2^2 + |D^s b(\tau)|_2^2).$$

Thanks to the lemma 5 and the results of theorem 1 in particular, the integral

$$\left| \int_0^t (I_1 + I_2 + I_3 + I_4 + I_5)(\tau) d\tau \right|$$

is majored, as in the first step, by :

$$K \left[\int_0^t (\chi_0^2(\tau) + \chi_s^2(\tau)) d\tau + a^\lambda(t)(\chi_0 + \chi_s) + a^\lambda(t)^2 \right].$$

Also, if we note

$$I_7(t) = \int \left(\gamma p_0 \frac{D^s(\tilde{p}^\lambda v^\lambda)}{\lambda} D^s w - \rho_0 \frac{D^s(u^\lambda v^\lambda)}{\lambda} D^s b \right) dx,$$

then,
$$|I_7(t)| \leq \frac{K}{\lambda} (\chi_0 + \chi_s) \leq a^\lambda(t) \cdot (\chi_0 + \chi_s).$$

Now, we have to estimate I_6 . Using (5.9) and (2.5), we get :

$$I_6(t) \leq \frac{1}{\lambda} \chi_s + (\chi_0 + \chi_s) \cdot \chi_s + \frac{1}{\lambda^2} |p_t^\infty|_{H^s} \chi_s.$$

Thus, we get the following inequality for χ_s :

$$\chi_s^2(t) \leq K \left[(\chi_0 + \chi_s) a^\lambda(t) + a^\lambda(t)^2 + \int_0^t (\chi_0^2 + \chi_s^2)(\tau) d\tau \right],$$

what, added (!) to (5.14), leads to a Gronwald's inequality verified by $\chi_0^2 + \chi_s^2$. Hence,

$$|w|_{H^s}^2 + |b|_{H^s}^2 = \chi_0^2 + \chi_s^2 \leq KM(t) \cdot a^\lambda(t)^2.$$

Finally, let us remark that :

$$|\lambda(p^\lambda - p_0) - q^\lambda|_{H^s}^2 \leq |b|_{H^s}^2 + \frac{1}{\lambda^2} |p^\infty|_{H^s}^2.$$

So, the theorem is proven.

Remark : As in paragraph 4, we can find a principal part of $u^\lambda - u^\infty - v^\lambda$, which, in fact, is the same than in the case : $\operatorname{div} u_0 = 0$.

A REMARK CONCERNING EULER'S EQUATIONS

In [2], Klainerman and Majda study the compressible Euler's equations

$$(E^\lambda) \quad \begin{cases} \rho^\lambda \left(\frac{\partial u^\lambda}{\partial t} + (u^\lambda \cdot \nabla) u^\lambda \right) = -\lambda^2 \nabla p^\lambda, \\ \frac{\partial p^\lambda}{\partial t} + u^\lambda \cdot \nabla p^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0, \\ u^\lambda(x, 0) = u_0^\lambda(x), \quad p^\lambda(x, 0) = p_0^\lambda(x), \end{cases}$$

with again : $p = A\rho^\gamma$, $\gamma > 1$.

First, they consider initial data :

$$u_0^\lambda \in H^s(\mathbb{R}^n), \quad (p_0^\lambda - p_0) \in H^s(\mathbb{R}^n) \quad \text{with} \quad s > \left[\frac{n}{2} \right] + 1.$$

Then, they obtain, on a finite time interval, estimations of the same type than the ones obtained in paragraph 2 (by completely different methods).

More precisely, they prove that there exists a finite time interval $[0, T]$, depending only on initial data, and a constant $\Delta_s > 0$, so that, for $\lambda \geq 1$, there exists a classical solution (u^λ, p^λ) in $C^1([0, T] \times \mathbb{R}^n)$ for the system (E^λ) , satisfying :

$$\forall t \in [0, T], \quad |u^\lambda|_{H^s} + |\lambda(p^\lambda - p_0)|_{H^s} \leq \Delta_s.$$

If the initial data verify in addition :

$$(5.15) \quad \begin{aligned} u_0^\lambda(x) &= u_0(x) + \frac{1}{\lambda} u_1(x), \quad \text{with} \quad \operatorname{div} u_0 = 0 \\ p_0^\lambda(x) &= p_0 + \frac{1}{\lambda^2} p_1(x), \quad p_0 = \text{Cte}, \quad (u_1, p_1) \in H^s, \end{aligned}$$

they obtain, as we did, estimates on derivatives in time of (u^λ, p^λ) .

So, they prove a weak convergence of the solutions (u^λ, p^λ) to the solution (u^∞, p^∞) of incompressible Euler's equations :

$$(E^\infty) \quad \begin{cases} \rho_0(u_t^\infty + (u^\infty \cdot \nabla) u^\infty) = -\nabla p^\infty, \\ \operatorname{div} u^\infty = 0, \quad u^\infty(x, 0) = u_0(x). \end{cases}$$

(this solution living on an interval $[0, T^*[$, see [10]).

Finally, introducing the supplementary condition :

$$\forall T_0 < T^*, \quad \forall t \in [0, T_0], \quad |p^\infty|_2 + |p_t^\infty|_2 \leq M(t),$$

they show the following strong convergence's result : there exists $\lambda(T_0)$ so that, for $\lambda \geq \lambda(T_0)$, the system (E^λ) with initial data (5.15) has a unic classical solution (u^λ, p^λ) verifying :

$$\forall t \leq T_0, \quad |u^\lambda - u^\infty|_{H^s} + \frac{1}{\lambda} |u_t^\lambda - u_t^\infty|_{H^{s-1}} \leq \frac{C}{\lambda},$$

$$\lambda |p^\lambda - p^\infty|_{H^s} + |p_t^\lambda|_{H^{s-1}} \leq \frac{C}{\lambda} \quad (C > 0).$$

They also show a principal part.

Their results and ours were so similar that we decided to study the initial layer's problem appearing in this case, if we no more suppose :

$$\text{Div } u_0 = 0, \quad \text{but : } u_0(x) = v_0(x) + \nabla \phi_0(x), \quad \text{with } \text{div } v_0 = 0.$$

Precisely, we get the :

PROPOSITION : *Let us consider the system (E^λ) with initial data :*

$$u^\lambda(x, 0) = v_0(x) + \nabla \phi_0(x) + \frac{1}{\lambda} u_1(x),$$

$$\text{div } v_0 = 0, \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x),$$

$$(v_0, u_1, p_1) \in [H^{s+1}(\mathbb{R}^n)]^3, \quad \phi_0 \in W^{1, s+n+2}(\mathbb{R}^n)$$

$$\text{and } s > \left[\frac{n}{2} \right] + 1 \quad (n \geq 2).$$

Let us suppose in addition that :

$$\forall T_0 < T, \quad \forall t \in [0, T_0], \quad |p^\infty(t)|_2 + |p_t^\infty(t)|_2 \leq M(t).$$

Then, there exists $\lambda(T_0) > 0$, so that :

$$\forall \lambda \geq \lambda(T_0), \quad \forall t \in [0, T_0],$$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leq \frac{C}{\sqrt{\lambda}}$$

if $n = 2$

$$\frac{C}{\lambda} (1 + \log(1 + \lambda t))$$

if $n = 3$

$$\frac{C}{\lambda} \left(1 + (1 + \lambda t)^{-\frac{n-3}{2}} \right)$$

if $n = 4$

where (v^λ, q^λ) is the solution of the waves equation :

$$\begin{cases} \rho_0 v_t^\lambda + \lambda \nabla q^\lambda = 0, \\ q_t^\lambda + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0, \\ v^\lambda(x, 0) = \nabla \phi_0(x), \quad q^\lambda(x, 0) = 0. \end{cases}$$

The demonstration of this result is exactly the same than the one of theorem 4 but, in this case, the initial layer's properties are well known. As a matter of fact, Klainerman proves in [8] the following property, which is here fundamental :

PROPOSITION : If $\phi_0 \in W^{1, s+n+1}$, we have the following $L^\infty - L^1$ estimate :

$$|v^\lambda(t)|_{W^{\infty, s}} \leq C (1 + \lambda t)^{-\frac{n-1}{2}} |\nabla \phi_0|_{W^{1, s+n}} \quad (\forall n \geq 2).$$

APPENDIX

Our purpose here is to study the decreasing with λ of $|D^s v^\lambda|_\infty$, where (v^λ, q^λ) is the solution of the following linear system :

$$(C^\lambda) \quad \begin{cases} \rho_0 \frac{\partial v^\lambda}{\partial t} - \nu \Delta v^\lambda = -\lambda \nabla q^\lambda, \\ \frac{\partial q^\lambda}{\partial t} + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0, \\ v^\lambda(x, 0) = \nabla \phi_0(x), \quad q^\lambda(x, 0) = 0. \end{cases}$$

The choice of the initial data $(v^\lambda(x, 0) = \nabla \phi_0(x))$, and the regularity of ϕ_0 , permit to write the solution (v^λ, q^λ) in the form $(\nabla \phi^\lambda, q^\lambda)$, where the couple $(\phi^\lambda, q^\lambda)$ verifies the following equations :

$$(D^\lambda) \quad \begin{cases} \rho_0 \frac{\partial \phi^\lambda}{\partial t} - \nu \Delta \phi^\lambda = -\lambda q^\lambda, \\ \frac{\partial q^\lambda}{\partial t} + \lambda \gamma p_0 \Delta \phi^\lambda = 0, \\ \phi^\lambda(x, 0) = \phi_0(x), \quad q^\lambda(x, 0) = 0. \end{cases}$$

We then obtain the following result :

THEOREM : Let us suppose that $\phi_0 \in W^{1, k+n+3}$ ($k \in \mathbb{N}$). Then, for λ large enough, the following estimates are verified :

$$|\phi^\lambda(\cdot, t)|_{W^{\infty, k}} \leq \frac{C}{(1 + \lambda t)} |\phi_0|_{W^{1, k+n+3}} \quad \text{if } n \geq 3,$$

$$|\phi^\lambda(\cdot, t)|_{W^{\infty, k}} \leq \frac{C}{\sqrt{1 + \lambda t}} |\phi_0|_{W^{1, k+5}} \quad \text{if } n = 2.$$

Remark : Since $W^{1, n}(\mathbb{R}^n) \subset H^{\left[\frac{n}{2}\right]}(\mathbb{R}^n)$, we also have :

$$\begin{aligned} \gamma p_0 p_0 |\nabla \phi^\lambda(\cdot, t)|_{H^h}^2 + |q^\lambda(\cdot, t)|_{H^h}^2 &\leq \\ &\leq \gamma p_0 p_0 |\nabla \phi_0|_{H^h}^2, \quad \text{for any } h \leq \left[\frac{n}{2}\right] + 2 + k. \end{aligned}$$

COROLLARY : If $\phi_0 \in W^{1, k+n+4}(\mathbb{R}^n)$, then :

$$|v^\lambda(\cdot, t)|_{W^{\infty, k}} \leq \frac{C}{(1 + \lambda t)} |\phi_0|_{W^{1, k+n+4}} \quad \text{if } n \geq 3,$$

$$|v^\lambda(\cdot, t)|_{W^{\infty, k}} \leq \frac{C}{\sqrt{1 + \lambda t}} |\phi_0|_{W^{1, k+6}} \quad \text{if } n = 2.$$

Remark : If we had chosen initial data under the shape :

$$v^\lambda(x, 0) = v_0(x) + \nabla \phi_0(x) \quad \text{with } \operatorname{div} v_0 = 0 \quad \text{and } v_0 \neq 0,$$

we couldn't have obtained these basic decreasing of v^λ results.

As a matter of fact, we would have obtained : $v^\lambda = w + \nabla \phi^\lambda$, where ϕ^λ is the solution of the system (D^λ) , and w the solution of the heat equation :

$$\begin{cases} w_t - \nu \Delta w = 0 \\ w(x, 0) = v_0(x). \end{cases}$$

w being independent of λ , there is no more decreasing with λ .

Proof of the theorem : The function ϕ^λ being a solution of the system (D^λ) , it verifies the following equation :

$$\begin{cases} \rho_0 \phi_{tt}^\lambda - \nu \Delta \phi_t^\lambda - \lambda^2 \gamma p_0 \Delta \phi^\lambda = 0, \\ \phi^\lambda(x, 0) = \phi_0(x), \quad \phi_t^\lambda(x, 0) = \frac{\nu}{\rho_0} \Delta \phi_0(x). \end{cases}$$

To make the calculations simpler, we shall suppose that :

$$\rho_0 = 1, \quad \nu = 2, \quad \gamma p_0 = 1.$$

Hence, let us consider ϕ^λ solution of

$$\begin{cases} \phi_{tt}^\lambda - 2 \Delta \phi_t^\lambda - \lambda^2 \Delta \phi^\lambda = 0, \\ \phi^\lambda(x, 0) = \phi_0(x), \quad \phi_t^\lambda(x, 0) = 2 \Delta \phi_0(x). \end{cases}$$

We then find that the Fourier Transform in x , $\hat{\phi}^\lambda$, of ϕ^λ verifies :

$$\begin{aligned}\hat{\phi}_t^\lambda + 2|\xi|^2 \hat{\phi}_t^\lambda + \lambda^2 |\xi|^2 \hat{\phi}^\lambda &= 0, \quad \xi \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \\ \hat{\phi}^\lambda(\xi, 0) &= \hat{\phi}_0(\xi), \quad \hat{\phi}_t^\lambda(\xi, 0) = -2|\xi|^2 \hat{\phi}_0(\xi).\end{aligned}$$

So we obtain ϕ^λ in the form :

$$\begin{aligned}\phi^\lambda(x, t) &= \int_{\mathbb{R}} e^{ix \cdot \xi} \hat{\phi}_0(\xi) d\xi \\ &= \int_{|\xi| < \lambda} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times \\ &\quad \times \left[\cos(t|\xi| \sqrt{\lambda^2 - |\xi|^2}) - \frac{|\xi|}{\sqrt{\lambda^2 - |\xi|^2}} \sin(t|\xi| \sqrt{\lambda^2 - |\xi|^2}) \right] d\xi \\ &+ \int_{|\xi| > \lambda} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times \\ &\quad \times \left[\operatorname{ch}(t|\xi| \sqrt{|\xi|^2 - \lambda^2}) - \frac{|\xi|}{\sqrt{|\xi|^2 - \lambda^2}} \operatorname{sh}(t|\xi| \sqrt{|\xi|^2 - \lambda^2}) \right] d\xi.\end{aligned}$$

So, we shall write :

$$\begin{aligned}\phi^\lambda(x, t) &= \int_{|\xi| < \sqrt{\lambda}} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times \cos(t|\xi| \sqrt{\lambda^2 - |\xi|^2}) d\xi \\ &+ \int_{\sqrt{\lambda} < |\xi| < \lambda} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times \\ &\quad \times \left[\cos(t|\xi| \sqrt{\lambda^2 - |\xi|^2}) - \frac{|\xi|}{\sqrt{\lambda^2 - |\xi|^2}} \sin(t|\xi| \sqrt{\lambda^2 - |\xi|^2}) \right] d\xi \\ &- \int_{|\xi| < \sqrt{\lambda}} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \frac{|\xi|}{\sqrt{\lambda^2 - |\xi|^2}} \sin(t|\xi| \sqrt{\lambda^2 - |\xi|^2}) d\xi \\ &+ \int_{|\xi| > \lambda} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times \\ &\quad \times \left[\operatorname{ch}(t|\xi| \sqrt{|\xi|^2 - \lambda^2}) - \frac{|\xi|}{\sqrt{|\xi|^2 - \lambda^2}} \operatorname{sh}(t|\xi| \sqrt{|\xi|^2 - \lambda^2}) \right] d\xi \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

(i) *Majoration of I_1 :*

This term represents, in a way, the « principal » part of $\phi^\lambda(x, t)$. Let us call S the waves equation's semi-group, and K the heat equation's Kernel.

Then, let us split up I_1 :

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \cos(t|\xi|\lambda) d\xi \\ &\quad - \int_{\sqrt{\lambda} < |\xi|} e^{ix \cdot \xi} e^{-t|\xi|^2} \hat{\phi}_0(\xi) \cos(t|\xi|\lambda) d\xi \\ &\quad + \int_{|\xi| < \sqrt{\lambda}} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \left[\cos t|\xi|\lambda \sqrt{1 - \frac{|\xi|^2}{\lambda^2}} - \cos t|\xi|\lambda \right] d\xi \\ &= I_5 + I_6 + I_7. \end{aligned}$$

We recognize in I_5 the following expression : $I_5 = S(\lambda t)(K * \phi_0)$.

Thanks to the properties of the solutions of the waves and heat equations, we deduce from that :

$$(A.1) \quad |I_5| \leq C \|K * \phi_0\|_{W^{1,n}} (1 + \lambda t)^{-\frac{n-1}{2}} \leq C \|\phi_0\|_{W^{1,n}} (1 + \lambda t)^{-\frac{n-1}{2}}.$$

Remark : In the case where $\nu = 0$, that is to say for Euler's equations, $\phi^\lambda(x, t)$ is reduced to integral I_5 , and we obtain :

$$\|\phi^\lambda(x, t)\|_\infty \leq C \|\phi_0\|_{W^{1,n}} (1 + \lambda t)^{-\frac{n-1}{2}}.$$

We are now going to estimate separately $I_2 + I_6$, $I_3 + I_7$ and I_4 .

For that, we shall need the following auxiliary results :

LEMMA :

$$(A.2) \quad \forall u \in [0, 1], \quad 1 - u \leq \sqrt{1 - u} \leq 1 - \frac{u}{2};$$

$$(A.3) \quad \forall u \geq 0, \quad \sin u \leq u, \quad \operatorname{sh} u \leq u \cdot e^u, \quad \operatorname{ch} u \leq e^u;$$

$$(A.4) \quad \forall u \geq 0, \quad (1 + u) \cdot e^{-u} \leq C \exp\left(-\frac{u}{2}\right).$$

(ii) *Majoration of $|I_2 + I_6|$.*

Using the inequalities (A.3) and (A.5), we easily obtain :

$$\begin{aligned} |I_2 + I_6| &\leq C \int_{\sqrt{\lambda} < |\xi|} e^{-|\xi|^2 t} |\hat{\phi}_0(\xi)| (1 + t|\xi|^2) d\xi \\ &\leq \int_{\sqrt{\lambda} < |\xi|} e^{-\frac{\lambda t}{2}} |\xi|^{n+1} |\hat{\phi}_0(\xi)| \frac{d\xi}{|\xi|^{n+1}}, \end{aligned}$$

that is to say :

$$(A.5) |I_2 + I_6| \leq C \exp\left(-\frac{\lambda t}{2}\right) |\phi_0|_{W^{1,n+1}} \leq C |\phi_0|_{W^{1,n+1}} (1 + \lambda t)^{-\frac{n-1}{2}}.$$

(iii) *Majoration of $|I_7| + |I_3|$.*

We can write :

$$|I_7| \leq C \int_{|\xi| < \sqrt{\lambda}} e^{-|\xi|^2 t} |\hat{\phi}_0(\xi)| \left| \sin \frac{t|\xi|\lambda}{2} \left(1 - \sqrt{1 - \frac{|\xi|^2}{\lambda^2}}\right) \right| \times \\ \times \left| \sin \frac{t|\xi|\lambda}{2} \left(1 - \sqrt{1 + \frac{|\xi|^2}{\lambda^2}}\right) \right| d\xi.$$

Thanks to the lemma, we deduce from that :

$$|I_7| \leq C \int_{|\xi| < \sqrt{\lambda}} e^{-|\xi|^2 t} |\hat{\phi}_0(\xi)| \frac{t|\xi|^3}{2\lambda} d\xi \\ \leq C \int_{|\xi| < \sqrt{\lambda}} \exp\left(-\frac{|\xi|^2 t}{2}\right) |\hat{\phi}_0(\xi)| \frac{|\xi|}{\lambda} d\xi \\ \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \frac{|\xi|^m}{|\xi|^{m-1}} \exp\left(-\frac{|\xi|^2 t}{2}\right) |\hat{\phi}_0(\xi)| d\xi \\ \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} (1 + |\xi|^a) |\hat{\phi}_0(\xi)| \exp\left(-\frac{|\xi|^2 t}{2}\right) \frac{d\xi}{|\xi|^{m-1}},$$

where $a = m$, if m is even, $a = m + 1$ if m is odd.

Choosing $m = n - 1$, we find :

$$|I_7| \leq \frac{C}{\lambda} |\phi_0|_{W^{1,n}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\xi|^2 t}{2}\right) \frac{d\xi}{|\xi|^{n-2}}.$$

So,

$$(A.6) \quad |I_7| \leq \frac{C}{\lambda t} |\phi_0|_{W^{1,n}}.$$

On the other hand, since $|\xi| < \sqrt{\lambda} \ll \lambda$, we get :

$$|I_3| \leq \int_{|\xi| < \sqrt{\lambda}} \exp(-|\xi|^2 t) |\hat{\phi}_0(\xi)| \frac{C|\xi|}{\lambda} d\xi.$$

So, as above :

$$(A.7) \quad |I_3| \leq \frac{C}{\lambda t} |\phi_0|_{W^{1,n}}.$$

(iv) *Majoration of I_4 .*

Thanks to the inequalities (A.2) and (A.3), we have :

$$\begin{aligned} |I_4| &\leq \int_{|\xi| > \lambda} \exp(-|\xi|^2 t) (1 + |\xi|^2 t) \times \\ &\quad \times \exp\left(|\xi|^2 t \sqrt{1 - \frac{\lambda^2}{|\xi|^2}}\right) |\hat{\phi}_0(\xi)| d\xi \\ &\leq \exp\left(-\frac{\lambda^2 t}{2}\right) (1+t) \int_{|\xi| > \lambda} (1 + |\xi|^2) |\xi|^{n+1} |\hat{\phi}_0(\xi)| \frac{d\xi}{|\xi|^{n+1}}. \end{aligned}$$

What finally gives the following inequality :

$$(A.8) \quad |I_4| \leq C \exp\left(-\frac{\lambda^2 t}{2}\right) (1+t) |\phi_0|_{W^{1,n+3}}.$$

(v) At last, let us remark that :

$$|\phi^\lambda|_\infty \leq |\phi^\lambda|_{H\left[\frac{n}{2}\right]^+} \leq |\phi_0|_{W^{1,n+2}}.$$

We then easily deduce from (A.1), (A.5), (A.6), (A.7) and (A.8) the following result :

$$\begin{aligned} |\phi^\lambda|_\infty &\leq \frac{C}{\sqrt{1+\lambda t}} |\phi_0|_{W^{1,n+3}} \quad \text{if } n \geq 3, \\ |\phi^\lambda|_\infty &\leq \frac{C}{1+\lambda t} |\phi_0|_{W^{1,5}} \quad \text{if } n = 2. \end{aligned}$$

In order to estimate the derivatives in x of ϕ^λ , we just have to do the same work after deriving the linear system (D^λ) .

So the theorem is proven.

BIBLIOGRAPHY

- [1] S. KLAINERMAN and A. MAJDA, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, C.P.A.M. 34 (1981) pp. 481-524.
- [2] S. KAINERMAN and A. MAJDA, *Compressible and incompressible fluids*, C.P.A.M. 35 (1982) pp. 629-651.
- [3] T. NISHIDA and A. MATSUMURA, *The initial value problem for the equations of motion of viscous and heat conductive gases*, J. Math. Kyoto Univ. 20-1 (1980) pp. 67-104.
- [4] A. LAGHA, *Limite des équations d'un fluide compressible lorsque la compressibilité tend vers 0*, Pré-pub. Math. Univ. Paris Nord, Fasc. n° 37.

- [5] R. TEMAN, *The evolution Navier-Stokes equations*, North-Holland (1977) pp. 427-443.
- [6] A. MAJDA, *Compressible fluid flow and systems of conservation laws in several space variables*, Univ. of California, Berkeley.
- [7] H. ADDED and S. ADDED, *Equations of Langmuir's turbulence and non linear Schrödinger equation, smoothness and approximation*, Pré-pub. Math. Univ. Paris Nord.
- [8] S. KLAINERMAN, *Global existence for non linear wave equations*, C.P.A.M. 33 (1980) pp. 43-101.
- [9] A. FRIEDMAN, *Partial differential equations*, Holt, Rinehart and Winston (1969).
- [10] T. KATO, *Non stationary flows of viscous and ideal fluids in $\mathbb{R}^3$* , Functional Analysis 9 (1972), pp. 296-305.