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Numerical methods with interface estimates for the porous medium equation

NUMERICAL METHODS WITH INTERFACE ESTIMATES
FOR THE POROUS MEDIUM EQUATION (*)

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Abstract. — *We provide a general basis, based on the weak truncation error, for proving L^∞ error bounds for the porous medium equation in one space dimension. We show how such bounds for the solution can lead to estimates for the interface of the support for the solution, and we apply this theory to a specific finite difference approximation to the differential equation.*

Résumé. — *Nous donnons une méthode générale, basée sur une troncation faible, pour obtenir des estimations L^∞ de l'erreur pour l'équation des milieux poreux en une dimension d'espace. Nous montrons comment de telles estimations permettent de localiser l'interface du support de la solution et nous appliquons cette théorie à une approximation de la solution de l'équation par différences finies.*

1. INTRODUCTION

We are concerned with numerical approximations to the so-called porous-medium equation [7],

$$(1.1) \quad \begin{cases} u_t = \phi(u)_{xx}, & x \in \mathbb{R}, \quad t > 0, \quad \phi(u) = u^m, \quad m > 1, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

We assume that the initial data $u_0(x)$ has bounded support, that $0 \le u_0 \le M$, and that $\phi(u_0)_x \in BV(\mathbb{R})$. It is well known that a unique solution $u(x, t)$ of (1.1) exists, and that u satisfies

$$(1.2) \quad \begin{aligned} & 0 \le u \le M, \quad u(., t) \text{ has bounded support, and} \\ & TV\phi(u(., t))_x \le TV\phi(u_0)_x. \end{aligned}$$

If the data has slightly more regularity, then this too is satisfied by the solution. Specifically, if m is no greater than two and u_0 is Lipschitz continuous, then $u(., t)$ is also Lipschitz ; if m is greater than two and $(u_0^{m-1})_x \in L^\infty(\mathbb{R})$, then $(u(., t)^{m-1})_x \in L^\infty(\mathbb{R})$ (see [4]). (This will follow from results presented here, also.) We also use the fact that the solution u is Hölder continuous in t [4].

As already remarked, if the nonnegative initial data u_0 has bounded support, then the solution $u(x, t)$ also has bounded support for all time ; this contrasts when $m = 1$ and (1.1) is the heat equation. It is therefore of interest that a numerical scheme for (1.1) be able to estimate not only the solution $u(x, t)$, but also the location of the boundary of the support of u.

Several numerical schemes that estimate the numerical interface have been proposed for the one-dimensional porous medium equation. Methods introduced by Tomoeda and Mimura [11] and Di Benedetto and Hoff [4] are based on the equation for the pressure $v = u^{m-1}$:

$$v_t = \frac{m}{m-1} v_x^2 + mvv_{xx} , \quad x \in \mathbb{R} , \quad t > 0 ,$$

$$v(x, 0) = v_0(x) = u_0^{m-1}(x) , \quad x \in \mathbb{R} .$$

Each method uses a finite difference scheme that is modified to track the estimated interface of the support. The true interface $z(t)$ at the right edge of the support satisfies the differential equation

(1.3) $$z_t = - \frac{m}{m-1} v_x(z(t) - 0, t) .$$

Di Benedetto and Hoff and Tomoeda and Mimura both use a numerical version of this condition to track the fronts.

The second type of method, introduced by Gurtin *et al.* [5], and analyzed by Hollig and Pilant [6], transforms the support of $u(x, t)$ to a fixed domain $[-1, 1]$ and solves numerically the transformed differential equation using finite elements. Using a technical assumption that ensures that $z'(t) > 0$ for all t, Hollig and Pilant have had great success in estimating both the position of the interface and the value of the function $u(x, t)$. They have also been able to show that for small time the interface is a C^∞ curve in x, t space.

There appear to be several difficulties with the underlying concepts of either front tracking or domain transformation when one attempts to apply them to problems in more than one space dimension. Both Rose [10] and Jerome [7] have introduced and analyzed finite-element methods for problems in several space dimensions without concern for estimating the interface of the support of the solution. We hope that the approach developed in this paper will eventually be applicable to several space dimensions.

In this paper we prove error bounds for the simplest finite-difference scheme based directly on (1.1) :

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \delta_x^2 \, \phi(U^n)_i \, , \quad n \geqslant 0 \, , \quad i \in \mathbb{Z} \, ,$$

(1.4)

$$U_i^0 = u_0(ih) \, , \qquad i \in \mathbb{Z} \, ,$$

where $\phi(u) = u^m$, h is the spatial mesh increment, Δt is the time step, and U_i^n is an approximation to $u(ih, n \, \Delta t)$. Our error bounds are of the form

$$\| u(., n \, \Delta t) - U^n(.) \|_{L^\infty(\mathbb{R})} \leqslant Ch^\beta \, , \quad 0 \leqslant n \leqslant N \, ,$$

for some N, where β depends on the Hölder exponents of continuity of u and u^h. Like several authors before us [1] [9], we make the trivial observation that if $Ch^\beta \leqslant \varepsilon$ and $U_i^k > \varepsilon$ for some $k \leqslant n$, then the point $(ih, n \, \Delta t)$ is unquestionably in the support of u. We therefore have an inner estimate for the support of $u(x, t)$, with a natural numerical boundary. Estimates for the difference between the numerical boundary and the true interface of the support, based on the differential equation satisfied by the interface and on the regularity properties of the *true* solution u, follow. (Previously, Nochetto [9] followed a similar program of deriving interface estimates from L^p bounds, but he required a certain global-in-time non-degeneracy assumption on the behavior of u near the boundary of its support that we do not assume. However, our results agree with his if we assume that the non-degeneracy assumption is satisfied locally in time.) Note that our interface estimate is not based on front tracking, but on a trivial post-processing of a numerical solution that may have rapidly increasing support.

We will use the following notation for what we call the weak truncation error. Let $\{u^h(x, t)\}_{0 < h \leqslant h_0}$ be a family of approximate solutions, each of which is assumed to be bounded and nonnegative. For given u^h, the weak truncation error E is the functional

$$(1.5) \quad E(u^h, w, T) = \int_{\mathbb{R}} u^h(x, .) \, w(x, .)|_0^T \, dx$$
$$- \int_0^T \int_{\mathbb{R}} (u^h w_t + \phi(u^h) \, w_{xx}) \, dx \, dt$$

defined on $X = \{ w \in C^2 : w(., t), w, w_t, w_{xx} \text{ are integrable} \}$. Of course, u is the unique solution of (1.1) if and only if $E(u, ., .) \equiv 0$.

The rest of the paper is as follows. In Section 2, the difference in L^∞ between an approximate solution u^h and the true solution u is bounded

in terms of the weak truncation error E. In Section 3, error estimates for the interface are proved. In Section 4, this theory is applied to the finite difference scheme (1.4).

2. $L^\infty(\mathbb{R})$ ERROR BOUNDS

The following theorem expresses the error of approximations u^h in terms of the weak truncation error E.

THEOREM 2.1 : *Let $\{u^h\}$ be a family of approximate solution satisfying (for $0 \le t \le T$)*

(a) $0 \le u^h(x, t) \le M, \quad x \in \mathbb{R}, t > 0,$

(b) *both u and u^h are Hölder-α in x for some $\alpha \in (0, 1 \wedge 1/(m - 1))$; u^h is right continuous in t ; and u^h is Hölder continuous in t on strips $\mathbb{R} \times (t^n, t^{n+1})$, with the set $\{t^n\}$ having no limit points ;*

(c) *there exists a positive function $\omega(h, \varepsilon)$ such that : whenever $\{w^\varepsilon\}_{0 < \varepsilon \le \varepsilon_0}$ is a family of functions in X for which*

$$(2.1) \qquad \|w^\varepsilon\|_\infty, \|w_x^\varepsilon(\cdot, t)\|_{L^1(\mathbb{R})} \le 1/\varepsilon$$

and

$$(2.2) \quad \begin{cases} \|w_x^\varepsilon\|_\infty, \|w_{xx}^\varepsilon(\cdot, t)\|_{L^1(\mathbb{R})}, \|w_t^\varepsilon(\cdot, t)\|_{L^1(\mathbb{R})}, \|w^\varepsilon(\cdot, t)\|_{L^1(\mathbb{R})} \quad and \\[2mm] \displaystyle\sup_{\substack{x \in \mathbb{R} \\ 0 \le t_1 < t_2 \le T}} \dfrac{|w^\varepsilon(x, t_2) - w^\varepsilon(x, t_1)|}{|t_2 - t_1|^{1/2}} \le 1/\varepsilon^2 , \end{cases}$$

then $\qquad\qquad |E(u^h, w^\varepsilon, T)| \le \omega(h, \varepsilon)$.

Then there is a constant $C = C(m, M, T)$ such that

$$(2.3) \quad \|u - u^h\|_{\infty, \mathbb{R} \times [0, T]}$$

$$\le C \left[\sup \left| \int_\mathbb{R} (u_0(x) - u^h(x, 0)) \, w(x, 0) \, dx \right| + \omega(h, \varepsilon) + \varepsilon^\alpha \right],$$

where the supremum is taken over all $w \in X$ that satisfy (2.1) and (2.2).

Proof: Let z be in X. Because $E(u, \cdot, \cdot) \equiv 0$, equation (1.5) implies that

$$(2.4) \quad \int_\mathbb{R} \Delta u z \, \big|_0^T \, dx = \int_0^T \int_\mathbb{R} \Delta u \, (z_t + \phi[u, u^h] \, z_{xx}) \, dx \, dt - E(u^h, z, t) ,$$

where $\Delta u = u - u^h$ and

$$\phi[u, u^h] = \frac{\phi(u) - \phi(u^h)}{u - u^h} .$$

Extend $\phi[u, u^h](x, t) = \phi[u, u^h](x, 0)$ for   negative t,   and
$\phi[u, u^h](x, t) = \phi[u, u^h](x, T)$ for $t > T$. Fix a point x_0 and a number
$\varepsilon > 0$. Let j_ε be a smooth function of x with integral 1 and support in
$[-\varepsilon, \varepsilon]$, and let J_δ be a smooth function of x and t with integral 1 and
support in $[-\delta, \delta] \times [-\delta, \delta]$; δ and ε are positive numbers to be specified
later. We choose $z = z^{\varepsilon\delta}$ to satisfy

(2.5)
$$\begin{cases} z_t + (\delta + J_\delta * \phi[u, u^h]) z_{xx} = 0 , & x \in \mathbb{R} , \quad 0 \le t \le T \\ z(x, T) = j_\varepsilon(x - x_0) . \end{cases}$$

Because the partial differential equation (2.5) is strictly parabolic with
smooth coefficients, the following results are direct consequences of
maximum principle arguments ; observe that all constants are independent
of ε and δ.

$$\left\| z^{\varepsilon\delta} \right\|_\infty \le \left\| j_\varepsilon \right\|_\infty \le C/\varepsilon$$

(2.6)
$$\left\| z_x^{\varepsilon\delta} \right\|_\infty \le \left\| j_\varepsilon' \right\|_\infty \le C/\varepsilon^2$$

$$\left\| z_x^{\varepsilon\delta}(\cdot, t) \right\|_1 \le \left\| j_\varepsilon' \right\|_1 \le C/\varepsilon$$

(2.7)
$$\left\| z_{xx}^{\varepsilon\delta}(\cdot, t) \right\|_1 \le \left\| j_\varepsilon'' \right\|_1 \le C/\varepsilon^2$$

Note also that (2.7) implies that for $\delta \le \delta_0$,

(2.8)
$$\left\| z_t^{\varepsilon\delta}(\cdot, t) \right\|_1 \le C/\varepsilon^2 .$$

The following simple argument shows that

(2.9)
$$\frac{\left| z^{\varepsilon\delta}(x, t_2) - z^{\varepsilon\delta}(x, t_1) \right|}{\left| t_2 - t_1 \right|^{1/2}} \le C/\varepsilon^2 .$$

By (2.6) and (2.8), for any positive H,

$$\left| z^{\varepsilon\delta}(x, t_2) - z^{\varepsilon\delta}(x, t_1) \right| = \frac{1}{H} \left| \int_x^{x+H} [z^{\varepsilon\delta}(y, t_2) - z^{\varepsilon\delta}(y, t_1)] dy \right| + O\left(\frac{H}{\varepsilon^2} \right)$$

$$\le \frac{1}{H} \int_x^{x+H} \int_{t_1}^{t_2} \left| z_t^{\varepsilon\delta}(y, t) \right| dt \, dy + O\left(\frac{H}{\varepsilon^2} \right)$$

$$\le \frac{1}{H} \cdot \frac{C}{\varepsilon^2} \cdot \left| t_2 - t_1 \right| + C \frac{H}{\varepsilon^2}$$

$$= \frac{C}{\varepsilon^2} \left(\frac{\left| t_2 - t_1 \right|}{H} + H \right)$$

$$= \frac{C}{\varepsilon^2} \left| t_2 - t_1 \right|^{1/2}$$

if $H = \left| t_2 - t_1 \right|^{1/2}$.

If we let C be the maximum of the above constants — still independent of ε and δ — the family $\left\{\dfrac{1}{C}\, z^{\varepsilon\delta}\right\}_{0 < \delta \leq \delta_0}$ satisfies (2.1) and (2.2). So, by assumption,

$$\left| E\left(u^h, \frac{1}{C}\, z^{\varepsilon\delta}, T\right) \right| \leq \omega(h, \varepsilon) \, ;$$

or, because E is linear in the test function,

$$\left| E(u^h, z^{\varepsilon\delta}, T) \right| \leq C\omega(h, \varepsilon) \, ,$$

where the constant C is independent of both ε and δ.

We now use this information to provide a pointwise bound for Δu. Equation (2.4) implies that

$$(j_\varepsilon * \Delta u)(x_0, T) = \int_{\mathbb{R}} \Delta u_0 \, z^{\varepsilon\delta}(., 0) \, dx$$
$$+ \int_0^T \int_{\mathbb{R}} \Delta u \left(\phi[u, u^h] - \delta - J_\delta * \phi[u, u^h]\right) z_{xx}^{\varepsilon\delta} \, dx \, dt - E(u^h, z^{\varepsilon\delta}, T) \, .$$

Using our inequalities, we can bound the left hand side of the preceding equation as

$$(2.10) \quad \left| (j_\varepsilon * \Delta u)(x_0, T) \right| \leq \left| \int_{\mathbb{R}} \Delta u_0 \, z^{\varepsilon\delta}(., 0) \, dx \right| + C\omega(h, \varepsilon)$$
$$+ C \int_0^T \int_{\mathbb{R}} \left| \phi[u, u^h] - \delta - J_\delta * \phi[u, u^h] \right| \left| z_{xx}^{\varepsilon\delta} \right| dx \, dt$$

where, again, the constants are independent of ε and δ. If we let δ vanish, the last term tends to zero. To see this, observe that, for fixed t,

$$\int_{\mathbb{R}} \left| \phi[u, u^h] - \delta - J_\delta * \phi[u, u^h] \right| \left| z_{xx}^{\varepsilon\delta} \right| dx$$
$$\leq \left\| \phi[u, u^h] - \delta - J_\delta * \phi[u, u^h] \right\|_{L^\infty(\mathbb{R})} \cdot C/\varepsilon^2 \to 0$$

because u and u^h are Hölder in x and locally Hölder in t. Furthermore, because the absolute value of each integral is bounded by C/ε^2 uniformly in δ, the double integral must tend to zero by the Lebesgue Dominated Convergence Theorem.

The conclusion of the theorem now follows from (2.10) and the fact that

$$\left| j_\varepsilon * \Delta u (x_0, t) - \Delta u (x_0, t) \right| \leq C\varepsilon^\alpha \, ,$$

which follows from our assumption (b). ∎

3. ESTIMATES FOR THE INTERFACE

In this section we will assume that the approximating family $\{u^h\}$ and the solution u of (1.1) satisfy the hypotheses, and hence the conclusion, of Theorem 2.1. Assume also that $\left| \int \Delta u_0 \, w(., 0) \, dx \right|$ and $\omega(h, \varepsilon)$ are bounded in terms of h and ε in such a way that the error bound becomes

$$(3.1) \qquad \| u - u^h \|_{L^\infty(\mathbb{R} \times [0, T])} \leqslant C_0 \, h^\beta$$

for some $\beta > 0$. For simplicity we assume that $u(., t)$ has bounded, connected support, with right hand interface curve $x = z(t)$. We fix Δt, let $t^n = n \, \Delta t$, and define an approximate right-hand interface curve $z^n \approx z(t^n)$ by

$$(3.2) \quad z^{n+1} = \inf \left\{ x \geqslant z^n : u^h(y, t^{n+1}) \leqslant 2 \, C_0 \, h^\beta, \; \forall y \geqslant x \right\}, \quad n \geqslant 0 \, .$$

The initial approximation z^0 may be defined as

$$z^0 = \inf \left\{ x : u^h(y, 0) \leqslant 2 \, C_0 \, h^\beta, \quad \forall y \geqslant x \right\},$$

or through some other method ; we require only that $z^0 \leqslant z(0)$. Note that the set in (3.2) is nonempty by (3.1), because $u(., t^{n+1})$ has compact support.

The approximation for the interface can be computed simply by noting that the right side approximate interface, for example, moves only to the right with time. Thus, after each time step one only has to examine a few mesh points to the right of the current interface to decide whether or not to increment the value of z^n to obtain z^{n+1}.

The results in this section are based upon arguments introduced by Di Benedetto and Hoff [4] ; we refer the reader to this paper for the regularity results that we use in Lemma 3.2.

LEMMA 3.1 : *If $z^0 \leqslant z(0)$, then $z^n \leqslant z(t^n)$.*

Proof : The proof is obvious.

LEMMA 3.2 : *Assume in addition to (1.2) that u_0 satisfies $(u_0^{m-1})_{xx} \geqslant - C_2$ as a distribution. Then there are constants $C_1 = C_1(m)$ and $C_3 = C_3(u_0)$ such that, for $s > 0$ and $t \geqslant \Delta t$,*

$$(3.3) \quad u^{m-1}(z(t) - s, t) \geqslant s C_1 \frac{z(t) - z(t - \Delta t)}{\Delta t} - \frac{1}{2} C_2 s^2 - C_3 \, \Delta t^{1/2}$$

Proof: Because $v = u^{m-1}$ is Hölder-1/2 in t,

$$v(z(t) - s, t) = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} v(z(t) - s, \tau) \, d\tau + O(\Delta t^{1/2})$$

$$= \frac{1}{\Delta t} \int_{t-\Delta t}^{t} v(z(\tau) - s, \tau) \, d\tau$$

$$+ \frac{1}{\Delta t} \int_{t-\Delta t}^{t} [v(z(t) - s, \tau) - v(z(\tau) - s, \tau)] \, d\tau + O(\Delta t^{1/2}).$$

Because v is Lipschitz in x for all time, and $z(t)$ is Lipschitz with

$$C_1 \dot{z}(t) = - v_x(z(t) - 0, t) \text{ a.e.},$$

the second integral is $O(\Delta t) = O(\Delta t^{1/2})$ for small Δt.

Because $v_{xx} \geq - C_2$ for x and t in the interior of the support of v,

$$v(z(\tau) - s, \tau) \geq v(z(\tau), \tau) - v_x(z(\tau), \tau) s - \frac{C_2}{2} s^2$$

$$= 0 + C_1 \dot{z}(\tau) s - \frac{C_2}{2} s^2 \text{ a.e.}$$

Thus,

$$v(z(t) - s, t) \geq \frac{1}{\Delta t} \int_{t-\Delta t}^{t} \left[C_1 \dot{z}(\tau) s - \frac{C_2}{2} s^2 \right] d\tau + O(\Delta t^{1/2})$$

$$\geq C_1 s \left[\frac{z(t) - z(t - \Delta t)}{\Delta t} \right] - \frac{C_2}{2} s^2 - C_3 \Delta t^{1/2}. \quad \blacksquare$$

The following theorem is our main result on estimating interface curves.

THEOREM 3.3 : *Assume that $\{u^h\}$ and u satisfy the hypotheses of Theorem 2.1 and the estimate (3.1), that $z^0 \leq z(0)$, and that $(u_0^{m-1})_{xx} \geq - C_2$ as a distribution. Then for sufficiently small Δt there is a constant $C = C(u_0, T, m)$ such that the approximations z^n satisfy*

$$(3.4) \qquad |z(t^n) - z^n|^2 \leq C[|z(0) - z^0|^2 + h^{(m-1)\beta} + \Delta t^{1/2}]$$

for $t \in [0, T]$.

Proof: Let $s^n = |z(t^n) - z^n| = z(t^n) - z^n$ by Lemma 3.1. The definition of z^n and (3.1) imply that

$$u(z(t^n) - s^n, t^n) \leq u^h(z(t^n) - s^n, t^n) + C_0 h^\beta$$

$$= u^h(z^n, t^n) + C_0 h^\beta \leq 3 C_0 h^\beta.$$

Therefore, from Lemma 3.2,

$$(3.5) \quad (3\,C_0\,h^\beta)^{m-1} \geq u\,(z\,(t^n) - s^n,\, t^n)^{m-1}$$

$$\geq C_1 \left[\frac{z\,(t^n) - z\,(t^{n-1})}{\Delta t} \right] s^n - \frac{C_2}{2}\,(s^n)^2 - C_3\,\Delta t^{1/2}$$

$$\geq C_1 \left[\frac{s^n - s^{n-1}}{\Delta t} \right] s^n - \frac{C_2}{2}\,(s^n)^2 - C_3\,\Delta t^{1/2}$$

(because $(z^n - z^{n-1})/(\Delta t) \geq 0$).

Rearranging the terms in this inequality shows that

$$(s^n)^2 \leq s^n s^{n-1} + C\,\Delta t\,[(s^n)^2 + h^{\beta(m-1)} + \Delta t^{1/2}]$$

$$\leq \frac{(s^n)^2}{2} + \frac{(s^{n-1})^2}{2} + C\,\Delta t\,[(s^n)^2 + h^{\beta(m-1)} + \Delta t^{1/2}]\,,$$

so that

$$(1 - 2\,C\,\Delta t)\,\frac{(s^n)^2}{2} \leq \frac{(s^{n-1})^2}{2} + C\,\Delta t\,[h^{\beta(m-1)} + \Delta t^{1/2}]\,,$$

or $\qquad \dfrac{(s^n)^2}{2} \leq (1 + C\,\Delta t)\,\dfrac{(s^{n-1})^2}{2} + C\,\Delta t\,[h^{\beta(m-1)} + \Delta t^{1/2}]\,.$

Solving this recurrence gives the statement of the theorem. ∎

We can improve this bound if we know a priori that $\dot{z}(t) \geq C > 0$ a.e. on a time interval $[t^{n-1}, t^n]$. It is known [2] that for some time interval $[0, \bar{t}\,]$, $\dot{z}(t) = 0$, and that after this time $\dot{z}(t) > 0$ ($\bar{t}$ may be zero). The time $\bar{t}$ is known as the waiting time. So for large time, the following corollary holds.

COROLLARY 3.4 : *If, in addition to the hypotheses to Theorem 3.3, $\dot{z}(t) \geq C > 0$ a.e. on $[t^{n-1}, t^n]$ $(n > 0)$ and*

$$|z^0 - z\,(0)|^2 \leq C\,[h^{(m-1)\beta} + \Delta t^{1/2}]\,,$$

then $\qquad |z^n - z\,(t^n)| \leq C\,[h^{(m-1)\beta} + \Delta t^{1/2}]\,.$

Proof: We see from (3.5) that with the extra hypothesis on $\dot{z}$,

$$(3.6) \qquad |s^n| \leq C\,(s^n)^2 + C\,(h^{(m-1)\beta} + \Delta t^{1/2})\,.$$

Our assumption on z^0 applied to (3.4) implies that

$$(s^n)^2 \leq C\,(h^{(m-1)\beta} + \Delta t^{1/2})\,.$$

The conclusion follows from substituting this into (3.6). ∎

Remark : Assuming the L^∞ bounds on Δu, Corollary 3.4 is implied by results in [9].

4. APPLICATION TO A PARTICULAR SCHEME

We analyze the simple scheme

$$(4.1) \quad \frac{U_k^{n+1} - U_k^n}{\Delta t} = \frac{\delta^2 \phi_k^n}{h^2} = \frac{\phi(U_{k-1}^n) - 2\phi(U_k^n) + \phi(U_{k+1}^n)}{h^2} \, ,$$

$$n \geqslant 0 \, , \quad k \in \mathbb{Z} \, ,$$

where h is the mesh spacing, Δt is the time step, and U_k^n is meant as an approximation to $u(kh, n\,\Delta t)$, where u is the solution of (1.1). We let $\phi_k^n = \phi(U_k^n)$. The following theorem summarizes the discrete regularity results that we will use.

THEOREM 4.1 : *If*

$$(4.2) \qquad\qquad 0 \leqslant U_k^0 \leqslant M \, ,$$

$$(4.3) \qquad\qquad \frac{U_{k+1}^0 - U_k^0}{h} = 0 \quad \textit{for large } |k| \, ,$$

$$(4.4) \qquad\qquad \sum_{k \in \mathbb{Z}} \frac{|\delta^2 \phi_k^0|}{h^2} \, h = V < \infty \, , \quad \textit{and}$$

$$(4.5) \qquad\qquad \frac{\Delta t}{h^2} \leqslant \frac{1}{2 \, mM^{m-1}} \, ,$$

then U_k^n is defined for all k and n, and

$$(4.6) \qquad\qquad 0 \leqslant U_k^n \leqslant M \, ,$$

$$(4.7) \qquad\qquad \sum_{k \in \mathbb{Z}} \frac{|\delta^2 \phi_k^n|}{h^2} \, h \leqslant V \, ,$$

$$(4.8) \qquad\qquad \left| \frac{\phi_{k+1}^n - \phi_k^n}{h} \right| \leqslant V \, ,$$

$$(4.9) \qquad\qquad \sum_{k \in \mathbb{Z}} \left| \frac{U_k^{n+1} - U_k^n}{\Delta t} \right| h \leqslant V \, , \quad \textit{and}$$

$$(4.10) \qquad \sum_{k \in \mathbb{Z}} \left| \frac{U_{k+1}^n - U_k^n}{h} \right| h \leqslant \sum_{k \in \mathbb{Z}} \left| \frac{U_{k+1}^0 - U_k^0}{h} \right| h \, .$$

Proof: This theorem is similar to results for numerical methods for hyperbolic conservation laws, and we refer the reader to Lucier [8], for example, for more detailed arguments.

We can write

$$(4.11) \qquad U_k^{n+1} = U_k^n - \frac{2\,\Delta t}{h^2}\,\phi(U_k^n) + \frac{\Delta t}{h^2}\,\phi(U_{k-1}^n) + \frac{\Delta t}{h^2}\,\phi(U_{k+1}^n)\,.$$

Because of (4.5), U_k^{n+1} is an increasing function of U_k^n, U_{k-1}^n and U_{k+1}^n. Therefore, $U^n \to U^{n+1}$ is an order preserving map of $L^1(\mathbb{Z})$ (or $L^\infty(\mathbb{Z})$) to itself : if $U_k^n \le V_k^n$ for all k, then $U_k^{n+1} \le V_k^{n+1}$ for all k. (4.6) follows immediately. It is also obvious that $\sum_{k\in\mathbb{Z}} U_k^{n+1} = \sum_{k\in\mathbb{Z}} U_k^n$, so that time-stepping also preserves the integral of U^n. A theorem of Crandall and Tartar [3] now implies that for every U^0 and V^0 satisfying (4.2) and (4.3),

$$\sum_{k\in\mathbb{Z}} |U_k^n - V_k^n| \le \sum_{k\in\mathbb{Z}} |U_k^0 - V_k^0|\,.$$

(4.7) and (4.9) follow by setting $V^0 = U^1$. (4.10) follows by setting $V_k^0 = U_{k+1}^0$. (4.8) is an immediate consequence of (4.7) and (4.3). ■
    We have the following estimate for the weak truncation error of the scheme.

    THEOREM 4.2 : *Assume that the hypotheses of Theorem 4.1 hold. Let* $u^h(.,t^n)$ *be the piecewise linear interpolant of* $\{U_k^n\}_{k\in\mathbb{Z}}$, *and set*

$$u^h(x,t) = u^h(x,t^n) \quad \text{for} \quad t^n \le t < t^{n+1}\,.$$

Then we can take $\omega(h,\varepsilon) = Ch/\varepsilon^2$ *in Theorem 2.1.*

    Proof: We write u for u^h and w for w^ε as introduced in Theorem 2.1. We must estimate $E(u,w,T)$. First, we assume that $T = t^N$, for if $t^N \le T < t^{N+1}$

$$\left| \int_{\mathbb{R}} uw \big|_{t^N}^T \, dx \right| \le \int_{\mathbb{R}} \int_{t^N}^T |u(x,t^N)\, w_t(x,s)|\, ds\, dx \le C\,\Delta t/\varepsilon^2 \le Ch^2/\varepsilon^2$$

$$\left| \int_{t^N}^T \int_{\mathbb{R}} uw_t \, dx\, dt \right| \le C\,\Delta t/\varepsilon^2 \le Ch^2/\varepsilon^2 ; \quad \text{and}$$

$$\left| \int_{t^n}^T \int_{\mathbb{R}} \phi w_{xx} \, dx\, dt \right| \le C\,\Delta t/\varepsilon^2 \le Ch^2/\varepsilon^2\,.$$

We defined H_j to be the continuous, piecewise linear « hat function » that is zero for $x_k \ne x_j = jh$ and is one for $x_k = x_j$, and let

$$I_j(x) = \int_{-\infty}^x H_j(s)\, ds = \begin{cases} 0\,, & \text{for} \quad x \le x_{j-1}\,, \\ (x - x_{j-1})^2/(2h)\,, & \text{for} \quad x_{j-1} \le x \le x_j\,, \\ h - (x - x_{j+1})^2/(2h)\,, & \text{for} \quad x_j \le x \le x_{j+1}\,, \\ h\,, & \text{for} \quad x_{j+1} < x\,. \end{cases}$$

Note also that

$$(4.12) \quad \sum_{n=1}^{N} \int_{\mathbb{R}} |w(x, t^n) - w(x, t^{n-1})| \, |u(x, t^n) - u(x, t^{n-1})| \, dx$$

$$\leqslant C \sum_{n=1}^{N} \Delta t^{1/2} \cdot \frac{1}{\varepsilon^2} \left(\sum_{k \in \mathbb{Z}} \left| \frac{U_k^n - U_k^{n-1}}{\Delta t} \right| h \right) \Delta t$$

$$\leqslant C \sum_{n=1}^{N} \frac{h}{\varepsilon^2} V \Delta t \leqslant C T h / \varepsilon^2 .$$

To begin, we have that

$$\int_{\mathbb{R}} \int_0^T u w_t \, dt \, dx = \sum_{n=0}^{N-1} \int_{\mathbb{R}} \int_{t^n}^{t^{n+1}} u w_t \, dt \, dx$$

$$= \sum_{n=0}^{N-1} \int_{\mathbb{R}} u(x, t^n) [w(x, t^{n+1}) - w(x, t^n)] \, dx$$

$$= \int_{\mathbb{R}} u(x, t^N) w(x, t^N) \, dx - \int_{\mathbb{R}} u(x, 0) w(x, 0) \, dx$$

$$- \sum_{n=1}^{N} \int_{\mathbb{R}} w(x, t^n) [u(x, t^n) - u(x, t^{n-1})] \, dx .$$

Thus,

$$\int_{\mathbb{R}} u w \big|_0^T \, dx - \int_{\mathbb{R}} \int_0^T u w_t \, dt \, dx$$

$$= \sum_{n=1}^{N} \int_{\mathbb{R}} w(x, t^n) [u(x, t^n) - u(x, t^{n-1})] \, dx$$

$$= \sum_{n=1}^{N} \int_{\mathbb{R}} w(x, t^{n-1}) [u(x, t^n) - u(x, t^{n-1})] \, dx + O(h/\varepsilon^2) \quad \text{by (4.12)}$$

$$= \sum_{n=1}^{N} \int_{\mathbb{R}} w(x, t^{n-1}) \left(\sum_{k \in \mathbb{Z}} \frac{U_k^n - U_k^{n-1}}{\Delta t} H_k(x) \right) dx \, \Delta t + O(h/\varepsilon^2)$$

$$= \sum_{n=1}^{N} \int_{\mathbb{R}} w(x, t^{n-1}) \left(\sum_{k \in \mathbb{Z}} \frac{\delta^2 \phi_k^{n-1}}{h^2} H_k(x) \right) dx \, \Delta t + O(h/\varepsilon^2)$$

$$= - \sum_{n=1}^{N} \int_{\mathbb{R}} w_x(x, t^{n-1}) \left(\sum_{k \in \mathbb{Z}} \frac{\delta^2 \phi_k^{n-1}}{h^2} I_k(x) \right) dx \, \Delta t + O(h/\varepsilon^2) .$$

From the definition of I_k we know that for $x_k \leqslant x \leqslant x_{k+1}$,

$$\sum_{j \in \mathbb{Z}} \frac{\delta^2 \phi_j}{h^2} I_j(x) = \sum_{j \leqslant k-1} \left(\frac{\delta^2 \phi_j}{h^2} h \right) + \frac{\delta^2 \phi_k}{h^2} I_k(x) + \frac{\delta^2 \phi_{k+1}}{h^2} I_{k+1}(x)$$

$$= \frac{\phi_k - \phi_{k-1}}{h} + \frac{\delta^2 \phi_k}{h^2} \left[h - \frac{(x - x_{k+1})^2}{2h} \right] + \frac{\delta^2 \phi_{k+1}}{h^2} \frac{(x - x_k)^2}{2h}$$

$$= \frac{\phi_{k+1} - \phi_k}{h} - \frac{\delta^2 \phi_k}{h^2} \frac{(x - x_{k+1})^2}{2h} + \frac{\delta^2 \phi_{k+1}}{h^2} \frac{(x - x_k)^2}{2h} .$$

Thus,

$$\int_{\mathbb{R}} uw \Big|_0^T dx - \int_{\mathbb{R}} \int_0^T uw_t \, dt \, dx = - \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} w_x(x, t^{n-1}) \cdot$$

$$\times \left[\frac{\phi_{k+1}^{n-1} - \phi_k^{n-1}}{h} + O(h) \left\{ \frac{\delta^2 \phi_k^{n-1}}{h^2} + \frac{\delta^2 \phi_{k+1}^{n-1}}{h^2} \right\} \right] dx \, \Delta t + O(h/\varepsilon^2) .$$

The term in braces is bounded by

$$Ch \|w_x\|_\infty \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} \left| \frac{\delta^2 \phi_k^{n-1}}{h^2} \right| h \, \Delta t \leqslant Ch/\varepsilon^2 .$$

Now let $\psi(., t)$ be the continuous, piecewise linear interpolant of $\phi(., t)$. On $[x_k, x_{k+1}]$, $\psi_x = (\phi_{k+1} - \phi_k) h$, and because of (4.7),

$$\int_{\mathbb{R}} |\psi_{xx}| \, dx \leqslant V ,$$

where the integral is really the total measure of ψ_{xx}. We then have

$$\int_{\mathbb{R}} uw \Big|_0^T dx - \int_{\mathbb{R}} \int_0^T uw_t \, dt \, dx$$

$$= - \sum_{n=1}^{N} \int_{\mathbb{R}} (w_x \psi_x)(x, t^{n-1}) \, dx \, \Delta t + O(h/\varepsilon^2)$$

$$= - \sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} (w_x \psi_x)(x, t) \, dx \, dt + O(h/\varepsilon^2)$$

since

$$\left| \sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \left[(w_x \psi_x)(x, t) - (w_x \psi_x)(x, t^{n-1}) \, dx \, dt \right] \right|$$

$$= \left| \sum_{n=1}^{N} \int_{\mathbb{R}} \psi_x(x, t^{n-1}) \left\{ \int_{t^{n-1}}^{t^n} [w_x(x, t) - w_x(x, t^{n-1})] \, dt \right\} dx \right|$$

$$\leq \left| \sum_{n=1}^{N} \int_{\mathbb{R}} \psi_{xx}(x, t^{n-1}) \int_{t^{n-1}}^{t^n} [w(x, t) - w(x, t^{n-1})] \, dt \, dx \right|$$

$$\leq \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} \frac{|\delta^2 \phi_k^n|}{h^2} h \int_{t^{n-1}}^{t^n} \frac{|t - t^{n-1}|^{1/2}}{\varepsilon^2} \, dt \, , \quad \text{by} \quad (2.2)$$

$$\leq \sum_{n=1}^{N} V \frac{C \, \Delta t^{3/2}}{\varepsilon^2} \quad \text{by} \quad (4.7)$$

$$\leq \sum_{n=1}^{N} C \cdot \frac{Ch}{\varepsilon^2} \cdot \Delta t = O(h/\varepsilon^2) \, ,$$

because $h = C \, \Delta t^{1/2}$. Thus

$$\int_{\mathbb{R}} uw \big|_0^T \, dx - \int_{\mathbb{R}} \int_0^T uw_t \, dt \, dx$$

$$= - \int_{\mathbb{R}} \int_0^T w_x \phi_x \, dt \, dx - \int_{\mathbb{R}} \int_0^T w_x (\phi - \psi)_x \, dt \, dx + O(h/\varepsilon^2) \, .$$

We must show that $\int_{\mathbb{R}} \int_0^T w_x (\phi - \psi)_x \, dt \, dx = O(h/\varepsilon^2)$.

First note that there is a constant $C = C(m)$ such that for all u, $v \geq 0$,

$$|u^m - v^m| \geq C |u - v| (u^{m-1} + v^{m-1}) \, .$$

Thus, on $[x_k, x_{k+1}]$,

$$|\phi(u)_x| = |(u^m)_x| = mu^{m-1}|u_x| = mu^{m-1} \left| \frac{U_{k+1} - U_k}{h} \right|$$

$$\leq m(U_k^{m-1} + U_{k+1}^{m-1}) \left| \frac{U_{k+1} - U_k}{h} \right|$$

$$\leq C \left| \frac{U_{k+1}^m - U_k^m}{h} \right| = C \frac{|\phi_{k+1} - \phi_k|}{h} \leq C \, .$$

Thus, since $\psi(. \, , t^{n-1})$ is the piecewise linear interpolant of $\phi(., t^{n-1})$, we have

$$\| (\phi - \psi)(. \, , t^{n-1}) \|_\infty \leq Ch \, ;$$

it follows that

$$\int_{\mathbb{R}} \int_0^T w_x (\phi - \psi)_x \, dt \, dx = \sum_{n=1}^N \int_{\mathbb{R}} \int_{t^{n-1}}^{t^n} w_{xx}(x, t)(\phi - \psi)(x, t^{n-1}) \, dt \, dx$$

$$\leq \sum_{n=1}^N \int_{t^{n-1}}^{t^n} \sup_t \|w_{xx}(\cdot, t)\|_1 \, \|(\phi - \psi)(\cdot, t^{n-1})\|_\infty \, dt$$

$$\leq \sum_{n=1}^N \frac{C}{\varepsilon^2} h \, \Delta t = Ch/\varepsilon^2 . \quad \blacksquare$$

The following lemma, which was proved with the help of Don French, shows that $u^h(\cdot, t) \in C^\alpha$ for the optimal value of α.

LEMMA 4.3 : *Assume that the hypotheses of Theorem 4.1 hold and that* $\{u^h\}$ *are constructed using the scheme* (4.1). *Then for all* $x_1, x_2 \in \mathbb{R}$, $t \in [0, T]$,

(4.13) $\left| u^h(x_2, t) - u^h(x_1, t) \right| \leq C \left| x_2 - x_1 \right|^{1/m} .$

If $m \in (1, 2]$ *and the initial values* $\{U_k^0\}$ *satisfy*

(4.14) $\left| U_{k+1}^0 - U_k^0 \right| \leq Lh$

and

(4.15) $U_k^0 \geq Lh$

then for all $x_1, x_2 \in \mathbb{R}$, $t \in [0, T]$,

(4.16) $\left| u^h(x_2, t) - u^h(x_1, t) \right| \leq L \left| x_2 - x_1 \right| .$

If $m > 2$ *and the initial values* $\{U_k^0\}$ *satisfy*

(4.17) $\left| (U_{k+1}^0)^{m-1} - (U_k^0)^{m-1} \right| \leq Lh$

and

(4.18) $(U_k^0)^{m-1} \geq Lh$

then for all $n \geq 0$ *and for all* $k \in \mathbb{Z}$,

(4.19) $\left| (U_{k+1}^n)^{m-1} - (U_k^n)^{m-1} \right| \leq Lh .$

Inequality (4.19) *implies that there is a constant* $C = C(m)$ *such that for all* $x_1, x_2 \in \mathbb{R}$, $t \in [0, T]$,

$$\left| u^h(x_2, t) - u^h(x_1, t) \right| \leq CL^{1/(m-1)} \left| x_2 - x_1 \right|^{1/(m-1)} .$$

Proof: From (4.8) we know that

$$|U_k^n - U_j^n| \leqslant C |\phi_k^n - \phi_j^n|^{1/m} \leqslant C (V |x_k - x_j|)^{1/m} .$$

(4.13) then follows immediately.

As for (4.16), we will show that for $n = 0$, $U_i^1 - U_{i-1}^1 \leqslant Lh$. A symmetric argument shows that $U_i^1 - U_{i-1}^1 \geqslant - Lh$. The result will follow by induction.

From (4.1) we know that for general n,

$$(4.20) \quad U_i^{n+1} - U_{i-1}^{n+1} = U_i^n - U_{i-1}^n + \frac{\Delta t}{h^2} ((\phi(U_{i+1}^n) - \phi(U_i^n)) -$$

$$- 2(\phi(U_i^n) - \phi(U_{i-1}^n)) + (\phi(U_{i-1}^n) - \phi(U_{i-2}^n))) .$$

Fix i and define $V_j = U_{i-1}^0 + (j - i + 1) Lh$ for all j. Equation (4.14) implies that $V_i \geqslant U_i^0$ and $V_{i+1} - V_i \geqslant U_{i+1}^0 - U_i^0$; therefore, because ϕ is convex,

$$\phi(V_{i+1}) - \phi(V_i) \geqslant \phi(U_{i+1}^0) - \phi(U_i^0) .$$

Because the mapping $U^n \to U^{n+1}$ is order preserving, substituting V_i for the first and third occurrences of U_i^0 in (4.20) will only increase the value of the right hand side of (4.20). Finally, because ϕ is increasing and $0 \leqslant V_{i-2} \leqslant U_{i-2}^0$ (from (4.15)), making the substitution V_{i-2} for U_{i-2}^0 again increases the right hand side of (4.20). Therefore,

$$(4.21) \quad U_i^1 - U_{i-1}^1 \leqslant V_i - V_{i-1} + \frac{\Delta t}{h^2} ((\phi(V_{i+1}) - \phi(V_i)) -$$

$$- 2(\phi(V_i) - \phi(V_{i-1})) + (\phi(V_{i-1}) - \phi(V_{i-2}))) .$$

Because the numbers V_j are evenly spaced, the quantity in parentheses in (4.21) equals $(Lh)^3 \phi'''(\xi)$ (for some $\xi \in [V_{i-2}, V_{i+1}]$), which is not positive because $1 < m \leqslant 2$. Therefore, $U_i^1 - U_{i-1}^1 \leqslant V_i - V_{i-1} = Lh$.

As for (4.19), we will show that $(U_{i+1}^{n+1})^{m-1} - (U_i^{n+1})^{m-1} \leqslant Lh$. Obviously we can assume that $U_{i+1}^{n+1} > U_i^{n+1}$. We first reduce the inequality to the special data $V_{i-1} = (y - \Delta)^{1/(m-1)}$, $V_i = y^{1/(m-1)}$, $V_{i+1} = (y + \Delta)^{1/(m-1)}$ and $V_{i+2} = (y + 2\Delta)^{1/(m-1)}$, where $\Delta = Lh$ and $U_i = y^{1/(m-1)}$. We are required to show that

$$(4.22) \quad \Delta \geqslant (U_{i+1} + \lambda(((U_{i+2})^m - (U_{i+1})^m) - (U_{i+1})^m + (U_i)^m))^{m-1} -$$

$$- (U_i + \lambda((U_{i+1})^m - 2(U_i)^m + (U_{i-1})^m))^{m-1} ,$$

where $\lambda = (\Delta t)/h^2$ and $\{U_k\}$ stands for $\{U_k^n\}$. Just like the argument for $1 < m \leqslant 2$, $V_{i+1} \geqslant U_{i+1}$ and $(V_{i+2})^m - (V_{i+1})^m \geqslant (U_{i+2})^m - (U_{i+1})^m$,

so making the latter replacement increases the right hand side of (4.22). Similarly, (4.22) increases when V_{i-1} replaces U_{i-1}. Finally, the derivative of the right hand side of (4.22) with respect to the remaining instances of U_{i+1}, is equal to

$$(m-1)(1 - \lambda m (U_{i+1})^{m-1})(U_{i+1} + \lambda (((U_{i+2})^m - (U_{i+1})^m)$$
$$- (U_{i+1})^m + (U_i)^m))^{m-2} - (m-1)\lambda m (U_{i+1})^{m-1}$$
$$\times (U_i + \lambda ((U_{i+1})^m - 2(U_i)^m + (U_{i-1})^m))^{m-2},$$

which is positive because $U_{i+1}^{n+1} > U_i^{n+1}$ and because of (4.5). Therefore, replacing the remaining instances of U_{i+1} with V_{i+1} increases the value of the right hand side of (4.22).

Thus, we are required to show that $F(y + \Delta) - F(y) \leq \Delta$, where

$$(4.23) \quad F(y) =$$
$$(y^{1/(m-1)} + \lambda ((y + \Delta)^{m/(m-1)} - 2 y^{m/(m-1)} + (y - \Delta)^{m/(m-1)}))^{m-1}.$$

It is therefore sufficient to show that $F'(y) \leq 1$ for $y \geq \Delta$. Taylor's Theorem shows that for any analytic function f,

$$f(y + \Delta) - 2 f(y) + f(y - \Delta) = 2 \sum_{k=1}^{\infty} \frac{f^{(2k)}(y)}{(2k)!} \Delta^{2k};$$

when $f(y) = y^{m/(m-1)} = y^{\alpha}$,

$$f^{(2k)}(y) = \alpha(\alpha - 1) \dots (\alpha - (2k - 1)) y^{\frac{1}{m-1} - (2k-1)}.$$

We can now rewrite (4.23) slightly more favorably; if we define the *positive* coefficients $b_{m,k} = 2 \lambda \Delta^{2k} \alpha \dots (\alpha - (2k - 1))/(2k)!$, then

$$F(y) = \left(y^{1/(m-1)} \left(1 + \sum_{k=1}^{\infty} b_{m,k} \left(\frac{1}{y} \right)^{2k-1} \right) \right)^{m-1}$$
$$= y \left(1 + \sum_{k=1}^{\infty} b_{m,k} y^{1-2k} \right)^{m-1},$$

and the series converges for $y \geq \Delta$. A calculation shows that

$$F'(y) = \left(1 + \sum_{k=1}^{\infty} b_{m,k} y^{1-2k} \right)^{m-2}$$
$$\times \left[1 + (m-1) \sum_{k=1}^{\infty} b_{m,k} \left(\frac{1}{m-1} + 1 - 2k \right) y^{1-2k} \right]$$
$$\leq \left(1 + \sum_{k=1}^{\infty} b_{m,k} y^{1-2k} \right)^{m-2} \left(1 - (m-2) \sum_{k=1}^{\infty} b_{m,k} y^{1-2k} \right).$$

Because $m > 2$, calculus shows that the function $(1 + a)^{m-2}(1 - (m - 2)a)$ takes its maximum value of 1 in the interval $[0, 1/(m - 2)]$ when $a = 0$; therefore the above bound for $F'(y)$ tends to its maximum value of 1 as y approaches infinity. This proves (4.19). The final inequality follows from the argument in the first paragraph of this proof. ∎

The following example shows that some condition like $U_i^0 \geqslant Lh$ is necessary for the above theorem to be true. Let $1 < m < 2$ and

$$U_i^0 = \begin{cases} 0 & \text{for} \quad i \leqslant 0, \\ h & \text{for} \quad i = 1, \\ 2h & \text{for} \quad i = 2, \end{cases}$$

with U_i^0 defined for $i > 2$ so that U^0 has Lipschitz constant 1 and otherwise satisfies the conditions of the lemma. Then

$$U_1^1 - U_0^1 = h - 0 + \frac{\Delta t}{h^2} ((2h)^m - 3h^m + 3 \cdot 0 - 0)$$

$$= h + \frac{\Delta t}{h^2} h^m (2^m - 3)$$

which is bigger than h if $2^m > 3$. It's not clear what the precise condition should be to guarantee that U^n is Lipschitz for $m \leqslant 2$. (Perhaps no condition is necessary if we allow controlled growth in time of the Lipschitz constant.)

We can summarize the results for our scheme as follows.

Case 1 : General m, error bound for u : Assume that u_0 has bounded variation and take $U_k^0 = u_0(kh)$. Because $(u_0^m)_{xx}$ is assumed to be a measure, Lemma 4.3 implies that both u and u^h are Hölder-$1/m$. In addition, the integral in (2.3) is bounded by

$$\int |u^0 - u(x, 0)| \, |w(x, 0)| \, dx \leqslant \|w(., 0)\|_{L^\infty(\mathbb{R})} \|\Delta u(., 0)\|_{L^1(\mathbb{R})}$$

$$\leqslant \frac{Ch}{\varepsilon}$$

Theorems 2.1 and 4.2 imply that

$$\|u - u^h\| \leqslant C \left[\frac{h}{\varepsilon} + \frac{h}{\varepsilon^2} + \varepsilon^{1/m} \right]$$

$$= Ch^{1/(2m+1)}$$

by taking $\varepsilon = h^{m/(2m+1)}$. Without further information, we have no error bounds for the interface because Theorem 3.3 assumes that $(u_0^{m-1})_{xx} + C_2$ is a positive distribution, hence a measure. At any rate, $\beta = 1/(2m + 1)$ in (3.1).

Case 2 : $m \in (1, 2]$, u_0 is Lipschitz, error bound for u : Assume $1 \leqslant m \leqslant 2$ and u_0 is Lipschitz continuous with Lipschitz constant L. Let $U_k^0 = \max (u_0(kh), Lh)$. Lemma 4.3 implies that $u^h(., t^n)$ is Lipschitz continuous for all positive n with the same Lipschitz constant. This choice of $\{U_k^0\}$ also satisfies (4.2) through (4.4). Because $\|w_t(., t)\|_{L^1(\mathbb{R})} \leqslant C/\varepsilon^2$, the integral in (2.3) is bounded by

$$\int |u^0 - u(x, 0)| |w(x, 0)| \, dx \leqslant \|w(., 0)\|_{L^1(\mathbb{R})} \|\Delta u(., 0)\|_{L^\infty(\mathbb{R})}$$

$$\leqslant \frac{Ch}{\varepsilon^2}.$$

Theorems 2.1 and 4.2 imply that

$$\|u - u^h\| \leqslant C \left[\frac{h}{\varepsilon^2} + \varepsilon \right]$$

$$= Ch^{1/3}$$

by taking $\varepsilon = h^{1/3}$.

Case 3 : $m \in (1, 2]$, u_0 Lipschitz, error bound for $z(t)$: Assume $(u_0^{m-1})_{xx} \geqslant -C$. The approximation u^h satisfies (3.1) with $\beta = 1/3$. Assume that z^0 is chosen to satisfy Theorem 3.3. Because $\Delta t^{1/4} = Ch^{1/2} \leqslant Ch^{(m-1)/6}$ for $m \in (1, 2]$, it follows that

$$|z^n - z(t^n)| \leqslant C [|z(0) - z^0| + h^{(m-1)/6}].$$

If, in addition, $|z^0 - z(0)| \leqslant Ch^{(m-1)/6}$, and $\dot{z}(t) \geqslant C > 0$ for $t \in (t^{n-1}, t^n)$, then

$$|z^n - z(t^n)| \leqslant Ch^{(m-1)/3}.$$

Case 4 : $m > 2$, (u_0^{m-1}) Lipschitz, error bounds for u and $z(t)$: Consider now when $m > 2$ and u_0^{m-1} is Lipschitz continuous with Lipschitz constant L. Lemma 4.3 shows that then $(U^n)^{m-1}$ is Lipschitz continuous with the same Lipschitz constant for all n. We now take advantage of the special form of $z^{\varepsilon\delta}$ to bound the integral in (2.3).

In (2.5) we can write

$$\phi[u, u^h] = \phi[u, u] + (u - u^h) \phi[u, u, u^h]$$

$$= mu^{m-1} + O(\|\Delta u\|_\infty) \phi''(\xi)/2 \quad \text{for some } \xi,$$

$$= mv + O(\|\Delta u\|_\infty), \quad \text{because } m > 2.$$

Thus, if we set $v_\delta = J_\delta * v$, $z^{\varepsilon\delta}$ satisfies

$$z_t = mv_\delta z_{xx} + \delta z_{xx} + O(\|\Delta u\|_\infty) z_{xx}.$$

Because (a) $z^{\varepsilon\delta}$ is nonnegative, (b) δ tends to zero while $z_{xx}^{\varepsilon\delta}$ is bounded in $L^1(\mathbb{R})$ independently of δ by (2.7), and (c) v_δ is uniformly Lipschitz continuous, one sees that

$$\|z^{\varepsilon\delta}(\cdot, T)\|_1 - \|z^{\varepsilon\delta}(\cdot, 0)\|_1$$
$$= -m \iint (v_\delta)_x z_x^{\varepsilon\delta} \, dx \, dt + O\left(\|\Delta u\|_\infty\right) \iint |z_{xx}^{\varepsilon\delta}| \, dx \, dt \, .$$

By the known bounds (2.6)-(2.7) for $z^{\varepsilon\delta}$, we conclude that

$$\|z^{\varepsilon\delta}(\cdot, 0)\|_1 \leq C\left[\frac{1}{\varepsilon} + \frac{\|\Delta u\|_\infty}{\varepsilon^2}\right].$$

Therefore, the integral in (2.3) is bounded by

$$\left|\int (u^0 - u(x, 0)) z^{\varepsilon\delta}(x, 0) \, dx\right| \leq \|z^{\varepsilon\delta}(\cdot, 0)\|_{L^1(\mathbb{R})} \|\Delta u(\cdot, 0)\|_{L^\infty(\mathbb{R})}$$

$$\leq C h^{1/(m-1)}\left[\frac{1}{\varepsilon} + \frac{\|\Delta u\|_\infty}{\varepsilon^2}\right].$$

Theorems 2.1 and 4.2 imply that

$$\|u - u^h\|_\infty \leq C h^{1/(m-1)}\left[\frac{1}{\varepsilon} + \frac{\|\Delta u\|_\infty}{\varepsilon^2}\right] + C\frac{h}{\varepsilon^2} + C\varepsilon^{1/(m-1)}.$$

Let's ignore the constants for a moment to consider the right hand side of this inequality. To hide the term $\|\Delta u\|_\infty$ on the left hand side, we require that $C h^{1/(m-1)}/\varepsilon^2 < 1$, or

(4.24) $\varepsilon > C h^{1/(2m-2)}.$

Balancing the sizes of the first and third remaining terms requires that $\varepsilon = h^{1/m}$, which violates (4.24). Balancing the second and third remaining terms gives $\varepsilon = h^{(m-1)/(2m-1)}$, which does satisfy (4.24) when $m \leq (3 + \sqrt{3})/2 \approx 2.366$. This value of ε gives an error bound of $O(h^{1/(2m-1)})$. It is easily seen that for m in this regime, the first term is smaller than the other two terms, so the bound holds for all three terms.

By bounding the integral in (2.3) with $h^{1/(m-1)}/\varepsilon^2$ (as in Case 2), setting $\varepsilon = h^{1/(2m-1)}$ yields the error bound $h^{1/[(2m-1)(m-1)]}$, which is smaller than $h^{1/(2m+1)}$ for $m < 5/2$. For other values of m, the error bound in Case 1 is still the best possible. Bounds for the interface error can now be determined in the usual way if $(u_0^{m-1}(x))_{xx} \geq -C > -\infty$.

These tricks give an error bound of $h^{1/3}$ when $m = 2$; we doubt that this is sharp. We believe that better bounds for the error could be achieved by using more precise estimates for the functions $z^{\varepsilon\delta}$.

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