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G. BARLES
B. PERTHAME

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DISCONTINUOUS SOLUTIONS OF DETERMINISTIC OPTIMAL STOPPING TIME PROBLEMS (*)

by G. BARLES (*) and B. PERTHAME (**)

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Abstract. — *We study optimal stopping time control problems with a (possibly) discontinuous stopping cost* ψ. *When* ψ *is upper semi-continuous, we show that the lower semi-continuous (l.s.c. in short) envellope of the value function is the unique l.s.c. viscosity solution of the associated variational inequality. We extend some technics, used in these problems, to prove stability results for general Hamilton-Jacobi Equations and to treat some particular exit time problems.*

Résumé. — *Nous étudions des problèmes de temps d'arrêt avec un coût d'arrêt* ψ *qui peut être discontinu. Quand* ψ *est semi-continu supérieurement, nous montrons que l'enveloppe semi-continue inférieure (s.c.i. en abrégé) de la fonction valeur est l'unique solution de viscosité s.c.i. de l'inéquation variationnelle associée. Nous étendons certaines techniques utilisées dans ces problèmes pour prouver des résultats de stabilité pour des équations générales de Hamilton-Jacobi et pour traiter certains problèmes de temps de sortie.*

INTRODUCTION

In this work, we are interested in the properties of possibly discontinuous optimal cost functions of deterministic control problem as viscosity solutions of first-order Hamilton-Jacobi (*H-J* in short) equations. In particular, we study optimal stopping time problems with discontinuous stopping cost. In order to be more specific, let us briefly describe the control problem : we consider a system which state is given by the solution y_x of

$$dy_x(s) + b(y_x(s), v(s))\, ds = 0\,, \quad y_x(0) = x \in \mathbb{R}^N \tag{1}$$

(*) Received in October 1986.

(*) Ceremade, Université Parix IX-Dauphine, place de Lattre de Tassigny, 75775 Paris Cedex 16.

(**) CMA, Ecole Normale Supérieure, 45, rue d'Ulm, 75230 Paris Cedex 05.

and the cost function is defined, for instance, by

$$J(x, v, \theta, \psi) = \int_0^\theta f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \psi(y_x(\theta))\, e^{-\lambda\theta}\,, \tag{2}$$

and the optimal cost function (the value function) is

$$u[\psi](x) = \operatorname{Inf}\ \{J(x, v, \theta, \psi), \theta \geqslant 0, v(\,.\,) \in L^\infty(\mathbb{R}^+\ ; V)\} \tag{3}$$

where b and f are given functions, λ is a fixed positive constant, θ is a non-negative number (the stopping time). $v(\,.\,) \in L^\infty(\mathbb{R}^+\ ; V)$ is the control, V being a compact metric space. The assumptions are detailed in the first part. The main point is that we assume only that ψ is a bounded function defined pointwise ; in particular, it may present discontinuities.

If ψ is bounded uniformly continuous, it is well-known that $u[\psi]$ is the unique uniformly continuous viscosity solution of

$$\operatorname{Max}\ \{H(x, u, Du)\ ; u - \psi\} = 0 \quad \text{in} \quad \mathbb{R}^N\,, \tag{VI}$$

where

$$H(x, t, p) = \operatorname{Sup}\ \{b(x, v)\,.\,p + \lambda t - f(x, v)\ ; v \in V\}\ . \tag{4}$$

Our aim is to obtain such a characterization. In fact, we prove a less precise but optimal result. Before detailing our results, let us explain our program :

(i) Since ψ is discontinuous, we must extend the notion of viscosity solution to (VI). Let us recall that this notion was introduced for continuous Hamiltonians by M. G. Crandall and P. L. Lions [5] (See also [3, 9, 11]). Let us point out that this definition is based, in the case of discontinuous solutions, on the lower semi-continuous (l.s.c. in short) and the upper semi-continuous (u.s.c. in short) envellope of the solution. In all the following, u_* (resp. u^*) will be the l.s.c. (resp. u.s.c.) envellope of u.

(ii) We have to show that $u[\psi]$ is viscosity solution of (VI). An additional difficulty to the uniformly continuous case is that the definition of viscosity solution deals with $u_*[\psi]$ and $u^*[\psi]$ and not with $u[\psi]$. So, we have either to identify $u^*[\psi]$ and $u_*[\psi]$ or to show that they satisfy respectively sub and superoptimality principles of dynamic programming (*cf.* P. L. Lions and P. E. Souganidis [13]).

(iii) We have to look at « uniqueness » (or characterization) property for the viscosity solution $u[\psi]$. In general, the discontinuous viscosity solution is not unique and we can prove the existence of a maximum and of a minimum solution. One reason is that the *H-J* equation is the same for the control problem (1)-(3) and for the relaxed problem (see Section I). But, the discontinuous optimal cost can be different for these two problems. Nethertheless, we can characterize, in some case, $u_*[\psi]$ as the unique l.s.c. solution of (VI).

Now, we detail our results. In the first part, we study the control problem. Essentially, we try to identify the l.s.c. and u.s.c. envelope of $u[\psi]$. This is possible for $u_*[\psi]$, since we prove that $u_*[\psi] = \hat{u}[\psi_*]$ (the value function for the relaxed stopping time problem). For $u^*[\psi]$, this is not possible since, in general, $u^*[\psi] < u[\psi^*]$. But, we show that if ψ satisfies

$$(\psi^*)_* = \psi_* \tag{5}$$

then

$$(u^*[\psi])_* = u_*[\psi]\,. \tag{6}$$

This « regularity » result will be used, in an essential way, to prove the uniqueness result for the l.s.c. solution. This section contains many counterexamples showing that our results are optimal.

The second section is devoted to study the properties of $u[\psi]$ as viscosity solution of (VI). First, we recall the notion of discontinuous viscosity solution for (VI). Then, the equality $u_*[\psi] = \hat{u}[\psi_*]$ allows us to adapt the standard methods used in the continuous case to prove that $u[\psi]$ is a viscosity solution of (VI). This property and the structure of the problem enables us to characterize $u_*[\psi]$ but, in general, not $u^*[\psi]$; more precisely, we prove that $u_*[\psi]$ is the minimum viscosity supersolution (and solution) and $u[\psi^*]$ is the maximum viscosity subsolution (and solution) of (VI). When ψ satisfies (5), we show that $u_*[\psi]$ is the unique l.s.c. solution of (VI) and so, we have an entirely satisfying result in this « regular » case.

In the third part, we examine some exit time problems which can be interpreted by considering an associated stopping time problem in $\mathbb{R}^N$. For that type of problem, defined by a technical assumption, we show that the value function of the relaxed control problem associated to the exit time of Ω is the unique l.s.c. viscosity solution of an Hamilton-Jacobi problem in Ω, with mixed boundary conditions. An application of such a result is, for example, the minimum exit time from Ω.

In the appendix, we extend the stability results for viscosity solutions obtained by M. G. Crandall and P. L. Lions [5] and H. Ishii [9]. Essentially, we show stability results for *any* sequence of viscosity sub and supersolutions and of Hamiltonians. Nethertheless, ou arguments are purely finite dimensional.

I. THE STOPPING TIME PROBLEM WITH DISCONTINUOUS STOPPING COST

In this Section, we investigate some general properties of discontinuous value functions for the stopping time problem with discontinuous stopping cost. Indeed, it seems to give a rather general situation ; for example, it covers some exit time problems (see Section III).

Denoting by u the value function we try to identify its lower semi-continuous (l.s.c.) version u_* and its upper semi-continuous (u.s.c.) version u^* defined by

$$\begin{cases} u_*(x) = \liminf\limits_{y \to x} u(y) , \\ u^*(x) = \limsup\limits_{y \to x} u(y) . \end{cases} \tag{I.1}$$

Our main motivation is that these functions play a special role when we want to use weak solutions of the Hamilton-Jacobi equation by extending Crandall-Lions [5] or Ishii [8, 9] definition of viscosity solutions (see Section II). Thus, we begin by giving some general inequalities between these functions (Subsection 1) and we show that they cannot be improved in general by indicating some counterexamples (Subsection 2). We also give further regularity results using relaxed controls. Finally, we recall the dynamic programming principle which will be used later on.

I.1. General inequalities

In this subsection we consider the stopping time problem in $\mathbb{R}^N$. Namely, we take functions $b(x, v)$ for $x \in \mathbb{R}^N$, $v \in V$. Here V is a compact set (the set of controls) and we assume that, for some constant C

$$\begin{cases} b \text{ and } f \text{ are continuous on } \mathbb{R}^N \times V ; \\ |b(x, v)| \leq C , \ |f(x, v)| \leq C , \quad \forall x \in \mathbb{R}^N , \quad \forall v \in V ; \\ |b(x, v) - b(y, v)| \leq C\,|x - y| , \quad \forall x, y \in \mathbb{R}^N , \quad \forall v \in V . \end{cases} \tag{I.2}$$

With this assumption, we may solve for any measurable $v(\,.\,) \in L^\infty(\mathbb{R}^+ ; V)$ the differential equation

$$dy_x(s) + b(y_x(s), v(s))\, ds = 0 , \quad y_x(0) = x , \tag{I.3}$$

and we define the cost function

$$J(x, v, \theta, \psi) = \int_0^\theta f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \psi(y_x(\theta))\, e^{-\lambda\theta} , \tag{I.4}$$

where λ is a fixed positive constant, ψ is a bounded function defined pointwise, and θ is a non-negative number (the stopping time). The optimal cost function of the system (I.3)-(I.4) is defined as $u[\psi]$ by

$$u[\psi](x) = \text{Inf } \{J(x, v, \theta, \psi) ; \theta \geq 0, v(\,.\,) \in L^{\infty}(\mathbb{R}^+ ; V)\} . \quad (\text{I.5})$$

PROPOSITION I.1 : *Under asumption (I.2) and if ψ is a bounded function defined pointwise, the following equalities hold :*

$$u_*[\psi] \leq u[\psi_*] \leq u[\psi] \leq u^*[\psi] \leq u[\psi^*] .$$

Remark : The couterexamples below show that all these inequalities may be strict.

Proof: To prove the first inequality, we remark that

$$u_*[\psi](x) \leq J_*(x, v, \theta, \psi) = J(x, v, \theta, \psi_*) , \quad \forall \theta, v(\,.\,) ,$$

this equality is proved in Lemma I.2 below ; therefore

$$u_*[\psi](x) \leq \text{Inf } \{J(x, v, \theta, \psi_*) , \quad \theta \geq 0 , \quad v(\,.\,) \in L^{\infty}(\mathbb{R}^+ ; V)\} = \\ = u[\psi_*](x) .$$

The second is clear since $\psi_* \leq \psi$. The third inequality is clear enough. To prove the last inequality, we notice that $u[\psi] \leq u[\psi^*]$ and thus, it is enough to show that $u[\psi^*]$ is u.s.c.. But we have

$$u[\psi^*] = \text{Inf } \{J(x, v, \theta, \psi^*) , \quad \theta \geq 0 , \quad v(\,.\,) \in L^{\infty}(\mathbb{R}^+ ; V)\} ,$$

and, since $J(x, v, \theta, \psi^*)$ is u.s.c., $u[\psi^*]$ is also u.s.c., proving the last inequality and Proposition I.1.

Remark : It follows from Proposition I.1 that we have $u_*[\psi] = u_*[\psi_*]$. We now state the Lemma I.2 that we have used in the proof above.

LEMMA I.2 : *We have*

$$J_*(x, v, \theta, \psi) = J(x, v, \theta, \psi_*) .$$

Proof: First, we notice that $J(x, v, \theta, \psi_*)$ is l.s.c. since

$$J(x, v, \theta, \psi_*) = \int_0^{\theta} f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \psi_*(y_x(\theta))\, e^{-\lambda\theta} ,$$

and $x \to y_x(\theta)$ is continuous. Thus, we have

$$J_*(x, v, \theta, \psi) \geq J(x, v, \theta, \psi_*) .$$

To prove the other inequality, we choose a sequence y_n such that $y_n \to y_x(\theta)$, $\psi(y_n) \to \psi_*(y(\theta))$ as n tends to infinity. Then, we may solve the O.D.E.

$$dy_n(s) + b(y_n(s), v(s))\, ds = 0\,, \quad y_n(\theta) = y_n\,.$$

Denoting $x_n = y_n(0)$, we have $x_n \to x$ as n tends to infinity. Therefore

$$\lim_{n \to \infty} J(x_n, v, \theta, \psi) = J(x, v, \theta, \psi_*)\,,$$

and the second inequality is proved, concluding the proof of Lemma I.2.

However, there is a general situation which allows to characterize $(u[\psi])_*$ as the value function of a stopping time problem. This is the case of relaxed controls that we briefly recall.

We introduce the set $P(V)$ of probability measures on V and we identify any control $w \in V$ with the Dirac mass δ_w. With this identification, we may write

$$f(x, w) = \int_V f(x, v)\, \delta_w(v)\,,$$

for any f continuous in v. This theory is develloped more precisely and used in [1, 10, 15, 16]. Here, let us only point out the following result. We set, for an obstacle ψ as before

$$\hat{u}[\psi](x) = \operatorname{Inf} \left\{ \int_0^\theta \int_V f(y_x(s), v)\, d\mu_s\, e^{-\lambda s}\, ds + \right.$$
$$\left. + \psi(y_x(\theta))\, e^{-\lambda\theta}\,;\ \mu_s \in P(V)\,;\ \theta \geq 0 \right\}\,,$$

where $y_x(\,.\,)$ is the relaxed trajectory given by

$$dy_x(s) + \left(\int_V b(y_x(s), v)\, d\mu_s \right) ds = 0, y_x(0) = x\,.$$

Then, we have the :

PROPOSITION I.3 : *With the assumptions of Proposition I.1 we have*

$$u_*[\psi] = \hat{u}[\psi_*]\,.$$

Proof: First, let us recall that, for uniformly continuous obstacle ψ, we have $\hat{u}[\psi] = u[\psi]$ (one nonstandard proof consisting in remarking that both are viscosity solutions of the same Hamilton-Jacobi Equation and thus, coincide by uniqueness results). Now, let us choose a sequence ψ_n of uniformly continuous obstacles such that ψ_n is nondecreasing and $\sup_n \psi_n = \psi_*$. Then, we can prove (see Section II) that $u[\psi_n]$ is nondecreasing and converges to $u_*[\psi]$; in the same way, $\hat{u}[\psi_n]$ converges to $\hat{u}_*[\psi]$ and thus, it remains to prove that $\hat{u}[\psi]$ is l.s.c., or in an equivalent way, that $\hat{u}[\psi_*]$ is l.s.c. To do so, we adapt classical arguments relying on the compactness of relaxed controls. Thus, let $x \in \mathbb{R}^N$ and x_n converging to x. For any n, we may choose $\varepsilon_n \to 0$, a stopping time $\theta(n) \to \theta$ as $n \to \infty$ and an optimal control $\mu^n(s)$, $\mu^n \to \mu$ weakly in $L^\infty(\mathbb{R}^+ ; P(V))$ (at least, this holds for a subsequence), such that

$$\hat{u}[\psi](x_n) = \varepsilon_n + \int_0^{\theta(n)} \int_V f(y_n(s), v)\, d\mu^n\, e^{-\lambda s}\, ds + \psi_*(y_n(\theta(n)))\, e^{-\lambda\theta(n)},$$

$$dy_n(s) + \left(\int_V b(y_n(s), v)\, d\mu^n \right) ds = 0 , \quad y_n(0) = x_n .$$

Thus, one can check that $y_n \to y$ when $n \to \infty$, uniformly on compacts sets ; hence, we have

$$dy(s) + \left(\int_V b(y(s), v)\, d\mu_s \right) ds = 0 , \quad y(0) = x ,$$

$$\liminf \hat{u}[\psi](x_n) = \int_0^{\theta} \int_V f(y(s), v)\, d\mu_s\, e^{-\lambda s}\, ds + \\ + \liminf \psi_*(y(\theta(n)))\, e^{-\lambda\theta} .$$

But, $\psi_*(y(\theta)) \leq \liminf \psi_*(y(\theta(n)))$, (if $\theta < \infty$; if not the last term is zero and we conclude). Therefore, we have proved that

$$\liminf \hat{u}[\psi](x_n) \geq J(x, \theta, \mu, \psi_*) \geq \hat{u}[\psi](x) ,$$

for some θ, μ and we have proved Proposition I.3.

Now, our purpose is to study the links between $u[\psi^*]$, $u^*[\psi]$ and $u_*[\psi]$, the question being : can we recover one from the other ? The answer is given by the

PROPOSITION I.4 : *With the above notations and assumptions, if ψ satisfies*

$$(\psi^*)_* = \psi_* , \tag{I.6}$$

then, we have

$$u_*[\psi^*] = (u^*[\psi])_* = u_*[\psi] . \tag{I.7}$$

Remark : The result (I.7) is false, in general, if we do not assume (I.6). It is also false that $u^*[\psi_*] = u^*[\psi]$, even if $\psi^* = (\psi_*)^*$. See Subsection 2.

Proof: From Proposition I.1, we have $u_*[\psi] \leqslant u[\psi_*]$; therefore

$$u_*[\psi] \leqslant u_*[\psi^*] \leqslant u[(\psi^*)_*] .$$

Thus, using (I.6), we get

$$u_*[\psi] \leqslant u_*[\psi^*] \leqslant u[\psi_*] ,$$

and we are done since $u_*[\psi_*] = u_*[\psi]$.

I.2. Counteramples

We give now the counterexamples announced in Subsection 1.

Exemple 1 : This example is to show that the first inequality of Proposition I.1 may be strict. More precisely, we give an example where $u[\psi_*]$ is not l.s.c. To this end, we consider in $\mathbb{R}^3$, the field b given by

$$b(x, v) = \begin{bmatrix} 1 \\ v \\ (\alpha + v^2)^{-1} + |x_2| \end{bmatrix}$$

where $v \in V = [-1 ; 1]$, $x = (x_1, x_2, x_3)$ are the coordinates of $x \in \mathbb{R}^3$. Here, α is a positive number. Finally, we take $f = 0$ and

$$\psi(x) = \begin{cases} 0 \text{ if } x_1 < 1 \\ 0 \text{ if } x_3 > (1 + \alpha)^{-1} \\ -1 \text{ in the other cases .} \end{cases}$$

We take the discount factor $\lambda = 0$ but this is not relevant for the example as we will see. Let us remark that ψ is l.s.c. and let us prove the :

LEMMA I.5 : *$u[\psi]$ is not l.s.c. at the point $x = 0$.*

Proof: First, we prove that $u[\psi](0) = 1$. Indeed, $J(x, T, v, \psi)$ can be 0 or -1. To obtain the value -1, we must take a stopping time T such that $x_1(T) = T \geqslant 1$, then we have

$$x_3(T) = \int_0^T \{(\alpha + v^2)^{-1} + |x_2|\} \, ds > (\alpha + 1)^{-1} ,$$

and thus $\psi(x(T)) = 0$ for any $T \geqslant 1$ and $u[\psi](0) = 0$. Now, we consider the point $x_\varepsilon = (0, 0, -\varepsilon)$, and the stopping time $T = 1$ and the control

$v(s) = +/-1$ such that $|x_2| \leq \varepsilon$, for all $s \in [0, 1]$, (this can be achieved very easily). Then, we have

$$x_1(1) = 1$$

$$x_3(1) = -\varepsilon + \int_0^1 \{(\alpha+1)^{-1} + |x_2|\} \, ds \leq (1+\alpha)^{-1}$$

and so, $u[\psi](x_\varepsilon) = -1$ and Lemma I.5 is proved. Let us finally remark that, if the discount factor is not zero, we have, by the same proof, $u[\psi](0) = 0$, $u[\psi](x_\varepsilon) \leq -e^{-\lambda}$ and we conclude in the same way.

Remark : A simple modification of this example proves that the value function of an *exit time problem* is not always l.s.c. In our example, we were interested in the problem of exit from

$$\Omega = \{\psi > -1\} .$$

Exemple 2 : Our second example concerns the last inequality of Proposition I.1. We claim that it is strict, in general, and that $u_*[\psi^*] \neq u[\psi_*]$, $u^*[\psi] \neq u[\psi^*]$ in general. We take for $x \in \mathbb{R}$, $b(x, v) = 1$, $f = 0$ and $\psi(x) = 0$ if $x \neq 0$, -1 if x. Thus, $u[\psi^*] = 0$, $u[\psi] = -1$ if $x \leq 0$, $u[\psi] = 0$ if $x > 0$ and our requirements are proved (again, if $\lambda > 0$, we have $u[\psi^*] = 0$ and $u[\psi](x) = -e^{\lambda x}$ if $X < 0$ and we conclude in the same way).

Exemple 3 : Our last example is to show a case when $(\psi_*)^* = \psi^*$, $(\psi^*)_* = \psi_*$ but $u^*[\psi] \neq (u_*[\psi])^*$. For $x \in \mathbb{R}^2$, we choose $b(x) = (1, 0)$, in the natural coordinates of $\mathbb{R}^2$, $f = 0$. Then, for $x = (x_1, x_2)$, we take $\psi(x) = -1$ if $x_1 \cdot x_2 > 0$, 0 otherwise. Again, we choose $\lambda = 0$ for the sake of simplicity and we have

$$u[\psi_*](x) = -1 \quad \text{if} \quad x_1 < 0 \quad \text{and} \quad x_2 \neq 0 .$$

But

$$u[\psi](x) = 0 \quad \text{if} \quad x_1 < 0 \quad \text{and} \quad x_2 = 0 .$$

Thus, $u^*[\psi](x) \neq (u_*[\psi])^*(x)$ if $x_1 < 0$ and $x_2 = 0$ and our claim is proved. (if $\lambda > 0$, we argue as in the example 2).

I.3. Dynamic programming principle

This subsection is to recall a general result in control theory which is the Dynamic Programming Principle. Its validity for continuous obstacle is wellknown (see [7, 10, 11, 13, 16]), let us only state it for discontinuous obstacle and refer the interested reader to the above reference for a proof.

We just remark that the continuity is no used in the proofs given in these references.

PROPOSITION I.6 : *Let b, f, λ satisfy the conditions of Subsect. 1 and let* ψ *be a bounded function defined pointwise. Then, for any* $T > 0$, *we have*

$$u[\psi](x) = \operatorname{Inf}\left\{\int_0^\theta f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \psi(y_x(\theta))\, e^{-\lambda\theta}\, 1_{(\theta < T)} + \right.$$
$$\left. + u[\psi](y_x(T))\, e^{-\lambda T}\, 1_{(\theta \geq T)}\, ; \theta \geq 0, v(\,.\,) \in L^\infty(\mathbb{R}^+\, ; V)\right\} .$$

This formula will be used later to prove the relation between the value functions $u[\psi]$ and the Hamilton-Jacobi equation.

II. CHARACTERIZATION OF THE COST FUNCTION BY HAMILTON-JACOBI EQUATION

The goal of this section is to show that the optimal cost function of the optimal stopping time problem described in the first Section is the unique viscosity solution of the variational inequality

$$\operatorname{Max}\,(H(x, u, Du)\, ; u - \psi) = 0 \quad \text{in} \quad \mathbb{R}^N\,, \tag{VI}$$

where

$$H(x, t, p) = \operatorname{Sup}\,\{b(x, v)\,.\,p + \lambda t - f(x, v)\, ; v \in V\}\ .$$

To do so, we have to extend to problems like (VI) the definition of viscosity solution introduced for continuous Hamiltonians in [5] and for some discontinuous Hamiltonians in [8, 12]. Then, we can prove the stability of the viscosity solution of (VI). The uniqueness of the discontinuous viscosity solution of (VI) only holds for obstacles satisfying the « regularity » property $(\psi^*)_* = \psi_*$ and its proof relies on the interpretation of the solution as the cost function for an optimal control problem.

II.1. The viscosity formulation of the dynamic programming principle

In this subsection we define the notion of viscosity solution of (VI), then we prove, via the dynamic programming, that the function u defined by (I.5) is a viscosity solution of (VI).

DEFINITION II.1 : *Let u be a locally bounded function u is said to be a viscosity subsolution of* (VI) *(resp. supersolution) if*

$$\begin{cases} \forall \phi \in C^1(\mathbb{R}^N), \text{ at each local maximum point } x_0 \text{ of } u^* - \phi, \text{ we have} \\ \qquad \operatorname{Max}\,(H(x_0, u^*(x_0), D\phi(x_0))\, ; u^*(x_0) - \psi^*(x_0)) \leq 0\ , \end{cases}$$

(*Resp.*

$$\begin{cases} \forall \phi \in C^1(\mathbb{R}^N), \textit{ at each local minimum point } x_0 \textit{ of } u_* - \phi, \textit{ we have} \\ \qquad \operatorname{Max} \left(H(x_0, u_*(x_0), D\phi(x_0))\, ;\, u_*(x_0) - \psi_*(x_0)\right) \geq 0) , \end{cases}$$

u is a viscosity solution of (VI) *if u is both a subsolution and a supersolution.*

Remark : When ψ is continuous, the definition of viscosity solution for (VI) is included in the one for first order Hamilton-Jacobi equations by considering the continuous Hamiltonian

$$H'(x, t, p) = \operatorname{Max} \left(H(x, t, p), t - \psi(x)\right) .$$

For discontinuous Hamiltonians, see [8, 12] and the Appendix.

THEOREM II.2 : *Let u be defined by (I.5). Under the assumptions of Propostion I.1, u is a viscosity solution of* (VI).

Proof: The proof is inspired by the corresponding one in [11]. First, we prove that u is a viscosity subsolution of (VI). Let $\phi \in C^1(\mathbb{R}^N)$ and x_0 be a local maximum point of $u^* - \phi$. Since $u \leq \psi$ in $\mathbb{R}^N$, it is clear that $u^*(x_0) \leq \psi^*(x_0)$ in $\mathbb{R}^N$. It remains to prove that

$$H(x_0, u^*(x_0), D\phi(x_0)) \leq 0 .$$

We fix a control such that $v(t) = v \in V$ and $\theta = \infty$. Then, for all $x \in \mathbb{R}^N$ and $T > 0$, we have by (I.8)

$$u(x) \leq \int_0^T f(y_x(t), v)\, e^{-\lambda t}\, dt + u(y_x(T))\, e^{-\lambda T} .$$

Hence

$$u^*(x) \leq \int_0^T f(y_x(t), v)\, e^{-\lambda t}\, dt + u^*(y_x(T))\, e^{-\lambda T} .$$

In particular, for $x = x_0$

$$u^*(x_0) \leq \int_0^T f(y_x(t), v)\, e^{-\lambda t}\, dt + u^*(y_x(T))\, e^{-\lambda T} .$$

It T is small enough, we have $u^*(y_x(T)) \leq \phi(y_x(T)) + (u^*(x_0) - \phi(x_0))$ because $|y_x(T) - x_0| \leq CT$. So, we obtain

$$u^*(x_0)(1 - e^{-\lambda T})/T \leq 1/T \int_0^T f(y_x(t), v)\, e^{-\lambda t}\, dt + \\ + e^{-\lambda T}[\phi(y_x(T)) - \phi(x_0)]$$

letting $T \to 0$, we get

$$\forall v \in V\ ,\quad b(x_0, v)\,.\,D\phi(x_0) + \lambda u^*(x_0) - f(x_0, v) \leq 0\ ,$$

which ends the first part of the proof. Now we prove that u is a viscosity supersolution of (VI). Let $\phi \in C^1(\mathbb{R}^N)$ and x_0 be a local minimum of $u_* - \phi$, then two cases are possible :

First case : $u_*(x_0) = \psi_*(x_0)$ and there is nothing to prove.

Second case : $u_*(x_0) < \psi_*(x_0)$. In this case we recall that $u_* = \hat{u}(\psi_*)$. Now, we claim that there exists $\varepsilon > 0$ such that

$$\hat{u}(\psi_*)(x_0) = \operatorname{Inf}\left\{\int_0^\theta \int_V f(y_x(t), v)\, e^{-\lambda t}\, d\mu_t(v)\, dt + \right.$$
$$\left. + \psi_*(y_x(\theta))\, e^{-\lambda\theta}\, ;\ \mu_t \in L^\infty(\mathbb{R}^+\ ; P(V)),\ \theta > \varepsilon\right\}\ .$$

If this claim is proved, we apply the Dynamic Programming Principle and for $T < \varepsilon$ deduce from (II.1) that

$$u_*(x_0) = \operatorname{Inf}\left\{\int_0^T \int_V f(y_x(t), v)\, e^{-\lambda t}\, d\mu_t(v)\, dt + \right.$$
$$\left. + u_*(y_x(T))\, e^{-\lambda T}\, ;\ \mu_t \in L^\infty(\mathbb{R}^+\ ; P(V))\right\}\ ,$$

and then we can conclude as in standard case. Now, let us prove (II.1). Let (θ^n, μ^n) be a minimizing sequence for $\hat{u}[\psi_*](x_0)$. If $\theta^n \to 0$

$$\int_0^{\theta^n} \int_V f(y_x(t), v)\, e^{-\lambda t}\, d\mu_t^n(v)\, dt \to 0, \quad \text{as} \quad n \to \infty\ ,$$

and since ψ_* is s.c.i.

$$\liminf_{n \to \infty} \psi_*(y_x^n(\theta^n))\, e^{-\lambda\theta^n} \geq \psi_*(x_0)\ .$$

Therefore

$$\hat{u}(\psi_*)(x_0) = \psi_*(x_0)\ ,$$

which contradicts the fact $u_*(x_0) < \psi_*(x_0)$. This proves (II.1) and Theorem II.2.

Remark : This proof is based on the suboptimality principle of Dynamic Programming (*cf.* [13]) satisfied by u^* and the fact that u_* is the value function of the relaxed control problem.

II.2. Uniqueness results for the variational inequality

By looking at the definition of viscosity solutions for (VI), it is clear that, in general, we are not able to find the solution $u[\psi]$ of the optimal stopping time problem from the variational inequality. The best we can do is to characterize $u_*[\psi]$ and $u^*[\psi]$ and this is the goal of this Section. But we begin by giving a stability result.

THEOREM II.3 : *Let* $(\psi_n)_{n\in\mathbb{N}}$ *(resp.* $(\varphi_n)_{n\in\mathbb{N}}$*) be a nonincreasing (resp. a nondecreasing) sequence of bounded uniformly continuous functions such that*

$$\operatorname{Inf}\,\{\psi_n\,;\,n\in\mathbb{N}\}=\psi^*\,,\quad (resp.\ \operatorname{Sup}\,\{\varphi_n\,;\,n\in\mathbb{N}\}=\psi_*)\,.$$

Then

$$u[\psi^*]=\operatorname{Inf}\,\{u[\psi_n]\,;\,n\in\mathbb{N}\}\,, \tag{II.2}$$

(resp.

$$\hat{u}[\psi_*]=\operatorname{Sup}\,\{u[\varphi_n]\,;\,n\in\mathbb{N}\})\,. \tag{II.3}$$

COROLLARY II.4 : *Let u be a bounded viscosity subsolution (resp. supersolution) of* (VI), *then*

$$u^*\leqslant u[\psi^*]\,, \tag{II.4}$$

(resp.

$$u_*[\psi_*]\leqslant u_*)\,. \tag{II.5}$$

This means that $u[\psi^*]$ is the maximum bounded viscosity subsolution of (VI) and $u_*[\psi_*]$ is the minimum bounded viscosity supersolution of (VI).

THEOREM II.5 : $U[\psi^*]$ *and* $u^*[\psi]$ *are viscosity solutions respectively of*

$$\operatorname{Max}\,(H(x,u,Du)\,;\,u-\psi^*)=0\quad\text{in}\quad\mathbb{R}^N\,, \tag{II.6}$$

and of

$$\operatorname{Max}\,(H(x,u,Du)\,;\,u-\psi_*)=0\quad\text{in}\quad\mathbb{R}^N\,, \tag{II.7}$$

Moreover, if ψ is u.s.c. (more generally if $(\psi^*)_*=\psi_*$*),* $u_*[\psi]$ *is the unique l.s.c. viscosity solution of* (VI).

This means that we are always able to identify $u_*[\psi]$ and u_* but no result concerns $u^*[\psi]$ which is not equal to $u[\psi]^*$ in general (see Section I).

Proof of Theorem II.3 : We leave the easy proof of (II.2) to the reader and we prove (II.3). First, since $\varphi_n \leqslant \psi_*$, we have $u[\varphi_n] \leqslant u[\psi_*]$ and since $u[\varphi_n]$ is continuous : $u[\varphi_n] \leqslant u_*[\psi_*]$. Hence

$$\operatorname{Sup}\{u[\varphi_n]\,;\, n \in \mathbb{N}\} \leqslant u_*[\psi_*]\,.$$

It remains to prove the opposite inequality. Let $x \in \mathbb{R}^N$, $\varepsilon > 0$. For all n, take (v^n, θ^n) such that

$$u[\varphi_n](x) + \varepsilon \geqslant \int_0^{\theta^n} f(y_x^n(t), v^n(t))\, e^{-\lambda t}\, dt + \varphi_n(y_x^n(\theta^n))\, e^{-\lambda\theta^n} \quad \text{(II.8)}$$

where y_x^n is the solution of

$$dy_x^n(t)/dt + b(y_x^n(t), v^n(t)) = 0\,; \quad y_x^n(0) = x\,.$$

First case: There exists a subsequence, still denoted $(\theta^n)_{n \in \mathbb{N}}$, of $(\theta^n)_{n \in \mathbb{N}}$, such that $\theta^n \to +\infty$. Since $\operatorname{Sup}\{u[\varphi_n](x)\,;\, n \in \mathbb{N}\}$ remains unchanged by this extraction, we deduce from (II.8)

$$u[\varphi_n](x) + \varepsilon \geqslant u[\psi_*](x) + [\varphi_n(y_x^n(\theta^n)) - \psi_*(y_x^n(\theta^n))]\, e^{-\lambda\theta^n}\,.$$

Since φ_n and ψ_* are bounded, letting $n \to \infty$, we obtain

$$\operatorname{Sup}\{u[\varphi_n](x)\,;\, n \in \mathbb{N}\} + \varepsilon \geqslant u[\psi_*](x) \geqslant u_*[\psi_*](x)\,.$$

Second case: The sequence θ^n is bounded. Since b is bounded $y_x^n(\theta^n)$ is bounded, so considering a subsequence, we may assume that $\theta^n \to \theta$ and $y_x^n(\theta^n) \to y$. Now, let us consider the trajectories x^n such that

$$dx^n(t)/dt + b(x^n(t), v^n(t)) = 0\,; \quad x^n(\theta^n) = y\,,$$

and let us define x_n by $x_n = x^n(0)$. The trajectory $x^n(\,.\,)$ is the same that the trajectory $y_x^n(\,.\,)$ defined by

$$dy_{x_n}(t)/dt + b(y_{x_n}(t), v^n(t)) = 0\,; \quad y_{x_n}(0) = x_n\,.$$

Moreover, since the sequence θ^n is bounded and since b is lipschitzian in x, we have

$$\left|y_x^n(t) - y_{x_n}(t)\right| \leqslant C\left|y_x^n(\theta^n) - y\right|\,,$$

in particular, for $t = 0$

$$|x_n - x| \leqslant C\left|y_x^n(\theta^n) - y\right|\,.$$

Now, using that f is lipschitzian in x, there exists ε_n such that

$$u[\varphi_n](x) + \varepsilon \geq \int_0^{\theta^n} f(y_x^n(t), v^n(t))\, e^{-\lambda t}\, dt + \varphi_n(y_x^n(\theta^n))\, e^{-\lambda\theta^n} - \varepsilon_n\,,$$

and $\varepsilon_n \to 0$ when $n \to \infty$. From the above inequality, we deduce

$$u[\varphi_n](x) + \varepsilon \geq u[\psi_*](x_n) + [\varphi_n(y_x^n(\theta^n)) - \psi_*(y)]\, e^{-\lambda\theta^n} - \varepsilon_n\,. \quad \text{(II.9)}$$

But, since the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is nondecreasing and since $y_x^n(\theta^n)$ converges to y, we have $\liminf_{n \to \infty} \varphi_n(y_x^n(\theta_n)) \geq \psi^*(y)$. Moreover $u[\psi^*](x_n) \geq u_*[\psi^*](x_n)$ and $u_*[\psi^*]$ is l.s.c., thus $\liminf_{n \to \infty} u[\psi^*](x_n) \geq u_*[\psi^*](x)$. Taking the limit in (II.9), we obtain

$$\operatorname{Sup}\,\{u[\varphi_n](x) + \varepsilon\,;\, n \in \mathbb{N}\} \geq u_*[\psi_*](x)\,.$$

In both cases, we have proved the inequality above, which is true for every $\varepsilon > 0$. Letting ε tend to 0, we conclude the proof of Theorem II.3.

Proof of Corollary II.4: First, we prove (II.4). Let $(\psi_n)_{n \in \mathbb{N}}$ be a nonincreasing sequence of bounded uniformly continuous functions such that

$$\operatorname{Inf}\,\{\psi_n\,;\, n \in \mathbb{N}\} = \psi^*\,.$$

Since $\psi_n \geq \psi^*$, u^* is a viscosity subsolution of

$$\operatorname{Max}\,\{H(x, w, Dw)\,;\, w - \psi_n\} = 0 \quad \text{in} \quad \mathbb{R}^N \quad \text{(II.10)}$$

and $u[\psi_n] \geq \psi^*$ is the viscosity solution of (II.9). By classical comparison results for first order H. J. equations, we have

$$u^* \leq u[\psi_n]\,. \quad \text{(II.11)}$$

Taking the infimum in n in (II.11) and using Theorem II.3, we conclude. We do not give the proof of (II.5) which is totally similar.

Proof of Theorem II.5: $u[\psi^*]$ is viscosity solution of (II.6) by Theorem II.2 In the same way, we already know that $u_*[\psi_*]$ is a viscosity supersolution of (II.7). It remains to prove that $u_*[\psi_*]$ is a viscosity subsolution of (II.7). Let $(\varphi_n)_{n \in \mathbb{N}}$ be a nondecreasing sequence of continuous functions such that

$$\operatorname{Sup}\,\{\varphi_n\,;\, n \in \mathbb{N}\} = \psi_*\,.$$

$u[\varphi_n]$ is a viscosity subsolution of

$$H(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N . \tag{II.12}$$

By standard result (*cf.* [9]), Sup $\{u[\varphi_n] \,;\, n \in \mathbb{N}\}$ is also a viscosity subsolution of (II.12). Moreover

$$u[\varphi_n] \leq \varphi_n \leq \psi_* \leq (\psi_*)^* .$$

Therefore, taking the supremum in n

$$u_*[\psi_*] \leq (\psi_*)^* ,$$

and so

$$(u_*[\psi_*])^* \leq (\psi_*)^* ,$$

finally, $u_*[\psi_*]$ is a viscosity subsolution of (II.7).

Now, we prove that $u_*[\psi_*]$ is the unique l.s.c. solution of (VI) when ψ is u.s.c. Let w be a l.s.c. solution of (VI). By Theorem 11.3, we have

$$u_*[\psi_*] \leq w \leq w^* \leq u[\psi] .$$

Therefore

$$u_*[\psi_*] \leq w_* \leq (w^*)_* \leq u_*[\psi] ,$$

but $w_* = w$ because w is l.s.c., and we have seen in the first part that $u_*[\psi] = u_*[\psi_*]$, so we conclude that

$$u_*[\psi_*] = w \quad \text{and} \quad (w^*)_* = w .$$

And Theorem II.5 is proved.

III. EXIT TIMES PROBLEMS LEADING TO STOPPING TIMES PROBLEMS WITH DISCONTINUOUS STOPPING COST

We want to present here some exit time problems which can be interpreted as stopping time problems with discontinuous stopping costs. In order to do so, let us first describe a typical problem of exit time.

The notations and assumptions of the following control problem are those of the first part. We consider, in addition, Ω a smooth bounded domain of $\mathbb{R}^N$. The state of the system is described by the solution y_x of

$$dy_x(s) + b(y_x(s), v(s))\, ds = 0 , \quad y_x(0) = x \in \Omega \tag{III.1}$$

and we define the cost function by

$$J(x, v) = \int_0^\tau f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \varphi(y_x(\tau))\, e^{-\lambda\tau}, \tag{III.2}$$

where τ is the first exit time from Ω. Our aim is to characterize the function

$$u(x) = \operatorname{Inf}\ \{J(x, v), v(\,.\,) \in L^\infty(\mathbb{R}^+ ; V)\}\ . \tag{III.3}$$

It is wellknown that, in general, u is not continuous and we are going to prove that, under some compatibility conditions, u_* is the unique viscosity solution of a Hamilton-Jacobi problem in Ω with mixed boundary condition on $\partial\Omega$. More precisely, we assume that

$$(H)\begin{cases} \text{there exist extensions of } b \text{ and } f \text{ to } \mathbb{R}^N \text{ satisfying (I.2) and } \psi \text{ a u.s.c.} \\ \text{bounded function in } \Omega^c \text{ which satisfy} \\ \qquad\qquad \psi = \psi_* = \varphi \quad \text{on} \quad \partial\Omega \\ \qquad \psi(x) \leqslant \displaystyle\int_0^T f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \psi(y_x(T))\, e^{-\lambda T}, \\ \text{for all } x \in \Omega^c, \text{ all controls } v(\,.\,) \in L^\infty(\mathbb{R}^+ ; V), \text{ all trajectories } y_x \text{ of} \\ \text{(III.1) in } \mathbb{R}^N \text{ and } y_x(T) \in \Omega^c \text{ or } T = +\infty\ . \end{cases}$$

In particular, this is satisfied for $f \geqslant 0$ and $\varphi = 0$ by taking $\psi = 0$.
We can state the following theorem

THEOREM III.1 : *Under the assumptions* (I.2) and (H), u_* *is the unique bounded l.s.c. (in* Ω*) viscosity solution of*

$$H(x, u, Du) = 0 \quad \text{in} \quad \Omega \tag{III.4}$$

$$\operatorname{Min}\ (H(x, u, Du)\ ; u - \varphi) \leqslant 0 \quad \text{on} \quad \partial\Omega \tag{III.5}$$

$$u_* = \varphi \quad \text{on} \quad \partial\Omega \tag{III.6}$$

where the equality and the inequality have to be understood in the viscosity sense and $H(x, t, p)$ *is given by* (4).

Remark : It is worth mentioning that u_* is the l.s.c. envellope of u in $\bar{\Omega}$ and not in Ω. We may have

$$u_*(x) < \liminf_{y \to x,\, y \in \Omega} u(y)\ .$$

Remark : Let us recall the meaning of (III.5). The definition of viscosity solution on the boundary was introduced and used for state-constraints problems by M. H. Soner [15] (See also I. Capuzzo-Dolcetta and P. L. Lions [2]).

u satisfies $H(x, u, Du) \leq 0$ at $x \in \partial\Omega$ in the viscosity sense if and only if

$$\begin{cases} \forall \phi \in C^1(\bar{\Omega}), \text{ if } x \text{ is a maximum point of } u^* - \phi \text{ in } \bar{\Omega}, \text{ we have} \\ \qquad H(x, u^*(x), D\phi(x)) \leq 0 . \end{cases}$$

The assumptions (H) is a technical assumption ; roughly speaking, it means that φ coincides on $\partial\Omega$ with a subsolution of the associated control problem in $\mathbb{R}^N$, which is continuous at each point of $\partial\Omega$. In particular, in Ω, this means that the exit time from Ω is the « best exit time », i.e. which gives the minimal value for u. More precisely, we have

$$u(x) = \operatorname{Inf} \left\{ \int_0^\theta f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \varphi(y_x(\theta))\, e^{-\lambda\theta} ; \right.$$
$$\left. v(\,.\,) \in L^\infty(\mathbb{R}^+ ; V),\ \tau_1 \leq \theta \leq \tau_2 ;\ y_x(\theta) \in \partial\Omega \right\}$$

where τ_1 is the first exit time from Ω and τ_2 from $\bar{\Omega}$. This type of formulation was considered by Quadrat [14]. Notice that the value function associated to relaxed controls is l.s.c. and so, is equal to u_* since f and φ are continuous.

Remark : It is an open problem to know what are the necessary and sufficient conditons on b, f, φ in Ω to have (H). For example, we do not know if the necessary condition

$$\varphi(x) \leq \int_0^T f(y_x(s), v(s))\, e^{-\lambda s}\, ds + \varphi(y_x(T))\, e^{-\lambda T},$$

for all $x \in \partial\Omega$, all trajectories y_x of (III.1) in $\bar{\Omega}$ and $y_x(T) \in \partial\Omega$ or $T = +\infty$, $v(\,.\,) \in L^\infty(\mathbb{R}^+ ; V)$, is sufficient or not.

Now, we turn to the *proof of Theorem* III.1.

The idea of the proof is to show that u is equal in Ω to an optimal cost function of a stopping time problem in $\mathbb{R}^N$ with an obstacle which satisfies (I.6). This allows us to show the equivalence between (III.4)-(III.5) and the variational inequality and to conclude by using Theorem II.5. First, let us extend ψ in Ω by setting

$$\psi(x) = C \quad \text{in} \quad \Omega$$

where C is a constant large enough, say

$$C = \lambda^{-1} \| f \|_{L^\infty(\mathbb{R}^N ; V)} + \| \varphi \|_{L^\infty(\partial\Omega)} + L .$$

Using (H), it is easy to see that the optimal cost function associated to the stopping time problem with the obstacle ψ coincides with u in Ω and with ψ in Ω^c. We will still denote this function by u. Hence, since ψ satisfies (I.6). Theorem II.5 implies that $(u_*)^* = u_*$ and u_* is the unique l.s.c. viscosity solution of

$$\text{Max } (H(x, u, Du)\,;\, u - \psi) = 0 \quad \text{in} \quad \mathbb{R}^N\,. \tag{VI}$$

Now, we investigate the properties of u in Ω. Since C is large enough, $u < \psi$ in Ω and so

$$H(x, u, Du) = 0 \quad \text{in} \quad \Omega\,.$$

In $\mathbb{R}^N$, $u \leq \psi$ and therefore $u_* \leq \psi_* = \varphi$ on $\partial\Omega$; but, as $u = \psi$ in Ω^c and since we know that $u_* = \hat{u}$ in Ω by Quadrat's remark, we have $u_* = \varphi$ on $\partial\Omega$. So, it remains to prove (III.5). Let $\phi \in C^1(\Omega)$ and let $x \in \partial\Omega$ be a maximum point of $u^* - \phi$ in Ω; two cases are possible :

a . $u^*(x) = \varphi(x)$, then (III.5) is satisfied.

b . $u^*(x) > \varphi(x)$, since $u^* = \psi$ in Ω^c and since the restriction of ψ to Ω^c is continuous on $\partial\Omega$, for any extension of ϕ to $\mathbb{R}^N$, x is a maximum point of $u^* - \phi$ in $\mathbb{R}^N$. This is a consequence of the jump of u^* toward $\partial\Omega$ at x. Therefore

$$H(x, u^*(x), D\phi(x)) \leq 0$$

which proves (III.5).

Conversely, let be v a bounded l.s.c. function which satisfy (III.4)-(III.6). By extending v by u_* outside Ω, we are going to prove that v is viscosity solution of (VI). By taking C large enough for ψ in Ω, we have $v \leq \psi$ in $\mathbb{R}^N$. The only difficulty is to prove that v is viscosity subsolution at the points of $\partial\Omega$. Let $\phi \in C^1(\mathbb{R}^N)$ and let $x \in \partial\Omega$ be a maximum point in $\mathbb{R}^N$ of $v^* - \phi$. If $v^*(x) > \varphi(x)$, since x is also a maximum point in Ω of $v^* - \phi$, (III.5) shows that $H(x, v^*(x), D\phi(x)) \leq 0$ and we conclude. If $v^*(x) = \varphi(x)$, since v is a supersolution of (VI), we have $v^* \geq u^*$ and thus $u^*(x) = \varphi(x)$ and x is also a maximum point of $u^* - \phi$ in $\mathbb{R}^N$. Finally, since u^* is viscosity subsolution of (VI), we obtain $H(x, u^*(x), D\phi(x)) \leq 0$, and again, we conclude. Finally, since u^* is the unique bounded l.s.c. viscosity solution of (VI), then

$$v = u_* \quad \text{in} \quad \mathbb{R}^N$$

and in particular in Ω. Let us finally point out that the proof of theorem III.1 shows that the mixed boundary condition (III.5) comes directly from the discontinuity of u on $\partial\Omega$.

Remark : In terms of control the interpretation of (III.5) is very clear. For $x \in \Omega$, let $\tau(x)$ be the minimal exit time of Ω with respect to the field b. For $x \in \partial\Omega$, two cases are possible :

(i) $\tau^*(x) = 0$; then, for any point in a neighbourhood of x, there exists a trajectory which goes out of Ω « immediately » and which is almost optimal by (H). So, u is continuous at x and $u(x) = \varphi(x)$.

(ii) $\tau^*(x) > 0$; then, there exists a sequence $x_n \to x$ such that

$$u^*(x_n) \to u^*(x) \geqslant \varphi(x)$$

and

$$\tau(x_n) \to \tau^*(x) .$$

Now, for n large enough, $\tau(x_n) \geqslant \tau^*(x)/2$ and using the Dynamic Programming Principle at x_n with $t \leqslant \tau^*(x)/2$, letting $n \to +\infty$, one shows easily that $u^*(x)$ satisfies a suboptimality inequality and so, u is viscosity subsolution of $H(x, u, Du) = 0$ at x.

APPENDIX

A STABILITY RESULT FOR GENERAL DISCONTINUOUS HAMILTONIANS

In this Appendix, we show a stability result for general H. J. Equations with discontinuous Hamiltonians which is inspired by the analogy presented in the remark following Definition II.1. Such problems has already been considered in [8], [12] with some particular discontinuities of $H(x, u, p)$ and for the evolution equation.

In all this part, we consider general discontinuities of H. Thus we look at the stationary equation and H is a locally bounded function of (x, u, p). First, we define viscosity solutions of

$$H(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N . \tag{A.1}$$

DEFINITION A.1 : *Let u be a locally bounded function in $\mathbb{R}^N$. We say that u is a viscosity subsolution (resp. supersolution) of* (A.1) *if*

$$\begin{cases} \forall \phi \in C^1(\mathbb{R}^N), \text{ at each local maximum point } x_0 \text{ of } u^* - \phi, \text{ we have :} \\ \qquad H_*(x_0, u^*(x_0), D\phi(x_0)) \leqslant 0 . \end{cases}$$

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$$\begin{cases} \forall \phi \in C^1(\mathbb{R}^N), \text{ at each local minimum point } x_0 \text{ of } u_* - \phi, \text{ we have :} \\ \qquad H^*(x_0, u_*(x_0), D\phi(x_0)) \geqslant 0 . \end{cases}$$

As we pointed it out in Section II, this definition is nothing but an extension of Crandall-Lions definition for continuous Hamiltonians. And Definition II.1 is a particular case of Definition A.1 taking

$$H(x, u, Du) = \text{Max } \{\text{Sup } (b(x, v) \,.\, Du - f(x, v) \,;\, v \in V) \,;\, u - \psi\} \,.$$

Now, we give a result concerning the stability properties of viscosity solutions. We only give the result for viscosity subsolutions ; the analogous result for viscosity supersolutions is true with easy adaptation.

THEOREM A.2 : *Assume that for all n, u_n^* is a viscosity subsolution of*

$$H_n(x, u_n^*, Du_n^*) = 0 \quad \text{in} \quad \mathbb{R}^N \,, \tag{A.2}$$

Set $u(x) = \limsup\limits_{n \to \infty, y \to x} u_n(y)$, $H(x, t, p) = \liminf\limits_{(n, y, s, q) \to (\infty, x, t, p)} H_n(y, s, q)$ *and assume that u and H are locally bounded. Then u is a viscosity subsolution of*

$$H(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N \,. \tag{A.3}$$

Proof of Theorem A.2 : Our proof is inspired by the corresponding one in [5], [9] and is based upon the following Lemma.

LEMMA A.3 : *Let u_n and u be as in Theorem* A.2. *Let $\phi \in C^1(\mathbb{R}^N)$ and x_0 be a local maximum point of $u - \phi$ (notice that u is u.s.c.). We assume that x_0 is the unique maximum point of $u - \phi$ in $B(x_0, r)$ for some $r > 0$. Then, let x_n be a sequence of maximum points of $u_n^* - \phi$ in $\overline{B(x_0, r)}$, then $x_n \to x_0$, $u_n^*(x_n) \to u(x_0)$ when $n \to \infty$.*

First, we prove Theorem A.2 by using this lemma. Let $\phi \in C^1(\mathbb{R}^N)$ and x_0 be a local maximum point of $u - \phi$. By a remark of M. G. Crandall and P. L. Lions [5], changing $\phi(x)$ in $\phi(x) + |x - x_0|^2$, we may assume that Lemma (A.3) holds. Thus, there exists a subsequence, still denoted x_n, of local maximum points of $u_n^* - \phi$ converging to x_0. Since u_n^* is a viscosity subsolution of (A.2) and since $x_n \in B(x_0, r)$ for n large enough, we have

$$H_{n*}(x_n, u_n^*(x_n), D\phi(x_n)) \leqslant 0 \,.$$

Moreover, we have

$$H(x_0, u(x_0), D\phi(x_0)) \leqslant \liminf H_{n*}(y, t, p)$$

as

$$n \to \infty, y \to x_0, t \to u(x_0), p \to D\phi(x_0) \,.$$

Therefore

$$H(x_0, u(x_0), D\phi(x_0)) \leq 0 .$$

And Theorem A.2 is proved.

Now, we prove Lemma A.3. Let x_0 and x_n be as in Lemma A.3 i.e.

$$\begin{aligned} u(x) - \phi(x) < u(x_0) - \phi(x_0), \quad \forall x \in B(x_0, r), x \neq x_0 , \\ u_n^*(x) - \phi(x) \leq u_n^*(x_n) - \phi(x_n), \quad \forall x \in B(x_0, r) . \end{aligned} \tag{A.5}$$

Extracting a subsequence, we may assume that x_n and $u_n^*(x_n)$ converge respectively to some point x_1 and some real β as $n \to \infty$. Thus, taking the lim sup as $n \to \infty$, $x \to y$ in (A.5), we obtain

$$u(y) - \phi(y) \leq \beta - \phi(x_1) \leq u(x_1) - \phi(x_1), \quad \forall y \in B(x_0, r) .$$

Since x_0 is the unique maximum point of $u - \phi$ in $B(x_0, r)$, this means that $x_1 = x_0$ and taking $y = x_0$, we have $\beta = u(x_0)$; and Lemma A.3 is proved.

Remark : One can deduce from this theorem that, under growth and monotonicity assumptions on H, (A.1) has a unique maximal subsolution which is also a supersolution — and in the same way, a unique minimal supersolution which is also a subsolution — . We do not prove this here since, with this generality, the definition (A.1) does not coincide with other classical definition of sub or supersolution (in D', for linear equation, for instance).

REFERENCES

[1] I. Capuzzo-Dolcetta and H. Ishii, *Approximate solutions of the Bellman Equations of deterministic control theory.* Applied Math. and Opt. 11, (1984), pp. 161-181.

[2] I. Capuzzo-Dolcetta and P. L. Lions, *Hamilton-Jacobi Equations and state-constraints problems* ; In preparation.

[3] M. G. Crandall, L. C. Evans and P. L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi Equations* ; Trans. Amer. Math. Soc., 282 (1984).

[4] M. G. Crandall, H. Ishii and P. L. Lions, *Uniqueness of viscosity solutions revisited* ; to appear.

[5] M. G. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi Equations* ; Trans. Amer. Math. Soc., *277* (1983).

[6] M. G. Crandall and P. L. Lions, *On existence and uniqueness of solutions of Hamilton-Jacobi Equations* ; Non Linear Anal. TMA. Vol. 10, N° 6 (1986).

[7] W. H. FLEMING and R. W. RISHEL, *Deterministic and stochastic optimal control.* Springer, Berlin 1975.

[8] H. ISHII, *Hamilton-Jacobi Equations with discontinuous Hamiltonians on arbitrary open subsets.*

[9] H. ISHII, *Perron's method for Hamilton-Jacobi Equations* ; to appear.

[10] E. B. LEE and L. MARKUS, *Foundations of optimal control theory,* J. Wiley, New York (1967).

[11] P. L. LIONS, *Generalized solutions of Hamilton-Jacobi Equations.* Pitman, 1982.

[12] P. L. LIONS and B. PERTHAME, *Remarks on Hamilton-Jacobi Equations with discontinuous time-dependent coefficients* ; Non Linear Anal. TMA. Vol. 11, n° 7 (1987).

[13] P. L. LIONS and P. E. SOUGANIDIS, *Differential games, optimal control and directional derivatives of viscosity solutions of Bellman's and Isaac's Equations* ; SIAM J. Control and Optimization, vol. 23, n° 4 (1985).

[14] J. P. QUADRAT, in Thèse d'Etat, Univ. Paris IX-Dauphine.

[15] M. H. SONER, *Optimal control problems with state-space constraints.* SIAM J. on Control and Optimisation. Vol. 24, n° 3, pp. 551-561 and Vol. 24, n° 4, pp. 1110-1122.

[16] J. WARGA, *Optimal control of differential and functionnal equations.* Academic press, (1972).