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HOMOGENIZATION LIMITS OF DIFFUSION EQUATIONS IN THIN DOMAINS (*)

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Communiqué par P G. CIARLET

Résume — *Ce travail étudie l'équation de diffusion linéaire pour un milieu composite dans un domaine n-dimensionnel fin*

On démontre que pour toute suite convergente de solutions (lorsque l'épaisseur du domaine tend vers zero) il existe une équation de diffusion effective satisfaite par la limite correspondante sur la section médiane Cette analyse ne nécessite aucune hypothèse sur la distribution géométrique des inhomogénéités.

Pour le cas d'une distribution horizontalement périodique de deux matériaux isotropes distincts, on détermine les bornes optimales indépendantes de la géométrie, pour la diffusivité effective du mélange lorsque l'épaisseur tend vers zéro.

Abstract. — *This is the study of the linear diffusion equation for a composite medium in a thin n-dimensional domain*

It is shown that, as the thickness approaches zero, any limit of solutions must necessarily satisfy a corresponding effective diffusion equation on the $(n-1)$*-dimensional mid-section This analysis does not require any periodicity assumptions about the geometry of the inhomogeneities*

For the case of a horizontally periodic mixture of two isotropic components, geometry independent optimal bounds are established for the effective diffusivity of the mixture in the limit as the thickness approaches zero

INTRODUCTION

Many solutions to optimal design problem involve the use of thin composite structures. A related mathematical problem is to derive consistent 1- or 2-dimensional models for these structures as limits of some 3-

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dimensional formulation. In a recent work [3], we have studied the behaviour of solutions to the equations of 3-dimensional linear elasticity for a composite material in a plate-like domain, in the limit as the plate thickness approaches zero. Without any assumptions about the geometry of the material inhomogeneities, we have shown that a limiting average vertical displacement must necessarily solve a fourth order linear elliptic boundary value problem on the plate midplane.

In this paper, we study a linear diffusion equation for a composite medium in the same thin plate-like domain. The reason for this is twofold : first, it allows us to illustrate some of the main ideas of [3] on a technically simpler problem ; secondly, we are able to characterize the limiting effective diffusivities (at least for a two component mixture of isotropic materials). In this second endeavor, we rely on the variational techniques recently presented in [5].

1. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

Let $\Omega_\varepsilon \subset \mathbb{R}^n$ denote the cylindrical domain $\omega \times (-\varepsilon/2, \varepsilon/2)$, where $\omega \subset \mathbb{R}^{n-1}$ is smooth and bounded (fig. 1). We write the independent variable $\underset{\sim}{x} \in \Omega_\varepsilon$ as $\underset{\sim}{x} = (x, x_n)$, with $x \in \omega$, $-\varepsilon/2 < x_n < \varepsilon/2$. Usually, latin indices will range from 1 to n, whereas greek indices will range from 1 to $n-1$; the summation convention applies when indices are repeated. We write ∂_i for differentiation with respect to x_i, $\partial/\partial x_i$, and ∂_{ij} for $\partial^2/\partial x_i\, \partial x_j$.

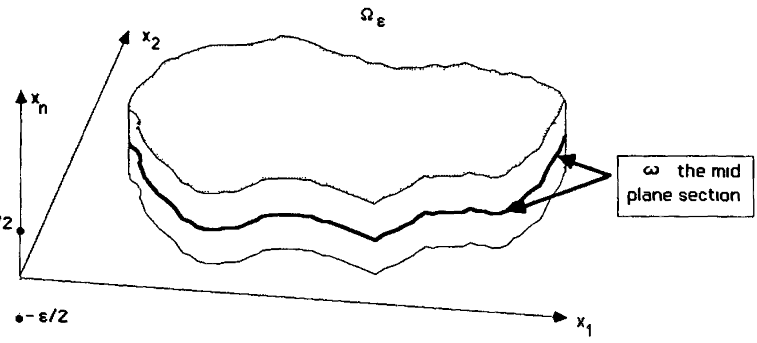

Figure 1.

The boundary of Ω_ε is divided into the following parts : $\partial\Omega_\varepsilon^+$ and $\partial\Omega_\varepsilon^-$, the top and bottom boundaries respectively, and $\partial_0\Omega_\varepsilon = \partial\omega \times (-\varepsilon/2, \varepsilon/2)$ the lateral boundary.

The linear diffusion problem consist in finding u_ε such that

(1) $$-\operatorname{div}_{\underline{x}}\left(\mathbf{a}_\varepsilon(\underline{x})\,\nabla_{\underline{x}} u_\varepsilon\right) = \mathscr{F}_\varepsilon \qquad \text{in } \Omega_\varepsilon$$

(2) $$\left(\mathbf{a}_\varepsilon(\underline{x})\,\nabla_{\underline{x}} u_\varepsilon\right) . \, \nu^{\pm} = \varepsilon g_\varepsilon^{\pm} \qquad \text{on } \partial\Omega_\varepsilon^{\pm}$$

(3) $$u_\varepsilon = 0 \qquad \text{on } \partial_0\Omega_\varepsilon \,.$$

Here $\nu^{\pm}$ denotes the outward unit normal $= (0, 0, \dots, \pm 1)$. The diffusivity matrices $\mathbf{a}_\varepsilon$ are spatially varying ; we assume that they are symmetric and that there exist positive real numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (independent of ε) such that

(4) $$\mathbf{a}_\varepsilon(\underline{x})_{ij}\,\eta_i\,\eta_j \geq \boldsymbol{\alpha}\, \|\eta\|^2 \quad \forall \eta \in \mathbb{R}^n, \quad \text{a.e. } \underline{x} \quad \text{in } \Omega_\varepsilon$$

(5) $$\left(\sum_i \left|\mathbf{a}_\varepsilon(\underline{x})_{ij}\,\eta_j\right|^2\right)^{\frac{1}{2}} \leq \boldsymbol{\beta}\, \|\eta\| \,.$$

(The presence of the extra ε in (2) is due to scaling considerations as will be clear later).

We introduce the space $\mathscr{W} = \left\{w \in H^1(\Omega_1),\ w\big|_{\partial_0\Omega_1} = 0\right\}$, and its dual space $\mathscr{W}'$, in terms of which hypotheses on $\mathscr{F}_\varepsilon$, $g_\varepsilon^{\pm}$ will be expressed.

Our first result is a compactness result much as in the theory of H- or Γ-convergence (*cf.* [4], [8]).

THEOREM 1 : *Let* $\{\varepsilon_k\}_{k=1}^{\infty}$ *be any given sequence converging to zero. There exists a subsequence* $\{\varepsilon_{k_l}\}_{l=1}^{\infty}$ *— for simplicity denoted* $\{\varepsilon_l\}_{l=1}^{\infty}$ *— and a field* $\mathscr{A}_0(x)$ *of symmetric* $n-1 \times n-1$ *matrices on* ω *such that whenever*

(6) $$g_\varepsilon^{\pm} \to g_0^{\pm} \quad \textit{in the space } H^{-\frac{1}{2}}(\omega), \textit{ and}$$

(7) $$\mathscr{F}_\varepsilon(x, \varepsilon y) \to \mathscr{F}_0(x, y) \quad \textit{in the space } \mathscr{W}' \,,$$

then the sequence $u_{\varepsilon_l}(x, \varepsilon_l\, y)$ *converges weakly in* $\mathscr{W}$ *to the solution* $u_0(x)$ *of the following* $n-1$ *dimensional problem :*

(8) $$\begin{cases} -\operatorname{div}_x\left(\mathscr{A}_0(x)\,\nabla_x u_0\right) = \bar{\mathscr{F}}_0 + g_0^{+} + g_0^{-} & \textit{in } \omega \\ u_{0\,|\,\partial\omega} = 0 \,. \end{cases}$$

Here, the notation $\bar{\mathscr{F}}_0$ *stands for the "y-average"* $\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{F}_0(x, y)\, dy$.

Furthermore, $\mathscr{A}_0(x)$ *satisfies the equivalent of inequalities* (4), (5) *on* ω *with the same* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$.

Remark : It is possible to prove a similar result for non-symmetric matrices $\mathbf{a}_\varepsilon(\underline{x})$, in which case $\mathscr{A}_0(x)$ is not symmetric but still satisfies (4) and (5) with the same $\boldsymbol{\alpha}$ but with $\boldsymbol{\beta}$ replaced by $\boldsymbol{\beta}^2/\boldsymbol{\alpha}$. □

If more detailed information is available concerning the spatial variation of $\mathbf{a}_\varepsilon$ and its dependence upon ε, then it may be possible to conclude that convergence holds for the *entire* sequence ε_k. One case in point is

$$\mathbf{a}_\varepsilon(\underline{x}) = \mathbf{a}(x/\varepsilon, x_n/\varepsilon)\,, \tag{9}$$

where $\mathbf{a}$ is periodic with period $\mathbf{P}$ in the first $n-1$ variables. It is possible to give a formula for $\mathscr{A}_0$ (here a *constant* matrix) in terms of certain " cell-problems " (*cf.* [1]).

The diffusivity $\mathbf{a}$ corresponds to a medium with two isotropic components if it has the form

$$\mathbf{a}(.) = (1-\chi(.))\,b_1\,\mathbb{I}_n + \chi(.)\,b_2\,\mathbb{I}_n \tag{10}$$

where $b_1 \leqslant b_2$ are two constants in the interval $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$, and χ is the characteristic function of some measurable subset of $\mathbf{Q} = \mathbf{P} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, continued periodically with period $\mathbf{P}$ in the first $n-1$ variables. The volume fraction of component b_2 is

$$\theta = 1/\mathrm{vol}\,(\mathbf{P}) \int_{\mathbf{Q}} \chi(X, y)\, dX\, dy = ⨍_{\mathbf{Q}} \chi(X, y)\, dX\, dy\,.$$

Our second result gives optimal bounds for the effective diffusivities that may be obtained from a horizontally periodic mixture of two isotropic components in the " thin domain " case :

THEOREM 2 : *If* $\mathbf{a}_\varepsilon(\underline{x})$ *is of the form* (9), (10), *then the corresponding* $n-1$ *dimensional diffusivity* $\mathscr{A}_0$ *satisfies*

$$\begin{cases} h\mathbb{I}_{n-1} \leqslant \mathscr{A}_0 \leqslant \mu\mathbb{I}_{n-1} \\ \mathrm{Tr}\,((\mathscr{A}_0 - b_1\,\mathbb{I}_{n-1})^{-1}) \leqslant (n-2)(\mu - b_1)^{-1} + (h - b_1)^{-1} \end{cases} \tag{11}$$

where $\mu = (1-\theta)\,b_1 + \theta b_2$ *and* $h = ((1-\theta)\,b_1^{-1} + \theta b_2^{-1})^{-1}$ *are the arithmetic and harmonic means of* b_1 *and* b_2 *respectively.*

Remark : The estimates (11) are optimal in the sense that equality is attained for specific composites. Furthermore, we claim that the closure of the set of symmetric matrices $\mathscr{A}_0$ that may be obtained from some $\mathbf{a}_\varepsilon$ satisfying (9), (10) and with volume fraction θ of component b_2 is exactly the set of symmetric matrices for which (11) holds. For a sketch of the proof of this, we refer to the end of section 5. □

We point out that our approach is not as general as that taken by Tartar [11] and Lurie-Cherkaev [7] for the n dimensional diffusion problem, since we do rely on periodicity to characterize $\mathscr{A}_0$. We conjecture that the results of Theorem 2 remain true pointwise almost everywhere for the " thin domain " limits $\mathscr{A}_0(x)$ of an arbitrary mixture of two isotropic components. The constant volume fraction θ should be replaced by the weak* limit of

$$1/\varepsilon \int_{-\varepsilon/2}^{\varepsilon/2} \chi_\varepsilon(x, x_n)\, dx_n ,$$

where χ_ε is the characteristic function of the set $\{\underline{x} : \mathbf{a}_\varepsilon(\underline{x}) = b_2\}$.

In principle, it should be possible to obtain a similar characterization of the effective rigidities that result from the mixture of two linearly elastic, isotropic materials in a thin domain. Formulas concerning the effect of horizontally periodic mixing are well-known (*cf.* [2], [6]) in this case. It should be pointed out that the technical difficulties of such a characterization may prove important. This is apparent in the partial results that have already been obtained for the equations of linear elasticity [9].

In an appendix, we prove the existence of first order correctors. This is a result of interest in itself, but it may also be useful in establishing the optimal bounds without assumptions about periodicity (at least, that is part of the approach taken in [11] for diffusion in a fixed domain).

2. THE RESCALED PROBLEM — A PRIORI ESTIMATES

In order to study the limiting behavior of u_ε, we rescale the problem to the fixed domain Ω_1, replacing x_n by εy, $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Let $U_\varepsilon(x, y) = u_\varepsilon(x, \varepsilon y)$, so that

$$\partial u_\varepsilon/\partial x_n(x, x_n) = (1/\varepsilon) \,.\, \partial U_\varepsilon/\partial y(x, x_n/\varepsilon) .$$

We also denote by $\mathcal{A}_\varepsilon(x, y) = \mathbf{a}_\varepsilon(x, \varepsilon y)$ and write ∂_y in place of ∂_n when applied to functions defined on Ω_1.

It is now easy to check that the variational formulation for problem (1), (2), (3), expressed in the new variables x, y is :

$$(12) \quad \int_{\Omega_1} \Big(\sum_{\alpha, \beta} \mathcal{A}_{\varepsilon, \alpha\beta}\, \partial_\alpha U_\varepsilon\, \partial_\beta V + 1/\varepsilon \Big(\sum_{\beta} \mathcal{A}_{\varepsilon, n\beta} (\partial_y U_\varepsilon\, \partial_\beta V + \\ + \partial_\beta U_\varepsilon\, \partial_y V)\Big) + 1/\varepsilon^2\, \mathcal{A}_{\varepsilon, nn}\, \partial_y U_\varepsilon\, \partial_y V \Big)\, dx\, dy \\ = \int_{\Omega_1} \mathscr{F}_\varepsilon(x, \varepsilon y)\, V(x, y)\, dx\, dy + \int_\omega g_\varepsilon^+(x)\, V\left(x, \frac{1}{2}\right) dx \\ + \int_\omega g_\varepsilon^-(x)\, V\left(x, -\frac{1}{2}\right) dx ,$$

which must holds for every V in $\mathscr{W}$, U_ε being itself in $\mathscr{W}$ (here the extra ε in front of the $g_\varepsilon^\pm$ disappears due to the change in variables). The integrals on the right-hand side of the equality represent the duality pairing between $\mathscr{W}'$, $\mathscr{W}$ and $H^{-\frac{1}{2}}(\omega)$, $H^{\frac{1}{2}}(\omega)$ respectively.

LEMMA 1: *Under hypotheses* (4) *and* (5), *the norms* $\|U_\varepsilon\|_{\mathscr{W}}$ *and* $\|1/\varepsilon\ \partial_y U_\varepsilon\|_{L^2(\Omega_1)}$ *are bounded by*

$$C\left(\|\mathscr{F}_\varepsilon(x, \varepsilon y)\|_{\mathscr{W}'} + \|g_\varepsilon^+\|_{H^{-\frac{1}{2}}(\omega)} + \|g_\varepsilon^-\|_{H^{-\frac{1}{2}}(\omega)}\right).$$

Proof: Inserting $V = U_\varepsilon$ in (12), and using the coerciveness of $\mathcal{A}_\varepsilon$, we see that

$$(13)\quad \|\nabla_x U_\varepsilon\|^2_{L^2(\Omega_1)} + \|1/\varepsilon\ \partial_y U_\varepsilon\|^2_{L^2(\Omega_1)} \leq$$
$$\leq 1/\alpha\left(\int_{\Omega_1} \mathscr{F}_\varepsilon(x, \varepsilon y)\, U_\varepsilon(x, y)\, dx\, dy + \int_\omega g_\varepsilon^+(x)\, U_\varepsilon\left(x, \frac{1}{2}\right) dx + \int_\omega g_\varepsilon^-(x)\, U_\varepsilon\left(x, -\frac{1}{2}\right) dx\right)$$

The desired result is now a consequence of the above inequality (13) in combination with the Poincaré inequality $(\|V\|_{\mathscr{W}} \leq C\|\nabla_{x,y} V\|_{L^2(\Omega_1)})$. □

If $\mathscr{F}_\varepsilon(x, \varepsilon y)$ and $g_\varepsilon^\pm$ are bounded in $\mathscr{W}'$ and $H^{-\frac{1}{2}}(\omega)$ respectively, uniformly as ε goes to 0, then it follows from Lemma 1 that U_ε is relatively weakly compact in the Hilbert space $\mathscr{W}$ and that $\{1/\varepsilon\ \partial_y U_\varepsilon\}$ and

$$\xi_{\varepsilon,\beta} \triangleq \sum_\alpha \mathcal{A}_{\varepsilon,\alpha\beta}\, \partial_\alpha U_\varepsilon + 1/\varepsilon\ \mathcal{A}_{\varepsilon,n\beta}\, \partial_y U_\varepsilon$$

are relatively weakly compact in $L^2(\Omega_1)$.

It is thus possible to find a subsequence $\{\varepsilon_l\}$ so that $\{U_{\varepsilon_l}\}$, $\{1/\varepsilon_l\ \partial_y U_{\varepsilon_l}\}$ and $\{\xi_{\varepsilon_l,\beta}\}$ all converge weakly. Due to the estimates of Lemma 1, we may use the same subsequence for any $\mathscr{F}_\varepsilon$, $g_\varepsilon^\pm$ satisfying (6), (7) with given limit $\mathscr{F}_0$; $g_0^\pm$. By a diagonalization argument we can now find a subsequence (still denoted $\{\varepsilon_l\}$ for simplicity) so that $\{U_{\varepsilon_l}\}$, $\{1/\varepsilon_l\ \partial_y U_{\varepsilon_l}\}$ and $\{\xi_{\varepsilon_l,\beta}\}$ converge provided (6) and (7) are satisfied for limits $\mathscr{F}_m$, $g_m^\pm$ in a countable subset of $\mathscr{W}' \times H^{-\frac{1}{2}}(\omega) \times H^{-\frac{1}{2}}(\omega)$. Since the latter product space is separable, the estimates of Lemma 1 now guarantee that this subsequence $\{\varepsilon_l\}$ leads to weakly convergent $\{U_{\varepsilon_l}\}$, $\{1/\varepsilon_l\ \partial_y U_{\varepsilon_l}\}$ and $\{\xi_{\varepsilon_l,\beta}\}$ for any $\mathscr{F}_\varepsilon$, $g_\varepsilon^\pm$ that converge strongly in the sense of (6) and (7).

In the following, $\{\varepsilon_l\}$ shall always refer to such a *universal* subsequence. Due to Lemma 1 again, the weak limit U_0 of $\{U_{\varepsilon_l}\}$ satisfies $\partial_y U_0 = 0$, hence it can be regarded as an element of $H_0^1(\omega)$. By inserting V, independent of y in (12), we get

$$\int_\omega \bar{\xi}_{\varepsilon_l, \beta}(x)\, \partial_\beta V(x)\, dx = \int_\omega (\bar{\mathscr{F}}_{\varepsilon_l}(x) + g_{\varepsilon_l}^+ + g_{\varepsilon_l}^-)\, V(x)\, dx\,.$$

It follows immediately from this that the weak limit $\bar{\xi}_{0,\beta}$ satisfies

$$\int_\omega \bar{\xi}_{0,\beta}(x)\, \partial_\beta V(x)\, dx = \int_\omega (\bar{\mathscr{F}}_0(x) + g_0^+ + g_0^-)\, V(x)\, dx\,, \tag{14}$$

or

$$-\operatorname{div}_x \bar{\xi}_0 = \bar{\mathscr{F}}_0 + g_0^+ + g_0^- \quad \text{in } \omega\,. \tag{14'}$$

The next two sections are devoted to constructing a symmetric $\mathscr{A}_0$ satisfying the equivalent of (4) and (5) in ω, and for which

$$\bar{\xi}_0 = \mathscr{A}_0(x)\, \nabla_x U_0\,.$$

In view of (14′), this will complete the proof of Theorem 1.

3. SOME AUXILIARY LEMMAS

The effective diffusivity $\mathscr{A}_0$, provided it exists at all, gives rise to an isomorphism between $H^{-1}(\omega)$ and $H_0^1(\omega)$ (the resolvent of the operator $-\operatorname{div}_x(\mathscr{A}_0 \nabla_x)$). It is thus natural to try and obtain $\mathscr{A}_0$ by constructing a candidate for the resolvent. Such a candidate may be obtained through a limiting process from the resolvents of the ε-dependent n-dimensional boundary value problems. We study the particular case when $\mathscr{F}_\varepsilon(x, x_n) = G(x) \in H^{-1}(\omega)$ and $g_\varepsilon^\pm = 0$. The corresponding solution to (1), (2), (3) is denoted V_ε, and its flux

$$\zeta_{\varepsilon,\beta} \triangleq \sum_\alpha \mathcal{a}_{\varepsilon_l, \alpha\beta}\, \partial_\alpha V_\varepsilon + 1/\varepsilon\; \mathcal{a}_{\varepsilon, n\beta}\, \partial_y V_\varepsilon\,.$$

$V_0 \in H_0^1(\omega)$ and $\zeta_0 \in L^2(\Omega_1)$ are the weak limits along the universal subsequence introduced at the end of the previous section.

In this section we show that $G \mapsto V_0$ is an isomorphism between $H^{-1}(\omega)$ and $H_0^1(\omega)$, and that $G \mapsto \bar{\zeta}_0$ is a continuous map from $H^{-1}(\omega)$ to $L^2(\omega)$.

LEMMA 2 : *For any G in $H^{-1}(\omega)$ the following hold :*

$$\|\nabla_x V_0\|^2_{L^2(\omega)} \leq C \int_\omega G V_0 \, dx \,, \quad and \quad \|G\|^2_{H^{-1}(\omega)} \leq C \int_\omega G V_0 \, dx \,.$$

Proof: Consider (12) with U_ε and V both replaced by V_ε :

$$(15) \quad \int_{\Omega_1} \Big(\sum_{\alpha\beta} \mathcal{A}_{\varepsilon,\alpha\beta} \, \partial_\alpha V_\varepsilon \, \partial_\beta V_\varepsilon + + 2 \sum_\alpha \mathcal{A}_{\varepsilon, n\alpha} (1/\varepsilon \, \partial_y V_\varepsilon) \, \partial_\alpha V_\varepsilon + \mathcal{A}_{\varepsilon, nn} (1/\varepsilon \, \partial_y V_\varepsilon)^2 \Big) \, dx \, dy = \int_\omega G(x) \, \bar{V}_\varepsilon(x) \, dx \,.$$

By coercivity we conclude that

$$(16) \qquad \alpha \, \|\nabla_x V_\varepsilon\|^2_{L^2(\Omega_1)} \leq \int_\omega G \bar{V}_\varepsilon \, dx \,.$$

Using the lower semi-continuity of the left-hand side in (16) and the weak continuity of the right-hand side we obtain the first inequality of Lemma 2 since $V_0 = \lim V_{\varepsilon_l}$ does not depend upon y.

Furthermore, one has $-\operatorname{div}_x \bar{\zeta}_\varepsilon = G$, as can be checked from equation (12) by taking a test function independent of y. In combination with (15) this gives

$$\|G\|^2_{H^{-1}(\omega)} \leq C \left\| \bar{\zeta}_\varepsilon \right\|^2_{L^2(\omega)} \leq C \, \|\zeta_\varepsilon\|^2_{L^2(\Omega_1)} \leq C \int_\omega G \bar{V}_\varepsilon \, dx \,.$$

Passing to the limit along the sequence ε_l we obtain the second inequality of Lemma 2. $\square$

LEMMA 3 : *The map $G \mapsto \mathbf{S}(G) = V_0$ is a symmetric isomorphism from $H^{-1}(\omega)$ onto $H^1_0(\omega)$, and $G \mapsto \mathbf{T}(G) = \bar{\zeta}_0$ is continuous from $H^{-1}(\omega)$ into $L^2(\omega)$.*

Proof: By Lemma 2 and the Poincaré inequality in $H^1_0(\omega)$, we conclude that $\|V_0\|_{H^1_0(\omega)} \leq C \, \|G\|_{H^{-1}(\omega)}$ so that the corresponding map $G \mapsto \mathbf{S}(G)$ is continuous from $H^{-1}(\omega)$ into $H^1_0(\omega)$. Since $\|\zeta_\varepsilon\|^2_{L^2(\Omega_1)} \leq C \int_\omega G \bar{V}_\varepsilon \, dx$, as seen in the previous proof, it now follows that

$$\|\mathbf{T}(G)\|^2_{L^2(\omega)} = \left\|\bar{\zeta}_0\right\|^2_{L^2(\omega)} \leq C \int_\omega G V_0 \, dx \leq C \, \|G\|_{H^{-1}(\omega)} \times \times \|V_0\|_{H^1_0(\omega)} \leq C \, \|G\|^2_{H^{-1}(\omega)} \,.$$

This proves the desired continuity property of **T**. From Lemma 2, we also have that $\|G\|^2_{H^{-1}(\omega)} \leq C \int_\omega GS(G)\,dx$, and by the Lax-Milgram lemma, we therefore conclude that **S** is an isomorphism. One easily checks that $\int_\omega GS(F)\,dx = \int_\omega FS(G)\,dx$ for any F, $G \in H^{-1}(\omega)$, i.e. **S** is symmetric. □

4. CONSTRUCTION OF $\mathscr{A}_0$

In this section we use the operators **S** and **T** to make explicit the relationship between $\bar{\xi}_0$ and $\nabla_x U_0$. We apply a simple form of the method of *compensated compactness* (*cf.* [8], [10]) which, here is just a judicious integration by parts. Let V_ε denote the solution of (1), (2), (3) with $\mathscr{F}_\varepsilon(x, x_n) = G(x)$ and $g_\varepsilon^\pm = 0$ as in the previous paragraph, and let ϕ be a smooth function of x *alone* with compact support in ω.

Consider the identity (12) with V replaced by ϕV_ε. A simple calculation gives

$$\text{(17)}\quad \int_{\Omega_1} \Big(\sum_{\alpha,\beta} \mathcal{A}_{\varepsilon,\alpha\beta}\,\partial_\alpha U_\varepsilon\,\partial_\beta V_\varepsilon + 1/\varepsilon\Big(\sum_\beta \mathcal{A}_{\varepsilon,n\beta}(\partial_y U_\varepsilon\,\partial_\beta V_\varepsilon + {}$$
$$+ \partial_\beta U_\varepsilon\,\partial_y V_\varepsilon)\Big) + 1/\varepsilon^2\,\mathcal{A}_{\varepsilon,nn}\,\partial_y U_\varepsilon\,\partial_y V_\varepsilon\Big)\,\phi\,dx\,dy$$
$$+ \int_{\Omega_1} \Big(\sum_{\alpha,\beta} \mathcal{A}_{\varepsilon,\alpha\beta}\,\partial_\alpha U_\varepsilon\,\partial_\beta\phi + 1/\varepsilon\Big(\sum_\beta \mathcal{A}_{\varepsilon,n\beta}\,\partial_y U_\varepsilon\,\partial_\beta\phi\Big)\Big)\,V_\varepsilon\,dx\,dy$$
$$= \int_{\Omega_1} \mathscr{F}_\varepsilon(x,\varepsilon y)\,V_\varepsilon(x,y)\,\phi(x)\,dx\,dy + \int_\omega g_\varepsilon^+(x)\,V_\varepsilon\Big(x,\frac{1}{2}\Big)\,\phi(x)\,dx$$
$$+ \int_\omega g_\varepsilon^-(x)\,V_\varepsilon\Big(x,-\frac{1}{2}\Big)\,\phi(x)\,dx\,.$$

Similarly, an exchange of the roles of U_ε and V_ε in the previous calculation gives

$$\text{(18)}\quad \int_{\Omega_1} \Big(\sum_{\alpha,\beta} \mathcal{A}_{\varepsilon,\alpha\beta}\,\partial_\alpha V_\varepsilon\,\partial_\beta U_\varepsilon + 1/\varepsilon\Big(\sum_\beta \mathcal{A}_{\varepsilon,n\beta}(\partial_y V_\varepsilon\,\partial_\beta U_\varepsilon + {}$$
$$+ \partial_\beta V_\varepsilon\,\partial_y U_\varepsilon)\Big) + 1/\varepsilon^2\,\mathcal{A}_{\varepsilon,nn}\,\partial_y V_\varepsilon\,\partial_y U_\varepsilon\Big)\,\phi\,dx\,dy$$
$$+ \int_{\Omega_1} \Big(\sum_{\alpha,\beta} \mathcal{A}_{\varepsilon,\alpha\beta}\,\partial_\alpha V_\varepsilon\,\partial_\beta\,\phi + 1/\varepsilon\Big(\sum_\beta \mathcal{A}_{\varepsilon,n\beta}\,\partial_y V_\varepsilon\,\partial_\beta\,\phi\Big)\Big)\,U_\varepsilon\,dx\,dy$$
$$= \int_{\Omega_1} G(x)\,U_\varepsilon(x,y)\,\phi(x)\,dx\,dy\,.$$

Subtracting (18) from (17) we get, due to the symmetry of $\mathcal{a}_{\varepsilon, \alpha\beta}$,

$$(19) \quad \int_{\Omega_1} \left(\sum_{\alpha, \beta} \mathcal{a}_{\varepsilon, \alpha\beta}\, \partial_\alpha U_\varepsilon\, \partial_\beta \phi + 1/\varepsilon \left(\sum_{\beta} \mathcal{a}_{\varepsilon, n\beta}\, \partial_y U_\varepsilon\, \partial_\beta \phi \right) \right) V_\varepsilon \, dx\, dy -$$

$$- \int_{\Omega_1} \left(\sum_{\alpha, \beta} \mathcal{a}_{\varepsilon, \alpha\beta}\, \partial_\alpha V_\varepsilon\, \partial_\beta \phi + 1/\varepsilon \left(\sum_{\beta} \mathcal{a}_{\varepsilon, n\beta}\, \partial_y V_\varepsilon\, \partial_\beta \phi \right) \right) U_\varepsilon \, dx\, dy$$

$$= \int_{\Omega_1} \mathcal{F}_\varepsilon(x, \varepsilon y)\, V_\varepsilon(x, y)\, \phi(x)\, dx\, dy + \int_\omega g_\varepsilon^+(x)\, V_\varepsilon\left(x, \frac{1}{2}\right) \phi(x)\, dx$$

$$+ \int_\omega g_\varepsilon^-(x)\, V_\varepsilon\left(x, -\frac{1}{2}\right) \phi(x)\, dx - \int_{\Omega_1} G(x)\, U_\varepsilon(x, y)\, \phi(x)\, dx\, dy\,.$$

When ε converges to zero along the « universal subsequence » $\{\varepsilon_l\}$, the right-hand side of (19) converges to $\int_\omega ((\bar{\mathcal{F}}_0 + g_0^+ + g_0^-)\, V_0 - G U_0)\, \phi\, dx$ because none of U_0, V_0, ϕ depend upon y.

Concerning the left-hand side of (19), it can be written as

$$\int_{\Omega_1} \sum_\beta \xi_{\varepsilon, \beta} \cdot \partial_\beta \phi V_\varepsilon \, dx\, dy - \int_{\Omega_1} \sum_\beta \zeta_{\varepsilon, \beta} \cdot \partial_\beta \phi U_\varepsilon \, dx\, dy\,.$$

Passing to the limit in these terms is easy since ξ_ε and ζ_ε converge weakly in $L^2(\Omega_1)$ to ξ_0 and ζ_0 respectively, and at the same time V_ε and U_ε converge strongly in $L^2(\Omega_1)$ to V_0 and U_0 respectively. As a result, we get

$$(20) \quad \int_\omega \sum_\beta \bar{\xi}_{0, \beta} \cdot \partial_\beta \phi V_0 \, dx - \int_\omega \sum_\beta \bar{\zeta}_{0, \beta} \cdot \partial_\beta \phi U_0 \, dx =$$

$$= \int_\omega ((\bar{\mathcal{F}}_0 + g_0^+ + g_0^-)\, V_0 - G U_0)\, \phi\, dx\,.$$

We already know that

$$(21) \qquad - \operatorname{div}_x \bar{\xi}_0 = \bar{\mathcal{F}}_0 + g_0^+ + g_0^-$$

as well as

$$(22) \qquad - \operatorname{div}_x \bar{\zeta}_0 = G\,.$$

A combination of (20), (21), and (22) gives

$$(23) \qquad \int_\omega \left(\sum_\beta \bar{\xi}_{0, \beta}\, \partial_\beta V_0 \right) \phi\, dx = \int_\omega \left(\sum_\beta \bar{\zeta}_{0, \beta}\, \partial_\beta U_0 \right) \phi\, dx$$

which, holding true for any test function ϕ, implies that

(24) $$\sum_\beta \bar{\xi}_{0,\beta}\, \partial_\beta V_0 = \sum_\beta \bar{\zeta}_{0,\beta}\, \partial_\beta U_0 \quad \text{a.e. in } \omega .$$

Pick an arbitrary subdomain $\omega' \subset\subset \omega$ and a basis vector e_α, and set $V_0(x) = x_\alpha\, \psi(x)$, where ψ is any smooth function with compact support in ω, and with $\psi \equiv 1$ on ω'. The corresponding $G = \mathbf{S}^{-1}(V_0) = \mathbf{S}^{-1}(x_\alpha\, \psi(x))$ is well defined by Lemma 3, and (24) shows that

(25) $$\bar{\xi}_{0,\alpha} = \sum_\beta \bar{\zeta}_{0,\beta}\, \partial_\beta U_0 = [\mathbf{T}(\mathbf{S}^{-1}(x_\alpha\, \psi))]\,(x)\,.\,\nabla_x U_0 \quad \text{a.e. in } \omega' .$$

This is in the form $\bar{\xi}_0 = \mathscr{A}_0\, \nabla_x\, U_0$ with the α-th row of $\mathscr{A}_0(x)$ given by

$$\mathscr{A}_0(x)_\alpha\big|_{\omega'} = \mathbf{T}[\mathbf{S}^{-1}(x_\alpha\, \psi)]\,(x)\big|_{\omega'}.$$

Note furthermore that this formula does not depend on the choice of ψ as long as $\psi \equiv 1$ on ω'.

In other words, $\mathbf{T}[\mathbf{S}^{-1}(x_\alpha\, \tilde{\psi})] = \mathbf{T}[\mathbf{S}^{-1}(x_\alpha\, \psi)]$ a.e. in ω' for any other $\tilde{\psi}$ in $\mathscr{D}(\omega)$ with $\tilde{\psi} \equiv 1$ on ω' (this follows from (25) by taking $U_0 = x_\alpha\, \tilde{\psi}$!). Since $\omega' \subset\subset \omega$ is arbitrary, $\mathscr{A}_0(x)$ is defined a.e. in the domain ω. In any subdomain ω', $\mathscr{A}_0\big|_{\omega'}$ is easily seen to depend on knowledge of the $\mathcal{A}_\varepsilon$'s only in an arbitrarily small neighborhood of $\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, and for ε in the « universal » subsequence $\{\varepsilon_l\}$. Writing (25) for $U_0 = x_\beta\, \psi(x)$, we get

$$\bar{\xi}_{0,\alpha}(x) = \mathscr{A}_0(x)_{\alpha\beta} = \bar{\zeta}_{0,\beta}(x) = \mathscr{A}_0(x)_{\beta\alpha}$$

a.e. in ω', consequently $\mathscr{A}_0$ is a symmetric matrix field.

So far, $\mathscr{A}_0(x)$ is only known to be in $L^2_{\mathrm{loc}}(\omega)$, we now verify that it is actually in $L^\infty(\omega)$. To do so, we consider (18) with U_ε replaced by V_ε itself ; we also use the fact that the $\mathcal{A}_\varepsilon$'s are symmetric and bounded by $\boldsymbol{\beta}$, (5), hence $1/\boldsymbol{\beta}\, \|\mathcal{A}_\varepsilon\, \eta\|^2_{\mathbb{R}^n} \leqslant (\mathcal{A}_\varepsilon\, \eta, \eta)$ for any η in $\mathbb{R}^n$. For any non negative ϕ

(26) $$1/\boldsymbol{\beta} \int_{\Omega_1} \|\zeta_\varepsilon(x, y)\|^2\, \phi(x)\, dx\, dy \leqslant$$
$$\leqslant \int_{\Omega_1} \Big(\sum_{\alpha,\beta} \mathcal{A}_{\varepsilon,\alpha\beta}\partial_\alpha V_\varepsilon\, \partial_\beta V_\varepsilon + 2/\varepsilon \Big(\sum_\beta \mathcal{A}_{\varepsilon,n\beta}\, \partial_y V_\varepsilon\, \partial_\beta V_\varepsilon\Big)$$
$$+ 1/\varepsilon^2\, \mathcal{A}_{\varepsilon,nn}\, \partial_y V_\varepsilon\, \partial_y V_\varepsilon\Big)\, \phi\, dx\, dy$$
$$= \int_{\Omega_1} G(x)\, V_\varepsilon(x, y)\, \phi(x)\, dx\, dy - \int_{\Omega_1} \sum_\beta \zeta_{\varepsilon,\beta}\, \partial_\beta\, \phi V_\varepsilon\, dx\, dy .$$

Using the weak lower semi-continuity of the left hand side of (26) we obtain in the limit as $\varepsilon_l \to 0$

$$(27) \qquad 1/\beta \int_\omega \left\|\bar{\zeta}_0(x)\right\|^2 \phi(x)\, dx \leq \int_\omega G(x)\, V_0(x)\, \phi(x)\, dx - \int_\omega \sum_\beta \bar{\zeta}_{0,\beta}\, \partial_\beta\, \phi V_0\, dx\,.$$

At the same time, $-\operatorname{div}_x \bar{\zeta}_0 = G$ in ω so that (27) becomes

$$(28) \qquad 1/\beta \int_\omega \left\|\bar{\zeta}_0(x)\right\|^2 \phi(x)\, dx \leq \int_\omega \bar{\zeta}_0(x) \cdot \nabla_x V_0(x)\, \phi(x)\, dx\,.$$

The estimate (28) holds for arbitrary non negative ϕ, and thus we conclude that $1/\beta \left\|\bar{\zeta}_0(x)\right\|^2 \leq \bar{\zeta}_0(x) \cdot \nabla_x V_0(x)$ a.e. in ω, which is equivalent to the statement that

$$\left(\sum_\alpha \left|\mathscr{A}_0(x)_{\alpha\beta}\, \eta_\beta\right|^2\right)^{\frac{1}{2}} \leq \beta \|\eta\| \quad \forall \eta \in \mathbb{R}^{n-1}\,, \quad \text{a.e. in } \omega\,.$$

A similar computation, making use of (4) instead of (5), yields

$$\alpha \int_\omega \|\nabla_x V_0(x)\|^2\, \phi(x)\, dx \leq \int_\omega \bar{\zeta}_0(x) \cdot \nabla_x V_0(x)\, \phi(x)\, dx$$

for every non negative ϕ, which implies coerciveness of $\mathscr{A}_0(x)$ with constant α a.e. in ω. This concludes our proof of Theorem 1. □

5. OPTIMAL BOUNDS FOR $\mathscr{A}_0$.

In this section, we are interested in describing as precisely as possible the set of values that the matrix-valued function $\mathscr{A}_0$ can take when mixing two isotropic materials with given volume fractions. The matrix $\mathcal{A}_\varepsilon(x, y)$ is therefore assumed to be of the following form : $(1 - \chi_\varepsilon(x, y))\, b_1\, \mathbb{I}_n + \chi_\varepsilon(x, y)\, b_2\, \mathbb{I}_n$, where $b_1 \leq b_2$ are two constants in the interval $[\alpha, \beta]$, and χ_ε is the characteristic function of a measurable subset of Ω_1. We restrict our study to the case where χ_ε is of the form $\chi_\varepsilon(x, y) = \chi(x/\varepsilon, y)$ with $\chi(X, y)$ the characteristic function of some measurable subset of $\mathbf{Q} = \mathbf{P} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, continued periodically with respect to X with period $\mathbf{P} = \left(-\frac{\lambda_1}{2}, \frac{\lambda_1}{2}\right) \times \left(-\frac{\lambda_2}{2}, \frac{\lambda_2}{2}\right) \times \cdots \times \left(-\frac{\lambda_{n-1}}{2}, \frac{\lambda_{n-1}}{2}\right)$ (here and in the following we use the letter X for the independent x-variable in $\mathbf{Q}$). The volume fraction of the component b_2 is $\theta = 1/\text{vol}\,(\mathbf{P}) \int_{\mathbf{Q}} \chi(X, y)\, dX\, dy =$

$\displaystyle -\!\!\!\!\!\!\int_Q \chi(X, y)\, dX\, dy$, that of the component b_1 being $1 - \theta$, of course. We start by restating Theorem 1 for the case of an $\mathcal{A}_\varepsilon$ which is periodic in the first $n-1$ variables (but not necessarily isotropic). This result was originally obtained in [1], Theorem 8.1.

THEOREM 1 : *Let* $\mathcal{A}_\varepsilon(x, y) = \mathcal{A}(x/\varepsilon, y)$ *where* $\mathcal{A}$ *is periodic in the first* $n-1$ *variables with period* $\mathbf{P}$. *Let* u_ε *denote the solution of* (1), (2) *and* (3). *If*

$$g_\varepsilon^\pm \to g_0^\pm \quad \text{in the space } H^{-\frac{1}{2}}(\omega), \text{ and}$$
$$\mathcal{F}_\varepsilon(x, \varepsilon_l\, y) \to \mathcal{F}_0(x, y) \quad \text{in the space } \mathcal{W}',$$

then the (entire) sequence $u_\varepsilon(x, \varepsilon y)$ *converges weakly in* $\mathcal{W}$ *to the solution* $u_0(x)$ *of the following* $n-1$ *dimensional problem with constant coefficients :*

$$\begin{cases} -\operatorname{div}_x (\mathcal{A}_0 \nabla_x u_0) = \bar{\mathcal{F}}_0 + g_0^+ + g_0^- & \text{in } \omega \\ u_{0 \mid \partial\omega} = 0 . \end{cases}$$

The symmetric matrix $\mathcal{A}_0$ *is given by*

$$(29)\quad \frac{1}{2}(\mathcal{A}_0 \eta, \eta) = \min \frac{1}{2} -\!\!\!\!\!\!\int_Q \Big(\sum_{\alpha, \beta} \mathcal{A}_{\alpha\beta}(\partial_\alpha \phi + \eta_\alpha)(\partial_\beta \phi + \eta_\beta) + {} + 2 \sum_\beta \mathcal{A}_{n\beta}(\partial_y \phi)(\partial_\beta \phi + \eta_\beta) + \mathcal{A}_{nn}(\partial_y \phi)(\partial_y \phi) \Big)\, dX\, dy$$

for every η *in* $\mathbb{R}^{n-1}$, *where the minimum is taken over the* ϕ*'s in* $H^1(\mathbf{Q})$ *which are periodic with respect to the first* $n-1$ *variables.* $\square$

Remarks : Solving the minimum problem in (29) yields a solution ϕ_η in terms of which $(\mathcal{A}_0 \eta, \eta)$ can be expressed explicitly. We prefer the variational formulation since it is more useful in the sequel.

The proof of Theorem 1′ may be obtained from the previous analysis by making the specific choice $(\langle \eta, x \rangle + \varepsilon \phi_\eta(x/\varepsilon, y))\, \Psi(x)$ for $V_\varepsilon(x, y)$ in section 4 (where Ψ is any smooth cut-off function).

- *Determination of the upper bound.*

The case of the upper bound is very simple. Since $\mathcal{A}(X, y)$ is isotropic, $= a(X, y)\, \mathbb{I}_n$, (29) leads to

$$\frac{1}{2}(\mathcal{A}_0 \eta, \eta) = \min_\phi \frac{1}{2} -\!\!\!\!\!\!\int_Q a(X, y) \left(\sum_{\alpha = 1}^{n-1} (\partial_\alpha \phi + \eta_\alpha)^2 + (\partial_y \phi)^2 \right) dX\, dy .$$

By choosing $\phi \equiv 0$ we get

$$(\mathcal{A}_0 \eta, \eta) \leq -\!\!\!\!\!\!\int_Q a(X, Y) |\eta|^2\, dX\, dy, \quad \eta \in \mathbb{R}^{n-1} .$$

For a of the form $a(X, y) = (1 - \chi(y))\, b_1 + \chi(y)\, b_2$, this immediately yields $\mathscr{A}_0 \leq \mu \mathbb{I}_{n-1}$ where $\mu = (1 - \theta)\, b_1 + \theta b_2$ is the arithmetic mean of b_1 and b_2. On the other hand, if $\chi \equiv \chi(y)$ is independent of X, then the minimum above is achieved for $\phi \equiv 0$, so that $\mathscr{A}_0$ is exactly $\mu \mathbb{I}_{n-1}$. In short, the upper bound is attained with any layering parallel to the midplane ω.

- *Determination of the lower bounds.*

It is clear that $b_1 \mathbb{I}_{n-1} \leq \mathscr{A}_0 \leq \mu \mathbb{I}_{n-1}$. It now suffices to prove

$$\operatorname{Tr}\left((\mathscr{A}_0 - b_1 \mathbb{I}_{n-1})^{-1}\right) \leq (n-2)(\mu - b_1)^{-1} + (h - b_1)^{-1}$$

since, in combination with the above inequalities, this implies that $\mathscr{A}_0 \geq h \mathbb{I}_{n-1}$. We use a variant of the Hashin-Shtrikman method as presented in [5].

For any fixed $\gamma \in (0, b_1)$ the formula (29) for $\mathscr{A}_0$ gives that

$$\begin{aligned} \frac{1}{2}(\mathscr{A}_0 \eta, \eta) &= \min_{\phi} \frac{1}{2} \fint_Q a(X, y) \left(\sum_{\alpha=1}^{n-1} (\partial_\alpha \phi + \eta_\alpha)^2 + (\partial_y \phi)^2 \right) dX\, dy \\ &= \min_{\phi} \frac{1}{2} \fint_Q \left((a(X, y) - \gamma) \left(\sum_{\alpha=1}^{n-1} (\partial_\alpha \phi + \eta_\alpha)^2 \right) \right. \\ &\quad \left. + a(X, y)(\partial_y \phi)^2 + \gamma \sum_{\alpha=1}^{n-1} (\partial_\alpha \phi + \eta_\alpha)^2 \right) dX\, dy\,. \end{aligned} \tag{30}$$

As a consequence of convex duality, we get

$$\begin{aligned} \frac{1}{2}(\mathscr{A}_0 \eta, \eta) = \min_{\phi} \max_{\sigma} \fint_Q & \left\{ \sum_{\alpha=1}^{n-1} \left(\sigma_\alpha \cdot (\partial_\alpha \phi + \eta_\alpha) \right. \right. \\ & \left. - \frac{1}{2} (a(X, y) - \gamma)^{-1} \sigma_\alpha^2 \right) + \sigma_n \cdot \partial_y \phi - \frac{1}{2} (a(X, y))^{-1} \sigma_n^2 \\ & \left. + \gamma/2 \sum_{\alpha=1}^{n-1} (\partial_\alpha \phi + \eta_\alpha)^2 \right\} dX\, dy\,, \end{aligned}$$

the latter max being over σ in $L^2(\mathbf{Q})^n$. Now we use the Mini Max inequality (here it is actually an equality), to obtain :

$$\begin{aligned} \frac{1}{2}(\mathscr{A}_0 \eta, \eta) \geq \sup_{\sigma} \inf_{\phi} \fint_Q & \left\{ \sum_{\alpha=1}^{n-1} \left(\sigma_\alpha \cdot (\partial_\alpha \phi + \eta_\alpha) \right. \right. \\ & \left. - \frac{1}{2} (a(X, y) - \gamma)^{-1} \sigma_\alpha^2 \right) + \sigma_n \cdot \partial_y \phi - \frac{1}{2} (a(X, y))^{-1} \sigma_n^2 \\ & \left. + \gamma/2 \sum_{\alpha=1}^{n-1} \left((\partial_\alpha \phi)^2 + 2\, \partial_\alpha \phi \cdot \eta_\alpha + (\eta_\alpha)^2 \right) \right\} dX\, dy\,. \end{aligned}$$

Note, however, that due to the periodicity, the integral

$$\fint_{\mathbf{Q}} \sum_{\alpha=1}^{n-1} \partial_\alpha \phi \cdot \eta_\alpha \, dX \, dy$$

is zero, so

$$\begin{aligned} \frac{1}{2}\left((\mathscr{A}_0 - \gamma)\, \eta, \eta\right) \geqslant \sup_{\sigma} \inf_{\phi} \fint_{\mathbf{Q}} & \left\{ \sum_{\alpha=1}^{n-1} \sigma_\alpha \cdot (\partial_\alpha \phi + \eta_\alpha) + \sigma_n \cdot \partial_y \phi \right. \\ & + \gamma/2 \sum_{\alpha=1}^{n-1} (\partial_\alpha \phi)^2 \\ & \left. - \frac{1}{2}\left((a(X, y) - \gamma)^{-1} \sum_{\alpha=1}^{n-1} \sigma_\alpha^2 + (a(X, y))^{-1} \sigma_n^2 \right) \right\} dX \, dy \, . \end{aligned}$$

If σ_n is not identically zero, the infimum in ϕ is $-\infty$, which is of no interest when computing the supremum in σ ; on the other hand if σ_n is zero and σ smooth enough, then the infimum in ϕ is achieved for ϕ_σ satisfying

$$(31) \qquad \begin{cases} -\gamma \nabla_x^2 \phi_\sigma = \operatorname{div}_x \sigma \\ \phi_\sigma \ \mathbf{P}\text{-periodic in } x \, . \end{cases}$$

If σ is only $L^2(\mathbf{Q})$, then ϕ_σ belongs only to $L^2(-\frac{1}{2}, \frac{1}{2}; H^1_{\mathrm{per}}(\mathbf{P}))$ and, therefore, is not admissible. In this case, the infimum is not achieved, however, by continuity, it still has the same expression in terms of ϕ_σ. The result is

$$\begin{aligned} \frac{1}{2}\left((\mathscr{A}_0 - \gamma)\, \eta, \eta\right) \geqslant \sup_{\sigma} \fint_{\mathbf{Q}} \sum_{\alpha=1}^{n-1} & \left(\frac{1}{2} \sigma_\alpha \, \partial_\alpha \phi_\sigma - \frac{1}{2} (a(X, y) - \gamma)^{-1} \sigma_\alpha^2 \right. \\ & \left. + \sigma_\alpha \, \eta_\alpha \right) dX \, dy \, , \end{aligned}$$

the supremum being taken over σ in $L^2(\mathbf{Q})$.

We now restrict our choice of σ's to the form $\sigma = \chi\xi$, where $\xi = (\xi_1, \ldots, \xi_{n-1}, 0)$ is a constant vector and χ is the characteristic functions of the set $\{a = b_2\}$. The result is :

$$\begin{aligned} \frac{1}{2}\left((\mathscr{A}_0 - \gamma)\, \eta, \eta\right) \geqslant \sup_{\xi} \fint_{\mathbf{Q}} \chi \sum_{\alpha=1}^{n-1} & \left(\frac{1}{2} \xi_\alpha \cdot \partial_\alpha \phi_\sigma + \xi_\alpha \, \eta_\alpha \right. \\ & \left. - \frac{1}{2} (b_2 - \gamma)^{-1} \xi_\alpha^2 \right) dX \, dy \, . \end{aligned}$$

In this inequality, we can let γ approach b_1 :

$$\text{(32)} \quad \frac{1}{2}((\mathscr{A}_0 - b_1)\eta, \eta) \geqslant \sup_{\xi} \int_{\mathbf{Q}} \chi \sum_{\alpha=1}^{n-1} \left(\frac{1}{2}\xi_\alpha \cdot \partial_\alpha \phi_\sigma\right) dX\,dy + \theta \sum_{\alpha=1}^{n-1} \left(\xi_\alpha \eta_\alpha - \frac{1}{2}(b_2 - b_1)^{-1} \xi_\alpha^2\right).$$

Consider now the (unique) periodic solution ψ of

$$\text{(33)} \qquad -\nabla_X^2 \psi = \chi - \fint_{\mathbf{P}} \chi(X, y)\, dX, \quad \fint_{\mathbf{P}} \psi(X, y)\, dX \equiv 0,$$

(y appears just as a parameter).

It is clear that ϕ_σ is equal to $b_1^{-1} \sum_{\beta=1}^{n-1} \partial_\beta \psi\, \xi_\beta$ and that $\frac{1}{2} \sum_{\alpha=1}^{n-1} \xi_\alpha\, \partial_\alpha \phi_\sigma = \frac{1}{2} b_1^{-1} \sum_{\alpha,\beta=1}^{n-1} \partial_{\alpha\beta}^2 \psi\, \xi_\alpha\, \xi_\beta$. Consequently, (32) gives :

$$\frac{1}{2}((\mathscr{A}_0 - b_1)\eta, \eta) \geqslant \fint_{\mathbf{Q}} \frac{1}{2} \chi b_1^{-1} \sum_{\alpha,\beta=1}^{n-1} \partial_{\alpha\beta}^2 \psi\, \xi_\alpha\, \xi_\beta\, dX\,dy + \theta \sum_{\alpha=1}^{n-1} \left(\xi_\alpha \eta_\alpha - \frac{1}{2}(b_2 - b_1)^{-1} \xi_\alpha^2\right), \quad \text{for any } \xi \text{ in } \mathbb{R}^{n-1}.$$

This can be rewritten as

$$\text{(34)} \quad \frac{1}{2}((\mathscr{A}_0 - b_1)\eta, \eta) - \theta \sum_{\alpha=1}^{n-1} \eta_\alpha \xi_\alpha \geqslant \geqslant \frac{1}{2} \sum_{\alpha,\beta=1}^{n-1} \xi_\alpha \xi_\beta \left(b_1^{-1} \fint_{\mathbf{Q}} \chi\, \partial_{\alpha\beta}^2 \psi\, dX\,dy - \theta (b_2 - b_1)^{-1} \delta_{\alpha\beta}\right),$$

which holds true for every pair of $n-1$ vectors η and ξ.

For fixed ξ, the minimum of the left hand side of (34) is achieved for $\eta = \theta(\mathscr{A}_0 - b_1)^{-1} \xi$, therefore

$$\text{(35)} \quad -\frac{1}{2}\theta^2((\mathscr{A}_0 - b_1)^{-1}\xi, \xi) \geqslant \geqslant \frac{1}{2} \sum_{\alpha,\beta=1}^{n-1} \xi_\alpha \xi_\beta \left(b_1^{-1} \fint_{\mathbf{Q}} \chi\, \partial_{\alpha\beta}^2 \psi\, dX\,dy - \theta (b_2 - b_1)^{-1} \delta_{\alpha\beta}\right).$$

Inequality (35) is between non negative symmetric matrices ; taking ξ to be each basis vector e_α successively and adding up, we get an inequality between the traces of these two matrices :

$$\theta^2 \operatorname{Tr}\left((\mathscr{A}_0 - b_1)^{-1}\right) \leqslant (n-1)\,\theta\,(b_2 - b_1)^{-1} - b_1^{-1} \fint_{\mathbf{Q}} \chi\, \nabla^2 \psi \, dX\, dy\,.$$

Because of (33) this becomes

$$(36)\quad \theta^2 \operatorname{Tr}\left((\mathscr{A}_0 - b_1)^{-1}\right) \leqslant$$
$$\leqslant (n-1)\,\theta\,(b_2 - b_1)^{-1} + b_1^{-1} \fint_{\mathbf{Q}} \chi \left(\chi - \fint_{\mathbf{P}} \chi(X, y)\, dX\right) dX\, dy\,.$$

To determine a bound which is independent of χ, we are left with the simple question of finding the maximum of the right-hand side of (36) with respect to χ subject to the constraint $\fint_{\mathbf{Q}} \chi(X, y)\, dX\, dy = \theta$. Setting $\rho(y) = \fint_{\mathbf{P}} \chi(X, y)\, dX$, we see that $\fint_{\mathbf{Q}} \chi\left(\chi - \fint_{\mathbf{P}} \chi(X, y)\, dX\right) dX\, dy$ is equal to $\theta - \int_{\left(-\frac{1}{2}, -\frac{1}{2}\right)} \rho(y)^2\, dy$; ρ is constrained by $0 \leqslant \rho \leqslant 1$, and $\int_{\left(-\frac{1}{2}, -\frac{1}{2}\right)} \rho(Y)\, dY = \theta$. It is straightforward to find that the extreme ρ is constant and equal to θ. The corresponding bound is as stated in Theorem 2 :

$$(37)\quad \operatorname{Tr}\left((\mathscr{A}_0 - b_1)^{-1}\right) \leqslant (n-1)\,\theta^{-1}(b_2 - b_1)^{-1} + \theta^{-1}\, b_1^{-1}(1-\theta) =$$
$$= (n-2)(\mu - b_1)^{-1} + (h - b_1)^{-1}\,,$$

where $\mu = (1-\theta)\, b_1 + \theta b_2$ is the arithmetic mean and $h = (1 - \theta/b_1 + \theta/b_2)^{-1}$ the harmonic mean of b_1 and b_2.

Since the extreme choice for ρ is a constant, one can expect to achieve this lower bound with χ's (and corresponding geometries) that are independent of y. In that case formula (29) for $\mathscr{A}_0$ becomes

$$\frac{1}{2}\,(\mathscr{A}_0\, \eta, \eta) = \min \frac{1}{2} \fint_{\mathbf{P}} a \sum_\alpha \left(\partial_\alpha \phi + \eta_\alpha\right)^2 dX\,,$$

the minimum being taken over $\phi \in H^1_{\text{per}}(\mathbf{P})$. This is the formula for the effective diffusivity of a rapidly varying periodic composite in $n-1$ variables, and it is well known that (37) is the optimal lower bound in that situation, being achieved for example by $n-1$ layering (*cf.* [11]).

• *Filling in the set between the bounds.*

For simplicity, we consider the case of $n = 3$. According to Theorem 2, the pair of eigenvalues of the matrix $\mathcal{A}_0$ must, for fixed volume fraction θ, lie inside or on the boundary of a shaded area like that shown in figure 2.

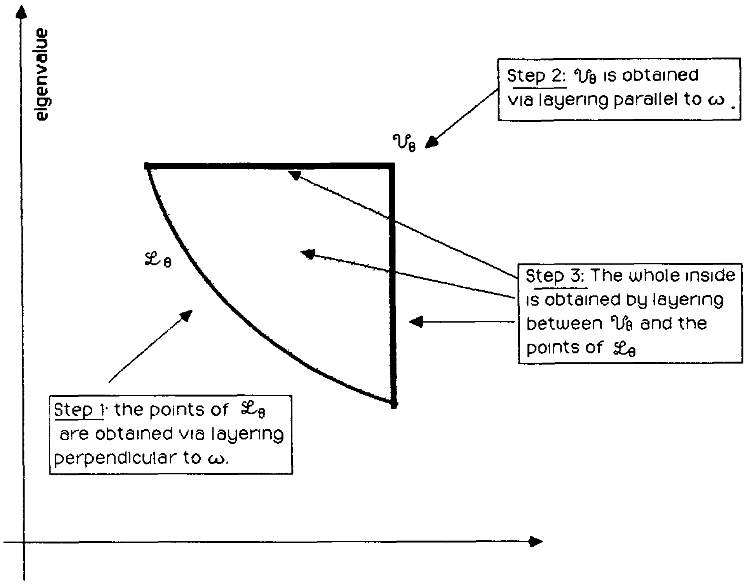

Figure 2.

The curve $\mathcal{L}_\theta$ corresponds to the lower bounds, the point $\mathcal{U}_\theta = (\mu, \mu)$ corresponds to the upper bound. The end points of the curve $\mathcal{L}_\theta$ are (h, μ) and (μ, h). As stated earlier, the curve $\mathcal{L}_\theta$ can be achieved by certain rank-2 composites (layers of layers perpendicular to ω as in *fig.* 3), the end points correspond to the degenerate case of one set of parallel layers. The matrix $\mu \mathbb{I}_2$ can be realized simply by a fixed double layer parallel to ω. Consider the « mix » of a matrix A with eigenvalues (h, μ) (the top end point of $\mathcal{L}_\theta$) with the matrix $\mu \mathbb{I}_2$ (the point $\mathcal{U}_\theta$), in proportions $1 - \rho$ and ρ. Note that this mix still has volume fraction $(1 - \rho)\theta + \rho\theta = \theta$ of material b_2. The eigendirections of A may be any orthogonal set of directions, but let the mixing consist of layers perpendicular to ω and parallel to the eigendirection of the largest eigenvalue μ of A. In this way, we obtain an effective matrix $\mathcal{A}_0$ which inherits the same eigendirections and has one eigenvalue μ and the other $\nu = ((1 - \rho)h^{-1} + \rho\mu^{-1})^{-1}$. It is clear that ν varies between h and μ as ρ varies between 0 and 1. This procedure therefore fills in all eigenvalues corresponding to the top

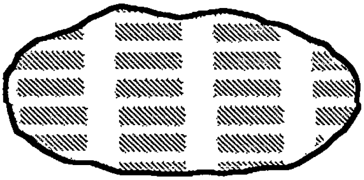

Figure 3. — Layering in two dimensions.

horizontal part of the boundary of the shaded domain. Similarly, we can fill in the right vertical boundary as well as the whole interior of the domain by layering between the matrix $\mu \mathbb{I}_2$ and every A on $\mathscr{L}_\theta$. In summary, we have sketched how to approximate any symmetric matrix whose eigenvalues lie inside or on the boundary of the shaded area by the effective diffusivity matrix of a periodic composite with two isotropic components.

A similar procedure would work for $n > 3$. For more details concerning the construction of composites by layering, we refer to [7] and [11]. □

6. APPENDIX : FIRST ORDER CORRECTORS

In this appendix, we reexamine the result of Theorem 1 with special emphasis on so-called correctors. This requires extra notation (following [10]).

For any subdomain $\omega' \subset\subset \omega$, consider a function $\psi(X)$ with compact support in ω and identically equal to 1 on ω'. Theorem 1 states that if we restrict ourselves to the usual « universal subsequence » $\{\varepsilon_l\}_{l=0}^{\infty}$, and consider the solution V_{ε_l} of the rescaled problem with right-hand side $\mathscr{F}_\varepsilon = G = \mathbf{S}^{-1}(x_\alpha \psi(x))$, $g_\varepsilon^\pm = 0$, then $\nabla_x V_{\varepsilon_l}$ converges weakly to $\nabla_x(x_\alpha \psi)$ in $L^2(\Omega_1)$. In particular, $\nabla_x V_{\varepsilon_l}$ converges weakly to e_α in $L^2\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$. This last result is independent of the choice of ψ, provided $\psi \equiv 1$ on ω'. In the following ∇^ε denotes the rescaled full gradient $(\nabla_x, 1/\varepsilon\, \partial_y)$. Let $\mathbf{P}_\alpha^{\varepsilon_l}(x, y)$ be the restriction of $\nabla^{\varepsilon_l} V_{\varepsilon_l}$ to $\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Covering ω (except for a set of measure zero) by a disjoint countable union of such subdomains, we may define $\mathbf{P}_\alpha^{\varepsilon_l}$ on the whole of Ω_1. Let $\mathbf{P}^{\varepsilon_l}$ be the $n \times (n-1)$-matrix-valued function whose α-th column is $\mathbf{P}_\alpha^{\varepsilon_l}$. It is clear that the family of subdomains used in the definition can be chosen so that $\mathbf{P}^{\varepsilon_l}$ converges weakly in $L^2\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ $\forall \omega' \subset\subset \omega$ (to a matrix the first $n-1$ rows of which equals $\mathbb{I}_{n-1}$). Note : here we also use that the sequence $\{1/\varepsilon_l\ \partial_y U_{\varepsilon_l}\}$ is weakly convergent. The main result of this section is :

PROPOSITION A.1 : *Let U_{ε_l} be the solution of the rescaled problem* (12) *with $U_{\varepsilon_l} \rightharpoonup U_0$. Then $\nabla^{\varepsilon_l} U_{\varepsilon_l} - \mathbf{P}^{\varepsilon_l} \nabla_x U_0$ converges to zero strongly in $L^1\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$, $\omega' \subset\subset \omega$, as ε_l goes to zero.*

Sketch of proof: Let g be in $\mathscr{D}(\omega)^{n-1}$. Let ω' be an arbitrary subdomain $\subset\subset \omega$ and let ϕ be in $\mathscr{D}^+(\omega)$ with $\phi \equiv 1$ on ω'.

Consider

$$\int_{\Omega_1} \phi \mathcal{A}_\varepsilon (\nabla^\varepsilon U_\varepsilon - \mathbf{P}^\varepsilon g) \cdot (\nabla^\varepsilon U_\varepsilon - \mathbf{P}^\varepsilon g)\, dx\, dy\,. \tag{A.1}$$

Each of the four terms obtained in the expansion of (A.1) can be evaluated and its limit along the sequence $\{\varepsilon_l\}$ can be determined, by integration by parts similar to that of section 4 :

$$\int_{\Omega_1} \phi (\mathcal{A}_\varepsilon \nabla^\varepsilon U_\varepsilon \cdot \nabla^\varepsilon U_\varepsilon)\, dx\, dy \to \int_\omega (\bar{\xi}_0 \cdot \nabla_x U_0)\, \phi\, dx = $$
$$= \int_\omega (\mathscr{A}_0 \nabla_x U_0 \cdot \nabla_x U_0)\, \phi\, dx,$$

$$\int_{\Omega_1} \phi (\mathcal{A}_\varepsilon \mathbf{P}^\varepsilon g \cdot \nabla^\varepsilon U_\varepsilon)\, dx\, dy = \int_{\Omega_1} \phi (\mathcal{A}_\varepsilon \nabla^\varepsilon U_\varepsilon \cdot \mathbf{P}^\varepsilon g)\, dx\, dy =$$
$$= \int_{\Omega_1} \phi \sum_\alpha g_\alpha (\mathcal{A}_\varepsilon \nabla^\varepsilon U_\varepsilon \cdot \mathbf{P}^\varepsilon e_\alpha)\, dx\, dy \to \int_\omega \phi \sum_\alpha g_\alpha (\mathscr{A}_0 \nabla_x U_0 \cdot e_\alpha)\, dx$$
$$= \int_\omega \phi (\mathscr{A}_0 \nabla_x U_0 \cdot g)\, dx = \int_\omega \phi (\mathscr{A}_0 g \cdot \nabla_x U_0)\, dx, \quad \text{and}$$
$$\int_{\Omega_1} \phi (\mathcal{A}_\varepsilon \mathbf{P}^\varepsilon g \cdot \mathbf{P}^\varepsilon g)\, dx\, dy \to \int_\omega \phi (\mathscr{A}_0 g \cdot g)\, dx$$

as ε goes to zero along the sequence $\{\varepsilon_l\}$.

Therefore we have :

$$\int_{\Omega_1} \phi \mathcal{A}_{\varepsilon_l} (\nabla^{\varepsilon_l} U_{\varepsilon_l} - \mathbf{P}^{\varepsilon_l} g) \cdot (\nabla^{\varepsilon_l} U_{\varepsilon_l} - \mathbf{P}^{\varepsilon_l} g)\, dx\, dy \to \tag{A.2}$$
$$\to \int_\omega \phi (\mathscr{A}_0 (\nabla_x U_0 - g) \cdot (\nabla_x U_0 - g))\, dx\,.$$

Choose g so that $\|\nabla_x U_0 - g\|_{L^2(\omega)} < \delta$. Based on (A.2) and the coercivity of $\mathcal{A}_\varepsilon$ we obtain that

$$\left\|\nabla^{\varepsilon_l} U_{\varepsilon_l} - \mathbf{P}^{\varepsilon_l} g\right\|_{L^2\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)} \leqslant \delta + C\|\nabla_x U_0 - g\|_{L^2(\omega)} \leqslant C\delta ,$$

for ε_l sufficiently small.

Consequently, for ε_l sufficiently small

$$\begin{aligned}\left\|\nabla^{\varepsilon_l} U_{\varepsilon_l} - \mathbf{P}^{\varepsilon_l} \nabla_x U_0\right\|_{L^1\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)} \leqslant \\ &\leqslant C\left\|\nabla^{\varepsilon_l} U_{\varepsilon_l} - \mathbf{P}^{\varepsilon_l} g\right\|_{L^2\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)} + \\ &\quad + \left\|\mathbf{P}^{\varepsilon_l} g - \mathbf{P}^{\varepsilon_l} \nabla_x U_0\right\|_{L^1\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)} \\ &\leqslant C\delta + \|\mathbf{P}^{\varepsilon_l}\|_{L^2\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)} \|g - \nabla_x U_0\|_{L^2(\omega')} \\ &\leqslant C\delta\end{aligned}$$

(the constant C depends on ω'). Since δ is arbitrary, this shows that

$$\left\|\nabla^{\varepsilon_l} U_{\varepsilon_l} - \mathbf{P}^{\varepsilon_l} \nabla_x U_0\right\|_{L^1\left(\omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)} \to 0$$

as ε_l goes to zero. ■

Denote by $\mathbf{Q}^\varepsilon$ the matrix-valued function of x defined by $\overline{\mathcal{A}_\varepsilon \mathbf{P}^\varepsilon}$. It is easy to check that the first $n-1$ rows of $\mathbf{Q}^{\varepsilon_l}$ converge weakly to $\mathscr{A}_0$ in $L^2(\omega')$, $\omega' \subset \omega$. Since

$$\overline{\mathcal{A}_\varepsilon(\nabla^\varepsilon U_\varepsilon - \mathbf{P}^\varepsilon \nabla_x U_0)} = \overline{\mathcal{A}_\varepsilon \nabla^\varepsilon U_\varepsilon} - \mathbf{Q}^\varepsilon \nabla_x U_0 ,$$

we obtain :

PROPOSITION A.2 : *Under the same hypotheses as in proposition* A.1 $\overline{\mathcal{A}_{\varepsilon_l} \nabla^{\varepsilon_l} U_{\varepsilon_l}} - \mathbf{Q}^{\varepsilon_l} \nabla_x U_0$ *converges to zero in* $L^1_{\mathrm{loc}}(\omega)$. ■

Remark : The functions $\mathbf{P}^\varepsilon$ and $\mathbf{Q}^\varepsilon$ are by no means unique. We chose one specific way of constructing this pair.

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