RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

A. C. DENEUVY

Theoretical study and optimization of a fluid-structure interaction problem

RAIRO – Modélisation mathématique et analyse numérique, tome 22, nº 1 (1988), p. 75-92.

<http://www.numdam.org/item?id=M2AN_1988__22_1_75_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 22, n° 1, 1988, p. 75 à 92)

THEORETICAL STUDY AND OPTIMIZATION OF A FLUID-STRUCTURE INTERACTION PROBLEM (*)

by A. C. DENEUVY [1]

Communicated by R. TEMAM

Abstract. — *In the present paper, the small harmonic vibrations of an elastoacoustic coupled system are under study. A symmetric variational formulation is presented, which particularly suits the model problem. The mathematical study is derived and the existence of a real spectrum of eigenvalues is proved. Then, the problem of designing the coupled structure such as to obtain as large a gap as possible in the eigenvalues spectrum is considered, in order to avoid resonance for a wide range of external excitation frequencies. An optimality criterion method is applied, using the structure thickness distribution as a control variable.*

Résumé. — *Dans cet article, on étudie les petites vibrations harmoniques d'un système couplé élasto-acoustique. On présente une formulation variationnelle symétrique particulièrement adaptée au cas étudié, et dont l'étude mathématique conduit à la démonstration de l'existence d'un spectre réel de valeurs propres. On considère ensuite le problème d'« optimum design » de la structure couplée, pour créer le plus grand trou possible dans le spectre des fréquences couplées. On utilise une méthode par critère d'optimalité, la variable de contrôle étant l'épaisseur de la structure.*

SECTION 1 : PRESENTATION

1.1. Presentation of the physical model

The present work studies the small harmonic vibrations of an enclosure which is completely filled up with fluid. The enclosure has a rectangular section in the plan referred to as (ox_1, ox_2) and is of infinite dimension in the perpendicular direction denoted by ox_3 (see *fig.* 1). Therefore the

(*) Received in December 1986, revised in April 1987.
The present work had been done within the framework of an A.N.R.T. « contrat CIFRE » between METRAVIB R. D. S. and Ecole Centrale de Lyon.
[1] C.R.N.S. U.A. 040740 Département M.I.S., Ecole Centrale de Lyon.

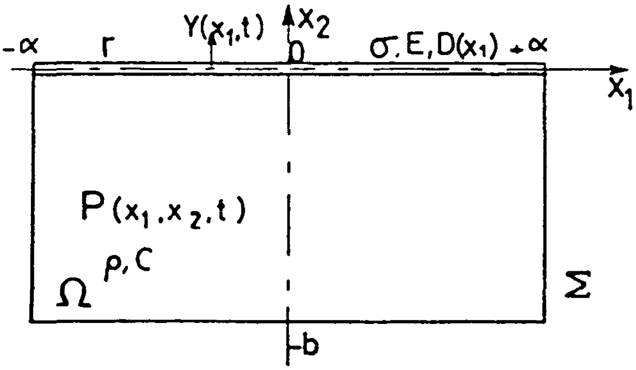

Figure 1.

corresponding state equations are those of a bidimensional elastoacoustic interaction problem in a bounded medium. The fluid is assumed to be ideal, irrotational and compressible. The structure is resolved into two parts :

— its top side is identified to a transversally vibrating plate, which is supposed to be thin, elastic, homogeneous and of variable thickness ;
— the rest of the structure is assumed to be rigid. Consequently, the (ox_1, ox_2) — section of the vibrating part, namely Γ, obeys the clamped-clamped beam equation.

Gravity effects are neglected and assumptions leading to linear equations are made (i.e. small displacements and perturbations, fixed geometry).

1.2. Notations

$\Omega =]-a, +a[\times]-b, 0[$ represents the bounded domain occupied by the fluid. Its boundary $\partial\Omega$ is splitted into two parts Γ and Σ, where $\Gamma =]-a, +a[\times \{0\}$ and $\Sigma = \partial\Omega - \Gamma$. $\vec{n}_\Gamma$ (resp. $\vec{n}_\Sigma$) represents the outward normal vector to Γ (resp. to Σ).

From now on, Γ will be identified to $]-a, +a[$.

ρ (resp. σ) is the constant density of the fluid (resp. of the structure).
c is the constant sound celerity in the fluid.
E is the constant Young modulus of the structure.
D is the variable plate thickness distribution.

Let ω be the harmonic pulsation for the coupled system which can be described by the following variables :

$p(x_1, x_2)\, e^{j\omega t}$ is the pressure field in the fluid ;

$y(x_1)\, e^{j\omega t}$ is the structure transversal deflection ;

$r(x_1)\, e^{j\omega t}$ is the dynamic reaction force .

N.B. : For the sake of brievity, the time dependence in $e^{j\omega t}$ will be omitted in the equations.

The present choice of variables deserves an explanation. It has been attempted, for a long time, to formulate interaction problems, in a symmetric way [1, 2, 3], by means of a restricted number of unknowns. In this optic, three-fields representations have been introduced (see [4, 5]). In the present paper, the basic idea, due to R. Ohayon [6], is to use a mixed description for the structure involving a dynamic dual variable and a scalar representation for the fluid, whereas, in the above mentionned papers, a mixed representation for the fluid was used, with a primal description for the structure. For many problems it might be more interesting to use the present description which needs two unknowns on the boundary and only one in the domain.

1.3. The governing equations

The hereabove hypothesis lead to the following set of equations :
The pressure field p obeys Helmholtz equation in Ω

$$\Delta p + \frac{\omega^2}{c^2} p = 0 , \tag{1}$$

with boundary interface conditions :

$$\partial p \mid \partial n \mid_{\Sigma} = 0 \quad \text{on} \quad \Sigma , \tag{2}$$

$$\partial p \mid \partial n \mid_{\Gamma} = \omega^2 \rho y \quad \text{on} \quad \Gamma , \tag{3}$$

the following compatibility condition, derived by a Green formula applied to (1) and (2), (3), must be fulfilled

$$\int_{\Omega} p \, d\Omega + \rho c^2 \int_{\Gamma} y \, d\gamma = 0 . \tag{4}$$

On the structure Γ, the dual variable is defined as the inertia reaction force :

$$r = \omega^2 \sigma D y , \tag{5}$$

the dynamic equilibrium equation is

$$\left(\frac{ED^3}{12} y'' \right)'' = r + p \mid_{\Gamma} , \tag{6}$$

and the clamping boundary conditions are

$$y(\pm a) = y'(\pm a) = 0 . \tag{7}$$

SECTION 2 : THEORETICAL STUDY

2.1. Definitions

Assume that function D, which represents the structure thickness distribution, belongs to the admissible set denoted by $\mathcal{U}_{ad}$ and defined by

$$\mathcal{U}_{ad} = \{D \in L^{\infty}(\Gamma) ; 0 < D_{\min} \leqslant D \leqslant D_{\max}, \text{ a.e. on } \Gamma\} ,$$

where $D_{\min}$ and $D_{\max}$ are given positive numbers.

The usual Sobolev spaces $L^2(\Omega)$, $L^2(\Gamma)$, $H^1(\Omega)$ and $H_0^2(\Gamma)$ will be used, endowed with natural Hilbert scalar product. The associated norms are respectively denoted by $|u|_{0,\Omega}$, $|u|_{0,\Gamma}$, $|u|_{1,\Omega}$ and $|u|_{2,\Gamma}$. Note that, in $H_0^2(\Gamma)$, the semi-norm $|u''|_{0,\Gamma}$ is equivalent to $|u|_{2,\Gamma}$ according to Korn's Lemma (*cf.* [7]). Last, let $L_0^2(\Omega)$ be defined by

$$L_0^2(\Omega) = \left\{u \in L^2(\Omega) ; \int_{\Omega} u \, d\Omega = 0\right\} .$$

2.2 Variational formulation

Define $\mathcal{C}$, [$\mathcal{C}$ standing for Coupling space], by :

$$\mathcal{C} = \Big\{X = (p, r, y) \in H^1(\Omega) \times L^2(\Gamma) \times H_0^2(\Gamma) ; \quad \int_{\Omega} p \, d\Omega + \rho c^2 \int_{\Gamma} y \, d\gamma = 0\Big\} .$$

$\mathcal{C}$ is a Hilbert space equipped with the natural scalar product $(p, q)_{1,\Omega} + (r, s)_{0,\Gamma} + (y'', z'')_{0,\Gamma}$ and the associated norm $\|X\|_{\mathcal{C}}$.

The variational formulation for equation (1) to (7) is given by :

Find ω^2 in $\mathbb{R}^*$ and $X = (p, r, y)$ in $\mathcal{C}$, $X \neq 0$, such that, for every $Y = (q, s, z)$ in $\mathcal{C}$,

$$\frac{1}{\rho} \int_{\Omega} \nabla p \, \nabla q \, d\Omega - \omega^2 \left\{ \frac{1}{\rho c^2} \int_{\Omega} pq \, d\Omega + \int_{\Gamma} yq \, d\gamma \right\} = 0 , \tag{8}$$

$$\int_{\Gamma} \frac{rs}{\sigma D} d\gamma - \omega^2 \int_{\Gamma} ys \, d\gamma = 0 , \tag{9}$$

$$- \int_{\Gamma} \frac{ED^3}{12} y'' z'' \, d\gamma + \int_{\Gamma} zp \, d\gamma + \int_{\Gamma} zr \, d\gamma = 0 . \tag{10}$$

N.B. : This formulation is symmetric in (p, r, y) versus (q, s, z).

In this section, the existence of a real discrete spectrum of eigenvalues for problem (8), (9), (10) is proved.

The variational formulation (8), (9), (10) is set into the form of a spectral problem in a constrained space, on which classical spectral analysis applies.

The most interesting point in that proof is to show off how to settle down a modal synthesis method which can be used to discretise the problem. This point is developed in another paper [9] by the author (see also [8]).

2.3 Decomposition of Coupling space $\mathscr{C}$

Let (p, r, y) be an element of $\mathscr{C}$. The pressure field p can be uniquely decomposed into :

— a « purely acoustic » part, denoted by $\tilde{p}$ and characterized by zero-mean over Ω ;

— and a « pneumatic » constant contribution denoted by $p_s(y)$ which depends upon the structure displacement y ; the value of this constant function is the pressure field mean over Ω.

So,

$$p = \tilde{p} - P_S(y) \tag{11}$$

(the minus sign in (11) has been chosen for convenience for later calculus).

As p belongs to $H^1(\Omega)$ and obeys compatibility condition (4), it can be deduced that :

$$\tilde{p} \text{ belongs to } H^1(\Omega) \cap L_0^2(\Omega)\,,$$

and that

$$P_S(y) = \frac{\rho c^2}{\text{meas } \Omega} \int_\Gamma y \, d\gamma\,. \tag{12}$$

(For the sake of brievity, the notation P_S will stand for the function and its constant value in Ω).

The « physical decomposition » (11) of p naturally induces a decomposition of the coupling space $\mathscr{C}$ into a direct sum of two subspaces denoted by $\mathscr{C}_{ST}$ and $\mathscr{C}_{AC}$:

— $\mathscr{C}_{ST}$, (ST for structure), is formed with the elements of $\mathscr{C}$ for which the pressure field is defined by relation (12) :

$$\mathscr{C}_{ST} = \{(p, r, y) \in \mathscr{C}, p = -P_S(y)\}\,.$$

Consequently, $\mathscr{C}_{AC}$, (AC for acoustic), is reduced to

$$\mathscr{C}_{AC} = \{(\tilde{p}, 0, 0), \tilde{p} \in H^1(\Omega) \cap L_0^2(\Omega)\}\,.$$

It can be checked that, for every $X = (p, r, y)$ in $\mathscr{C}$, the following relation is satisfied :

$$|p|_{0,\Omega}^2 = |\tilde{p}|_{0,\Omega}^2 + |P_S(y)|_{0,\Omega}^2 \tag{13}$$

(i.e., $\tilde{p}$ and $P_S(y)$ are orthogonal in $L^2(\Omega)$).

Therefore, $\mathscr{C}_{ST}$ and $\mathscr{C}_{AC}$ are orthogonal subspaces in $\mathscr{C}$.

2.4. Translated variationnal formulation

An other implication of decomposition (11) is presented here.

Instead of using the pressure field p to describe the fluid, one can just use its purely acoustic part $\tilde{p}$, its pneumatic part $P_S(y)$ being totally determined by the knowledge of y. This permits to get rid of compatibility condition (4) in the Coupling space. The translated variationnal formulation writes :

Find ω^2 in $\mathbb{R}^*$ and $(\tilde{p}, r, y)$ in $(H^1(\Omega) \cap L_0^2(\Omega)) \times L^2(\Gamma) \times H_0^2(\Gamma)$, $(\tilde{p}, r, y) \neq (0, 0, 0)$, such that, for every (q, s, z) in

$$(H^1(\Omega) \cap L_0^2(\Gamma)) \times L^2(\Gamma) \times H_0^2(\Gamma) ,$$

$$\frac{1}{\rho} \int_\Omega \nabla\tilde{p}\, \nabla\tilde{q}\, d\Omega - \omega^2 \left[\frac{1}{\rho c^2} \int_\Omega \tilde{p}\tilde{q}\, d\Omega + \int_\Omega y\tilde{q}\, d\gamma \right] = 0 , \tag{14}$$

$$\int_\Gamma \frac{rs}{\sigma D}\, d\gamma - \omega^2 \int_\Gamma ys\, d\gamma = 0 , \tag{15}$$

$$- \int_\Gamma \frac{ED^3}{12} y''\, z''\, d\gamma - \int_\Gamma P_S(y)\, z\, d\gamma + \int_\Gamma z\tilde{p}\, d\gamma + \int_\Gamma zr\, d\gamma = 0 . \tag{16}$$

The actual pressure field p is, of course, calculated afterwards, owing to relation

$$p = \tilde{p} - P_S(y), \text{ where } P_S(y) = \frac{\rho c^2}{\text{meas } \Omega} \int_\Gamma y\, d\gamma .$$

Equation (16) reveals the part played by the structure displacement y :

y is an auxiliary variable for problem (14), (15), (16). Indeed, for every given $(\tilde{p}, r)$ in $(H^1(\Omega) \cap L_0^2(\Gamma)) \times L^2(\Gamma)$, there exists a unique y in $H_0^2(\Gamma)$ defined by equation (16). This result derives from Lax-Milgram's theorem.

Furthermore, it can be checked that y obeys the estimation :

$$|y''|_{0,\Gamma} \leq c \left\{ |\nabla\tilde{p}|_{0,\Omega} + |r|_{0,\Gamma} \right\} \tag{17}$$

where c is a strictly positive constant.

To prove (17), one has to use Deny-Lion's result (see [10]) saying that the gradient semi-norm $|\nabla\tilde{p}|_{0,\Omega}$ is a norm in $(H^1(\Omega) \cap L_0^2(\Omega))$ equivalent to $|\tilde{p}|_{1,\Omega}$.

y is also solution to an energy minimization problem :

$J(y) = \mathrm{Inf}\ \{J(z), z \in H_0^2(\Gamma)\}$ where functional J is defined as follows

$$J(z) = \frac{1}{2}\left[\int_\Gamma \frac{ED^3}{12}(z'')^2\,d\gamma + \int_\Gamma P_S(z)\,z\,d\gamma\right] - \int_\Gamma \tilde{p}z\,d\gamma - \int_\Gamma rz\,d\gamma\,.$$

Notes :

J represents the structure dissipated energy,

$\int_\Gamma \frac{ED^3}{12}(z'')^2\,d\gamma$ represents the elastic deformation energy,

$\int_\Gamma P_S(z)\,z\,d\gamma$ represents the pneumatic deformation energy,

$\int_\Gamma z\tilde{p}\,d\gamma$ represents the acoustic stress work,

$\int_\Gamma zr\,d\gamma$ represents the inertia stress work .

A Green operator G may be associated to equation (16)

$$G : H^1(\Omega) \cap L_0^2(\Omega) \times L^2(\Gamma) \to H_0^2(\Gamma)$$
$$(\tilde{p}, r) \to y = G(\tilde{p}, r)\,.$$

Operator G enables a condensation for the problem under study, by eliminating y from equations (14) and (15), without loosing the symmetry in $\{(\tilde{p}, r), (\tilde{q}, s)\}$.

2.5. Spectral formulation

Though the initial formulation in (p, r, y) is resumed in this section, the abovementionned results are exploited. As y is an auxiliary variable, its determinating equation (10) is now treated as a constraint equation. Hence, a restricted Coupling space is introduced : denote by $\mathscr{C}^*$ the subspace of $\mathscr{C}$ defined as follows

$$\mathscr{C}^* = \{X = (p, r, y) \in \mathscr{C}\ ;\quad \forall z \in H_0^2(\Gamma),\ \int_\Gamma \frac{ED^3}{12}y''\,z''\,d\gamma = \int_\Gamma zp\,d\gamma + \int_\Gamma zp\,d\gamma\right\}\,.$$

Result 1 : $\mathscr{C}^*$ is a Hilbert space equipped with the energy scalar product

$$\frac{1}{\rho}\int_\Omega \nabla p\, \nabla q\, d\Omega + \int_\Gamma \frac{rs}{\sigma D}\, d\gamma + \int_\Gamma \frac{ED^3}{12}\, y''\, z''\, d\gamma + \frac{\rho c^2}{\text{meas}\ \Omega}\int_\Gamma y\, d\gamma \int_\Gamma z\, d\gamma$$

and the associated norm, denoted by $\|X\|_{\mathscr{C}^*}$ is equivalent to $\|X\|_{\mathscr{C}}$.

Proof of Result 1 : (see [8])

• Resume the direct decomposition of $\mathscr{C}$ into

$$\mathscr{C} = \mathscr{C}_{AC} \oplus \mathscr{C}_{ST} \quad \text{(see section 2.3)}\,.$$

• $\mathscr{C}_{AC} = \{(\tilde{p}, 0, 0), \tilde{p} \in H^1(\Omega) \cap L_0^2(\Omega)\}$ is a Hilbert space endowed with the scalar product $\frac{1}{\rho}\int_\Omega \nabla p\, \nabla q\, d\Omega$, and the associated norm is equivalent to $\|X\|_{\mathscr{C}}$ in $\mathscr{C}_{AC}$.

• $\mathscr{C}_{ST} = \left\{(p, r, y) \in \mathscr{C}\ ; p = -P_S(y) = -\frac{\rho c^2}{\text{meas}\ \Omega}\int_\Gamma y\, d\gamma\right\}$
is a Hilbert space endowed with the scalar product

$$\int_\Gamma \frac{rs}{\sigma D}\, d\gamma + \int_\Gamma \frac{\mathrm{ED}^3}{12}\, y''\, z''\, d\gamma + \frac{\rho c^2}{\text{meas}\ \Omega}\int_\Gamma y\, d\gamma \int_\Gamma z\, d\gamma\,,$$

and the associated norm is equivalent to $\|X\|_{\mathscr{C}}$ in $\mathscr{C}_{ST}$.

• Consequently, $\|X\|_{\mathscr{C}^*}$ is equivalent to $\|X\|_{\mathscr{C}}$ in $\mathscr{C}$.

• Finally, remark that $\mathscr{C}^*$ is closed in $\mathscr{C}$ as $\mathscr{C}^*$ is the Kernel of a bilinear continuous form on $\mathscr{C}$. □

Now, consider the two bilinear forms :

$$A(X, Y) = \frac{1}{\rho}\int_\Omega \nabla p\, \nabla q\, d\Omega + \int_\Gamma \frac{rs}{\sigma D}\, d\gamma$$

and

$$B(X, Y) = \frac{1}{\rho c^2}\int_\Omega pq\, d\Omega + \int_\Gamma \frac{ED^3}{12}\, y''\, z''\, d\gamma\,.$$

Result 2 :

The spectral problem

$$\left\{\begin{array}{l} \text{Find } \omega^2 \text{ in } \mathbb{R}^* \text{ and } X \text{ in } \mathscr{C}^*, X \neq 0\,, \\ \text{such that, for every } Y \text{ in } \mathscr{C}^* : \\ A(X, Y) = \omega^2\, B(X, Y) \end{array}\right. \tag{18}$$

is equivalent to variational problem $\{(8), (9), (10)\}$

Proof of result 2 :

• Take a solution ω^2 and X in $\mathscr{C}$ to variational problem (8), (9), (10). Constraint equation (10) being satisfied, X belongs to $\mathscr{C}^*$.

Now, add equations (8) and (9). There comes :

$$\frac{1}{\rho}\int_\Omega \nabla p\, \nabla q\, d\Omega + \int_\Gamma \frac{rs}{\sigma D}\, d\gamma = \\ = \omega^2\left\{\frac{1}{\rho c^2}\int_\Omega pq\, d\Omega + \int_\Gamma yq\, d\gamma + \int_\Gamma ys\, d\gamma\right\} \tag{19}$$

for every $Y = (q, s, z)$ in $\mathscr{C}$.

If Y belongs to $\mathscr{C}^*$, one can write that

$$\int_\Gamma yq\, d\gamma + \int_\Gamma ys\, d\gamma = \int_\Gamma \frac{ED^3}{12} y''\, z''\, d\gamma\,. \tag{20}$$

Therefore, equation (19), written for every Y in $\mathscr{C}^*$, becomes :

$$\frac{1}{\rho}\int_\Omega \nabla p\, \nabla q\, d\Omega + \int_\Gamma \frac{rs}{\sigma D}\, d\gamma = \omega^2\left\{\frac{1}{\rho c^2}\int_\Omega pq\, d\Omega + \int_\Gamma \frac{ED^3}{12} y''\, z''\, d\gamma\right\}\,,$$

and so, $\{\omega^2, X\}$ is a solution to spectral problem (18).

• Reciprocally, take a solution ω^2 in $\mathbb{R}^*$ and X in $\mathscr{C}^*$ to spectral problem (18). Equation (10) is satisfied, as X belongs to $\mathscr{C}^*$. Check now, that X obeys also equations (8) and (9). For that, write equation (18), for peculiar elements Y of $\mathscr{C}^*$: first, take $Y = (0, s, z)$ in $\mathscr{C}^*$ and check that equation (8) is satisfied, then take $Y = (q, 0, z)$ in $\mathscr{C}^*$ and check that equation (9) is satisfied.

Those verifications present no difficulties. □

THEOREM : *The spectral problem : find ω^2 in $\mathbb{R}^*$ and X in $\mathscr{C}^*$, $X \neq 0$, such that for every Y in $\mathscr{C}^*$, $A(X, Y) = \omega^2 B(X, Y)$, admits a denumerable sequence of real, strictly positive eigenvalues :*

$$0 < \omega_1^2 \leqslant \omega_2^2 \leqslant \cdots \leqslant \omega_n^2 \leqslant \cdots \to +\infty\,.$$

The associated eigenvectors, denoted by X^n, form a complete basis in $\mathscr{C}^*$, which is orthonormal for the scalar product $B(X, Y)$:

$$A(X^n, X^m) = \omega_n^2\, B(X^n, X^m) = \omega_n^2\, \delta_{nm}\,. \tag{21}$$

Proof of theorem :

• The bilinear form $A(X, Y) = \dfrac{1}{\rho}\displaystyle\int_\Omega \nabla p\, \nabla q\, d\Omega + \int_\Gamma \frac{rs}{\sigma D}\, d\gamma$ is continuous, symmetric and coercive on $\mathscr{C}^*$.

Check the last point : Remind that

$$\|X\|^2_{\mathscr{C}^*} = \frac{1}{\rho}\int_\Omega \nabla p^2\, d\Omega + \int_\Gamma \frac{r^2}{\sigma D}\, d\gamma + \int_\Gamma \frac{ED^3}{12}\,(y'')^2\, d\gamma + \\ + \frac{\rho c^2}{\operatorname{meas}\Omega}\left(\int_\Gamma y\, d\gamma\right)^2 ,$$

From estimation (17) (section 2.4)

$$|y''|_{0,\Gamma} \leq C\left\{|\nabla\tilde{p}|_{0,\Omega} + |r|_{0,\Gamma}\right\} .$$

Deduce that :

$$|y''|^2_{0,\Gamma} \leq 2\, C_2\left\{|\nabla p|^2_{0,\Omega} + |r|^2_{0,\Gamma}\right\} . \tag{22}$$

So, equation (22) permits to deduce that

$$\int_\Gamma \frac{ED^3}{12}\, y''^2\, d\gamma + \frac{\rho c^2}{\operatorname{meas}\Omega}\left(\int_\Gamma y\, d\gamma\right)^2 \leq C_2\left\{\frac{1}{\rho}\int_\Omega \nabla p^2\, d\Omega + \int_\Gamma \frac{r^2}{\sigma D}\, d\gamma\right\}$$

where C_2 is a strictly positive constant.

Then, equation (23) leads to the wanted conclusion :

$$A(X, X) \geq C_3\|X\|^2_{\mathscr{C}^*} . \tag{24}$$

• It is trivial that $B(X, Y) = \dfrac{1}{\rho c^2}\displaystyle\int_\Omega pq\, d\Omega + \int_\Gamma \frac{ED^3}{12}\, y''\, z''\, d\gamma$ defines a scalar product in $\mathscr{C}^*$, the associated norm being equivalent to the norm $\left\{|p|^2_{0,\Omega} + |y''|^2_{0,\Gamma}\right\}^{1/2}$.

• Let there $\mathscr{A}$ be the linear continuous mapping in $\mathscr{C}^*$ associated to $A(X, Y)$.

$\mathscr{A}$ is defined, for every X and Y in $\mathscr{C}^*$, by :

$$B(\mathscr{A}X, Y) = A(X, Y) , \tag{25}$$

$\mathscr{A}$ is selfadjoint, positive and invertible.

To be in the classical spectral analysis framework (see [11] for instance), one has to check that $\mathscr{A}^{-1}$ is compact in $\mathscr{C}^*$ for the topology associated to $B(X, Y)$, and this is the only non-immediat point of the proof.

Let there be a sequence, bounded in $\mathscr{C}^*$. A subsequence, denoted by X^n, $X^n = (p^n, r^n, y^n)$, can be extracted and weakly converges in $\mathscr{C}^*$.

Let there be $X = (p, r, y)$ its weak limit, such that

$$p^n \text{ weakly tends to } p \text{ in } H^1(\Omega) , \tag{26}$$

$$r^n \text{ weakly tends to } r \text{ in } L^2(\Gamma)\,, \tag{27}$$

$$y^n \text{ weakly tends to } y \text{ in } H_0^2(\Gamma)\,. \tag{28}$$

The injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact according to Rellich's theorem. From (26), deduce that :

$$p^n \text{ strongly tends to } p \text{ in } L^2(\Gamma)\,. \tag{29}$$

In the same way, the injection of $H_0^2(\Gamma)$ into $H^1(\Gamma)$ is continuous, and the injection of $H^1(\Gamma)$ into $L^2(\Gamma)$ is compact. From (28), deduce that

$$y^n \text{ strongly tends to } y \text{ in } L^2(\Gamma)\,. \tag{30}$$

The trace operator is continuous from $H^1(\Omega)$ into $L^2(\Gamma)$. From (26), deduce that

$$p^n|_\Gamma \text{ weakly tends to } p|_\Gamma \text{ in } L^2(\Gamma)\,. \tag{31}$$

Now, from results (30) and (31), deduce that

$$\int_\Gamma y^n p^n \, d\gamma \text{ tends to } \int_\Gamma yp \, d\gamma \tag{32}$$

and from results (27) and (30), deduce that

$$\int_\Gamma y^n r^n \, d\gamma \text{ tends to } \int_\Gamma yr \, d\gamma\,. \tag{33}$$

Write that X^n and X belong to $\mathscr{C}^*$. There comes the identities

$$\int_\Gamma \frac{ED^3}{12} (y^{n''})^2 \, d\gamma = \int_\Gamma y^n p^n \, d\gamma + \int_\Gamma y^n r^n \, d\gamma \tag{34}$$

and

$$\int_\Gamma \frac{ED^3}{12} y''^2 \, d\gamma = \int_\Gamma yp \, d\gamma + \int_\Gamma yr \, d\gamma\,. \tag{35}$$

Finally, from (32), (33), (34), (35), deduce that

$$y^n \text{ strongly tends to } y \text{ in } H_0^2(\Gamma)\,. \tag{36}$$

Results (26) and (36) permit to conclude that X^n strongly tends to X for the topology associated to $B(X, Y)$.

The classical spectral analysis theory permits to conclude. □

SECTION 3 : THE OPTIMIZATION PROBLEM

3.1. Formulation of the optimization problem

In this section, we intend to maximize the gap between two consecutive coupled eigenvalues ω_{N-1} and ω_N of spectral problem (21), (see section 2.5), for a given frequential order N. This optimum design problem is simplified as domain variations are controled by a lone design function, the structure thickness distribution D.

The problem can be summed up by the following scheme :

For a given control variable D, the system is governed by the state equation whose solutions, the state variables, permit to define the criterion to minimize in an admissible control set $\mathcal{U}_{ad}$:

$$\begin{array}{l} \qquad\qquad\qquad D \\ \qquad\qquad\qquad \downarrow \\ A(D)(X^i(D), .) = \omega_i^2(D)\, B(D)(X^i(D), .) \\ \qquad\qquad\qquad \downarrow \\ X^i(D), \quad \omega_i^2(D), \quad i \in \mathbb{N}^* \\ \qquad\qquad\qquad \downarrow \\ g(D) = G(D\,; X^i(D), \omega_i^2(D)) \\ \qquad\quad = \omega_{N-1}(D) - \omega_N(D) . \end{array}$$

Note that similar problems have been studied by M. P. Bendsøe and N. Olhoff for beams [12], shallow arches [13] and plates [14] in *vacuo*. As far as we know, optimization for a coupled fluid-structure system had never been looked at before.

According to Taylor and Bendsøe [15], a bound formulation is used to avoid cumbersome difficulties due to the non-differentiability of multimodal eigenvalues.

The trick consists in introducing two artificial variables $\omega(D)$ and $\beta(D)$, respectively middle-point and radius of the interval defined by $[\omega_{N-1}(D)\,;\,\omega_N(D)]$, and stating the problem as the minimization of the criterion defined by

$$j(D) = J(D\,; X^i(D), \omega_i^2(D), \omega(D), \beta(D)) = -\beta(D) . \qquad (37)$$

This minimization is submitted to suitable constraints of three types : design constraints, artificial constraints and state constraints.

The design constraints are imposed for technological reasons, and, in the present case, they are also necessary to ensure the existence of an optimum (see [16]). Control variable D is searched in the space $\mathscr{U}_{ad}$ of piecewise continuous functions on Γ and obeys the following two constraints :

$$0 < D_{\min} \leqslant D \,, \tag{38}$$

$$\int_{\Gamma} D \, d\gamma = v \,, \tag{39}$$

where $D_{\min}$ and v are given positive numbers.

The artificial constraints relate the bound-criterion $j(D) = -\beta(D)$ to the initial problem of minimizing $g(D) = \omega_{N-1}(D) - \omega_N(D)$, by excluding any eigenvalues $\omega_i(D)$ from control interval :

$$\text{for } i \leqslant N-1 \,, \quad \omega_i^2(D) \leqslant (\omega(D) - \beta(D))^2 \,, \tag{40}$$

and

$$\text{for } i \geqslant N \,, \quad \omega_i^2(D) \geqslant (\omega(D) + \beta(D))^2 \,. \tag{41}$$

Last, the state problem is related as a constraint :

$$\text{for } i = 1 \quad \text{to} \quad +\infty \,, \quad A(D)(X^i(D), X) = \omega_i^2(D)\, B(D)(X^i(D), X) \tag{42}$$

for every X in $\mathscr{C}^*$ ($cf.$ section 2.5) ; and for every $j \leqslant i$,

$$B(D)(X^i(D), X^j(D)) = \delta_{ij} \,. \tag{43}$$

Inequality constraint (38), (resp. (40) and (41), is relaxed by means of a slack variable denoted by $d(D)$ (resp. $\sigma_i(D)$, $i \leqslant N-1$ and $\sigma_i(D)$, $i \geqslant N$) and defined by

$$d^2(D) = D - D_{\min} \tag{44}$$

$$(\text{resp.} \quad \sigma_i^2(D) = (\omega(D) - \beta(D))^2 - \omega_i^2(D) \,, \quad \text{for} \quad i \leqslant N-1 \tag{45}$$

and

$$\sigma_i^2(D) = \omega_i^2(D) - (\omega(D) + \beta(D))^2 \,, \quad \text{for} \quad i \geqslant N) \,. \tag{46}$$

Now, for the sake of brievity, implicit dependance of all variables upon control D shall be omitted.

3.2. Lagrangian of the problem

To introduce the Lagrangian functionnal $\mathscr{L}$ for the present optimization problem, a multiplier is associated to each constraint. Namely,

— the function α to relaxe minimum thickness constraint (44),

— the scalar $\bar{v}$ to volume constraint (39),

— the scalars η_i to relaxed artificial constraints (45) and (46),
— the $\bar{X}^i$, elements of $\mathscr{C}^*$, to state problem (42), and
— the scalars τ_{ij} to orthonormality constraint (43).

$\mathscr{L}$ is defined as follows :

$$\begin{aligned}\mathscr{L} = & -\beta + \int_\Gamma (d^2 - D + D_{\min})\,\alpha\, d\gamma + \left(\int_\Gamma D\, d\gamma - v\right)\bar{v} \\ & + \sum_{i=1}^{N-1} \{\sigma_i^2 - (\omega - \beta)^2 + \omega_i^2\}\,\eta_i + \sum_{i=N}^{\infty} \{\sigma_i^2 + (\omega + \beta)^2 - \omega_i^2\}\,\eta_i \\ & + \sum_{i=1}^{\infty} \Big\{A(X^i, \bar{X}^i) - \omega_i^2 B(X^i, \bar{X}^i) \\ & + \sum_{j=1}^{i} (B(X^i, X^j) - \delta_{ij})\,\tau_{ij}\Big\}\,. \end{aligned} \tag{47}$$

$\mathscr{L}$ depends upon design variable D, artificial variables ω and β, state variables (X^i, ω_i^2), slack variables d and σ_i, and Lagrangian multipliers α, $\bar{v}$, η_i, $\bar{X}^i$ and τ_{ij}. In order to obtain necessary conditions satisfied by any optimum solution, stationarity of $\mathscr{L}$ with respect to all its variables is written.

First, note that derivation of $\mathscr{L}$ with respect to multipliers gives back all the constraints (38) to (43).

Then, derivation with respect to slack variables α (resp. σ_i) enables to derive activity conditions for inequality constraints (38) (resp. (40) and (41)).

From $\dfrac{\partial \mathscr{L}}{\partial d} = \alpha d = 0$, deduce that :

$$\text{on} \quad \Gamma_u = \{x \in \Gamma \,;\, d(x) \neq 0\}\ ,$$

constraint (38) is inactive as

$$D(x) > D_{\min} \quad \text{and} \quad \alpha(x) = 0 \quad \text{and} \quad \text{on}\, \Gamma_c = \{x \in \Gamma\,;\quad d(x) = 0\}\ ,$$

constraint (38) is active as $D(x) = D_{\min}$.

From $\dfrac{\partial \mathscr{L}}{\partial \sigma_i} = \eta_i\, \sigma_i = 0$, $i = 1, +\infty$, deduce the following conditions :

— for $i = \underline{N}$ to $N - 1$, $\omega_i^2 = (\omega - \beta)^2$, where $\underline{m} = N - \underline{N}$ is the multiplicity order of the eigenvalue equal to $(\omega - \beta)^2$;

— for $i = N$ to $\bar{N}$, $\omega_i^2 = (\omega + \beta)^2$, where $\bar{m} = \bar{N} - N + 1$ is the multiplicity order of the eigenvalue equal to $(\omega + \beta)^2$;

— else, constraints (40) and (41) are inactive and $\eta_i = 0$ for $i = 1$ to $\underline{N} - 1$ and $i = \bar{N} + 1$ to $+\infty$.

Derivation with respect to artificial variables ω and β permits to determine the last unknown multipliers η_i, $i = \underline{N}$ to $\overline{N}$. It is found out that

$$\text{for} \quad i = \underline{N} \quad \text{to} \quad N-1 , \quad \eta_i = \underline{\eta} = \frac{1}{4\,\underline{m}(\omega - \beta)} ,$$

and

$$\text{for} \quad i = N \quad \text{to} \quad \overline{N} , \quad \eta_i = \overline{\eta} = \frac{1}{4\,\overline{m}(\omega - \beta)} .$$

Note that necessarily $\underline{m}$, (resp. $\overline{m}$), is greater or equal than 1, as constraint (40), (resp. (41)), is always active for $i = N - 1$, (resp. $i = N$).

Derivation with respect to state variables (X^i, ω_i^2, $i \in \mathbb{N}^*$) leads to write the so-called « co-state equation ». Let us mention the main results obtained (more details can be found in [8]) :

— Orthonormality condition (43) is not a real constraint to the problem, as $\tau_{ij} = 0$ for every i and j.

— The costate variables $\bar{X}^i$ are proportional to the state variables X^i :

$$\bar{X}^i = \eta_i X^i \quad \text{for} \quad i \leqslant N - 1 ,$$

$$\bar{X}^i = -\eta_i X^i \quad \text{for} \quad i > N .$$

Note that $\bar{X}^i$ is equal to zero, as soon as $i \leqslant \underline{N}$ or $i - \overline{N}$. Hence, only the eigenvectors associated to the eigenvalues equal to $(\omega - \beta)^2$ and $(\omega + \beta)^2$ should be computed.

Last, derivation with respect to design variable D gives the « optimality criterion » (see [8]).

Let us recall that eigenvector X^i stands for $X^i = (p^i, r^i, y^i)$ (see section 2.5), hence $\bar{X}^i$ stands for $\bar{X}^i = (\bar{p}^i, \bar{r}^i, \bar{y}^i)$; and that

$$A(X, Y) = \frac{1}{c^2} \int_\Omega \nabla p \, \nabla q \, d\Omega + \int_\Gamma \frac{rs}{\sigma D} \, d\gamma$$

and

$$B(X, Y) = \frac{1}{\rho c^2} \int_\Omega pq \, d\Omega + \int_\Gamma \frac{ED^3}{12} y'' z'' \, d\gamma ,$$

where $X = (p, q, r)$ and $Y = (q, s, z)$ belong to $\mathscr{C}^*$.

Thanks to all the above results, the optimality criterium can be finally reduced to the following equations.

$$\bar{v} = \sum_{i=\underline{N}}^{N-1} \frac{r_i^2(x)}{\sigma \underline{m}(\omega - \beta)} - \sum_{i=N}^{\bar{N}} \frac{r_i^2(x)}{\sigma \bar{m}(\omega + \beta)} +$$
$$+ 3\, D^4(x) \left\{ \sum_{i=\underline{N}}^{N-1} \frac{E(\omega - \beta)}{12\, \underline{m}} (y_i''(x))^2 - \sum_{i=N}^{\bar{N}} \frac{E(\omega + \beta)}{12\, \bar{m}} (y_i''(x)^2) \right\}$$

for almost every x on Γ_u. Of course, $D(x) = D_{\min}$ on Γ_c.
Lagrangian multiplier $\bar{v}$ can be explicitely determined :

$$\bar{v} = \frac{1}{w} \sum_{i=\underline{N}}^{\bar{N}} \{-4\, \eta_i\, \omega_i^2 + E(i) + 3\, \omega_i^2\, F(i)\} \tag{49}$$

where $w = v - D_{\min}$ meas Γ_c,

$$E(i) = \frac{1}{D_{\min}} \int_{\Gamma_c} \frac{r_i\, \bar{r}_i}{\sigma}\, d\gamma + \frac{1}{\rho} \int_{\Omega} \nabla p_i\, \nabla \bar{p}_i\, d\Omega$$

and

$$F(i) = D_{\min}^3 \int_{\Gamma_c} E \frac{y_i''\, \bar{y}_i''}{12}\, d\gamma + \frac{1}{\rho c^2} \int_{\Omega} p_i\, \bar{p}_i\, d\Omega$$

for $i = \underline{N}$ to $\bar{N}$.

3.3 Numerical resolution

From section 3.2, it can be deduced that every optimum solution necessarily satisfies the following problem :

Find $(X^i, \omega_{N-1}, \omega_N, \underline{N}, \bar{N}, \bar{m}, \omega, \beta, \underline{\eta}, \bar{\eta}, \bar{v}, \mathrm{D}, \Gamma_c, \Gamma_u)$ such that :

$$A(X^i, X) - \omega_{N-1}\, B(X^i, X) = 0\,, \quad i = \underline{N}, N-1\,;$$
$$A(X^i, X) - \omega_N\, B(X^i, X) = 0\,, \quad i = \underline{N}, \bar{N}\,;$$
$$\underline{m} = N - \underline{N}\,; \bar{m} = \bar{N} - N + 1\,;$$
$$\omega = \frac{\omega_N + \omega_{N-1}}{2}\,; \quad \beta = \frac{\omega_N - \omega_{N-1}}{2}\,;$$
$$\underline{\eta} = \frac{1}{4\, \underline{m}(\omega - \beta)}\,; \quad \bar{\eta} = \frac{1}{4\, \bar{m}(\omega + \beta)}\,;$$

$\bar{v}$ defined by equation (49),

$$D(x) = D_{\min}\,, \quad \text{for} \quad x \quad \text{in} \quad \Gamma_c\,,$$

$D(x)$ implicitely defined by equation (48), for x in Γ_u.

The above system has been solved by an efficient iterative algorithm, using a modal synthesis method for the state problem. Full details can be found in reference [8] and [9].

The author wants to express all her thanks to R. Ohayon, J. F. Maitre and F. Musy for their valuable help.

BIBLIOGRAPHY

[1] M. A. HAMDI, Y. OUSSET, G. VERCHERY, 1978, *A Displacement method for the analysis of vibrations of coupled Fluid-Structure systems.* Int. Journal for Num. Meth. in Eng., vol. 13, n° 1, pp. 139-150.

[2] C. FELIPPA, 1986, *Some aspects of the Symmetrization of the contained Compressible Fluid-Vibration eigenproblem in Innovating Numerical Methods in Engineering.* Springer-Verlag Berlin. Heidelberg. New York. Tokyo.

[3] R. OHAYON, 1979, *Formulation variationnelle symétrique du problème des vibrations harmoniques par couplage des principes primal et dual.* Application aux oscillations de systèmes couplés fluide-structure. La Recherche Aérospatiale, n° 3 (mai-juin), pp. 207-211.

[4] J. BOUJOT, 1986, *Mathematical formulation of fluid-structure interactions problems.* To appear in M²AN.

[5] H. BERGER, J. BOUJOT, R. OHAYON, 1975, *On a spectral problem in vibration mechanics computation of elastic tanks partially filled with liquids.* Journal of Mathematical analysis and applications, 51, pp. 272-298.

[6] R. OHAYON, 1984, *Transient and modal analysis of bounded medium fluid-structure Problems.* In Numerical Methods for transient and coupled problems. Venice.

[7] G. DUVAUT, J. L. LIONS, 1972, *Les inéquations variationnelles en mécanique et en physique.* Dunod, Paris.

[8] A. C. DENEUVY, 1986, *Etude d'un problème de conception optimale avec critère sur les fréquences pour un système couplé fluide-structure.* Thèse. Ecole Centrale de Lyon.

[9] A. C. DENEUVY, 1987, *A new modal synthesis method for a fluid-structure problem.* Resolution and Optimization. (To appear).

[10] J. DENY, J. L. LIONS, 1953, *Les espaces de type Beppo-Levi.* Ann. Inst. Fourier (Grenoble), 5, pp. 305-370.

[11] H. BREZIS, 1983, *Analyse fonctionnelle (Théorie et Applications).* Masson, Paris.

[12] M. P. BENDSØE, N. OLHOFF, 1983, *A method of design against vibration resonance of beams and shafts.* First IASTED International Symposium on applied control and identification, Copenhagen Denmark, June.

[13] N. OLHOFF, R. PLAUT, 1983, *Bimodal optimization of vibrating shallow arches.* Int. J. structures, vol. 19, n° 5, pp. 663-670.

[14] N. OLHOFF, 1973, *On singularities, Local Optima and Formation of stiffners in optimal design of plates.* Report n° 56, September DC.AMN. The technical University of Denmark.

[15] J. E. TAYLOR, M. P. BENDSOE, 1984, *An interpretation for Min-Max structural design Problems including a Method for relaxing constraints.* Int. J. Solids Structures, vol. 20, n° 4, pp. 301-314.

[16] M. P. BENDSØE, 1984, *Existence proofs for a class of plate optimization problems. Lecture notes in Control and Information Sciences*, vol. 59, Springer, pp. 773-779).