RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

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RAIRO – Modélisation mathématique et analyse numérique, tome 22, nº 2 (1988), p. 243-250.

<http://www.numdam.org/item?id=M2AN_1988__22_2_243_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 22, n° 2, 1988, p. 243 à 250)

ON THE APPLICATION OF MIXED FINITE ELEMENT METHODS TO THE WAVE EQUATIONS (*)

by Tunc GEVECI [1]

Abstract — *The convergence of certain semidiscrete approximation schemes based on the « velocity-stress » formulation of the wave equation and spaces such as those introduced by Raviart and Thomas is discussed The discussion also applies to similar schemes for the equations of elasticity*

Resumé. — *La convergence de certains schémas d'approximation semi-discrète basés sur la formulation « vitesse-contrainte » de l'équation d'onde et d'espace tel que ceux introduits par Raviart et Thomas est discuté La discussion s'applique également pour les schémas similaires aux équations d'élasticité*

1. THE « VELOCITY-STRESS » FORMULATION OF THE WAVE EQUATION AND A SEMIDISCRETE VERSION

Let us consider the following initial-boundary value problem for the wave equation :

$$\begin{aligned} &D_t^2 u(t,x) - \Delta u(t,x) = f(t,x)\,, \quad t>0\,, \quad x \in \Omega \subset \mathbb{R}^2\,, \\ &u(t,x) = 0\,, \quad t>0\,, \quad x \in \Gamma\,, \\ &u(0,x) = u_0(x)\,, \quad D_t u(0,x) = v_0(x)\,, \quad x \in \Omega\,, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain with boundary Γ, and f, u_0, v_0 are given functions. Introducing the « stress » $\sigma = \nabla u$, (1.1) may be reformulated as

$$\begin{aligned} &D_t^2 u(t,x) - \operatorname{div} \sigma(t,x) = f(t,x)\,, \quad t>0\,, \quad x \in \Omega\,, \\ &\sigma(t,x) = \nabla u(t,x)\,, \quad t>0\,, \quad x \in \Omega\,, \\ &u(t,x) = 0\,, \quad t>0\,, \quad x \in \Gamma\,, \\ &u(0,x) = u_0(x)\,, \quad D_t u(0,x) = v_0(x)\,, \quad x \in \Omega\,. \end{aligned} \tag{1.2}$$

(*) Received in October 1986.

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We use the notation of Johnson and Thomée [12] :

$$V = L_2(\Omega)\,, \quad H = \{\chi \in L_2(\Omega)^2 : \operatorname{div}\chi \in L_2(\Omega)\}\,.$$

Using Green's formula

$$\int_\Omega u \operatorname{div}\chi\, dx = \int_\Gamma u\chi \,.\, n\, ds - \int_\Omega \nabla u \,.\, \chi\, dx$$

where n is the unit exterior normal to Γ, a Galerkin version of (1.2) is to seek $u(t) \in V$, $\sigma(t) \in H$, $t > 0$, satisfying

$$\begin{aligned} &(D_t^2 u(t), w) - (\operatorname{div}\sigma(t), w) = (f(t), w)\,, \quad w \in V\,,\\ &(\sigma(t)\,\chi) + (u(t), \operatorname{div}\chi) = 0\,, \quad \chi \in H\,,\\ &u(0) = u_0\,, \quad D_t u(0) = v_0\,, \end{aligned} \tag{1.3}$$

where the parentheses denote the appropriate inner products (L_2-inner product in V, $L_2(\Omega)^2$-inner product in H). If $V_h \subset V$ and $H_h \subset H$ are finite dimensional subspaces, such as the spaces introduced by Raviart and Thomas [13], and by Brezzi, Douglas, Jr. and Marini [6], a semidiscrete version of (1.3) seeks $u_h(t) \in V_h$, $\sigma_h(t) \in H_h$, $t > 0$, satisfying

$$\begin{aligned} &(D_t^2 u_h(t), w_h) - (\operatorname{div}\sigma_h(t), w_h) = (f(t), w_h)\,, \quad w_h \in V_h\,,\\ &(\sigma_h(t), \chi_h) + (u_h(t), \operatorname{div}\chi_h) = 0\,, \quad \chi_h \in H_h\,,\\ &u_h(0) = u_{0,h}\,, \quad D_t u_h(0) = v_{0,h}\,, \end{aligned} \tag{1.3$_h$}$$

where $u_{0,h}$, $v_{0,h} \in V_h$ are approximations to u_0 and v_0, respectively.

Johnson and Thomée [12] have discussed the parabolic counterpart of (1.3_h). The analysis of convergence of (1.3_h) can be carried out along similar lines, parallel to Baker and Bramble [4], for example. (1.3_h) is treated essentially as a non-conforming « displacement » model for the wave equation (1.1). The purpose of this note is to discuss the convergence of the « velocity-stress » models based on pairs of spaces (V_h, H_h) such as those in [6], [12], [13]. Thus, defining $v = D_t u$, $v_h = D_t u_h$, (1.3) and (1.3_h) are transformed, respectively, to

$$\begin{aligned} &(D_t v(t), w) - (\operatorname{div}\sigma(t), w) = (f(t), w)\,, \quad w \in V\,,\\ &(D_t \sigma(t), \chi) + (v(t), \operatorname{div}\chi) = 0\,, \quad \chi \in H\,,\\ &v(0) = v_0\,, \quad \sigma(0) = \sigma_0 = \nabla u_0\,, \end{aligned} \tag{1.4}$$

where $v(t) \in V$, $\sigma(t) \in H$, $t \geq 0$, and

$$(D_t v_h(t), w_h) - (\operatorname{div}\sigma_h(t), w_h) = (f(t), w_h), w_h \in V_h\,,$$

$$(1.4_h) \qquad (D_t\,\sigma_h(t), \chi_h) + (v_h(t), \operatorname{div} \chi_h) = 0\,, \quad \chi_h \in H_h\,,$$
$$v_h(0) = v_{0,h}\,, \quad \sigma_h(0) = \sigma_{0,h}\,,$$

where $v_h(t) \in V_h$, $\sigma_h(t) \in H_h$, $t \geqslant 0$.

We now list the basic features of the space V_h, H_h which lead to a straightforward analysis of the convergence of v_h to v and σ_h to σ :

(H.1) There exists a linear operator $\Pi_h : H \to H_h$ such that

$$(1.5) \qquad (\operatorname{div} \Pi_h\,\chi, w_h) = (\operatorname{div} \chi, w_h) \quad \forall w_h \in V_h\,, \quad \chi \in H\,,$$

$$(1.6) \qquad \|\Pi_h\,\chi - \chi\| \leqslant C h^s \|\chi\|_s \quad \text{for} \quad 1 \leqslant s \leqslant r\,, \quad r \geqslant 2$$

($\| \;\|$ is the $L_2(\Omega)^2$-norm, and $\|.\|_s$ is the $H^s(\Omega)^2$-norm).

(H.2) There exists a linear operator $P_h : V \to V_h$ such that

$$(1.7) \qquad (P_h\,v, \operatorname{div} \chi_h) = (v, \operatorname{div} \chi_h) \quad \forall \chi_h \in H_h\,, \quad v \in V\,,$$

$$(1.8) \qquad \|P_h\,v - v\| \leqslant C h^s \|v\|_s\,, \quad 1 \leqslant s \leqslant r\,, \quad r \geqslant 2$$

($\| \;\|$ is the $L_2(\Omega)$-norm, and $\| \;\|_s$ is the $H^s(\Omega)$-norm, and, as usual, C denotes a generic constant which depends only on the data and on the particular discretization scheme).

If Ω is a polygonal domain and V_h, H_h are the Raviart-Thomas spaces [12], [13], or if these spaces are the pairs introduced in the paper by Brezzi, Douglas, Jr., and Marini [6], $\operatorname{div} \chi_h \in V_h$, and P_h can be taken to be the L_2-projection. For an example of a pair (V_h, H_h) satisfying the above hypotheses (with $r = 2$), where P_h is not the L_2-projection, we refer the reader to the paper by Johnson and Thomée [12]. We would also like to point out that (H.1) and (H.2) are valid for the mixed method that has been introduced by Arnold, Douglas, Jr., and Gupta [3] to approximate solution of plane elasticity problems. Our analysis is readily adapted to the corresponding (genuine) velocity-stress formulation of the time-dependent problem.

We can now state and prove our convergence result :

THEOREM : *If u is the solution of* (1.1), $v = D_t\,u$, $\sigma = \nabla u$, *and if the pair* $\{v_h, \sigma_h\}$ *is the solution of* (1.4_h), *under the hypotheses* (H.1) *and* (H.2) *we have, for* $1 \leqslant s \leqslant r$, $r \geqslant 2$,

$$(1.9) \quad \|v_h(t) - v(t)\| + \|\sigma_h(t) - \sigma(t)\| \leqslant C\,(\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\|) + \\ + C h^s \left(\|v_0\|_s + \|\sigma_0\|_s + \int_0^t (\|D_\tau\,v(\tau)\|_s + \|D_\tau\,\sigma(\tau)\|_s)\,d\tau \right).$$

Proof : Let us denote by X the space $V \times H$, the elements of which will be designated as $\xi = \{v, \sigma\}$ or $\zeta = \{w, \chi\}$ and set

$$((\xi, \zeta)) = (v, w) + (\sigma, \chi),$$
$$|||\xi||| = \sqrt{((\xi, \xi))}.$$

Let $X_h = V_h \times H_h$ be equipped with $((.\,,.))$ and the induced norm $|||\cdot|||$. Elements of X_h will be designated as $\xi_h = \{v_h, \sigma_h\}$ or $\zeta_h = \{w_h, \chi_h\}$.
We define the bilinear form $a(.\,,.)$ on X by

$$a(\xi, \zeta) = -(\operatorname{div} \sigma, w) + (v, \operatorname{div} \chi) \tag{1.10}$$

for $\xi = \{v, \sigma\}$, $\zeta = \{w, \chi\}$.
We can now express (1.4) as

$$((D_t\, \xi(t), \zeta)) + a(\xi(t), \zeta) = (f(t), w), \quad \zeta \in X \tag{1.11}$$
$$(\xi(t) = \{v(t), \sigma(t)\}, \zeta = \{w, \chi\}),$$

and we can express (1.4_h) as

$$((D_t\, \xi_h(t), \zeta_h)) + a(\xi_h(t), \zeta_h) = (f(t), w_h), \quad \zeta_h \in X_h \tag{1.11_h}$$
$$(\xi_h(t) = \{v_h(t), \sigma_h(t)\}, \zeta_h = \{w_h, \chi_h\}).$$

Let us define $\underline{P}_h\, \xi = \{P_h\, v, \Pi_h\, \sigma\}$ for $\xi = \{v, \sigma\} \in X$, and observe that

$$a(\underline{P}_h\, \xi, \zeta_h) = a(\xi, \zeta_h), \quad \zeta_h \in X_h \tag{1.12}$$

by (H.1) ((1.5)) and (H.2) ((1.7)).
Therefore we obtain from (1.11)

$$\begin{aligned} ((D_t\, \underline{P}_h\, \xi(t), \zeta_h)) &+ a(\underline{P}_h\, \xi(t), \zeta_h) \\ &= (f(t), w_h) + ((\underline{P}_h\, D_t\, \xi(t) - D_t\, \xi(t), \zeta_h)), \quad \zeta_h \in X_h. \end{aligned} \tag{1.13}$$

Setting $\varepsilon_h(t) = \underline{P}_h\, \xi(t) - \xi_h(t)$, (1.11_h) and (1.13) yield

$$\begin{aligned} ((D_t\, \varepsilon_h(t), \zeta_h)) &+ a(\varepsilon_h(t), \zeta_h) = \\ &= ((\underline{P}_h\, D_t\, \xi(t) - D_t\, \xi(t), \zeta_h)), \quad \zeta_h \in X_h. \end{aligned} \tag{1.14}$$

Let us define $\Lambda_h : X_h \to X_h$ by

$$((\Lambda_h\, \xi_h, \zeta_h)) = a(\xi_h, \zeta_h), \quad \xi_h, \zeta_h \in X_h. \tag{1.15}$$

Since

$$a(\xi_h, \zeta_h) = -a(\zeta_h, \xi_h), \quad \xi_h, \zeta_h \in X_h, \tag{1.16}$$

as is readily seen (*cf.* (1.10)), Λ_h is shew-adjoint,

$$((\Lambda_h \xi_h, \zeta_h)) = -((\xi_h, \Lambda_h \zeta_h)), \quad \xi_h, \zeta_h \in X_h, \tag{1.17}$$

and $-\Lambda_h$ generates the unitary group $e^{-t\Lambda_h}$. In particular,

$$|||e^{-t\Lambda_h}\xi_h(0)||| = |||\xi_h(0)|||, \quad t \in \mathbb{R}. \tag{1.18}$$

Let us denote by $\underline{P}_h^0 : X \to X_h$ the projection with respect $((.\,,.))$.

We can now express (1.14) as

$$D_t\,\varepsilon_h(t) + \Lambda_h\,\varepsilon_h(t) = \underline{P}_h^0(\underline{P}_h\,D_t\,\xi(t) - D_t\,\xi(t)) \tag{1.19}$$

so that

$$\varepsilon_h(t) = e^{-t\Lambda_h}\,\varepsilon_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h}\,\underline{P}_h^0(\underline{P}_h\,D_\tau\,\xi(\tau) - D_\tau\,\xi(\tau))\,d\tau. \tag{1.20}$$

(1.18) and (1.20) yield the estimate

$$|||\varepsilon_h(t)||| \leqslant |||\varepsilon_h(0)||| + \int_0^t |||\underline{P}_h\,D_\tau\,\xi(\tau) - D_\tau\,\xi(\tau)|||\,d\tau \tag{1.21}$$

($\underline{P}_h^0$ is the $((.\,,.))$-projection).

(1.21) is readily translated to

$$\begin{aligned}
&\|P_h\,v(t) - v_h(t)\| + \|\Pi_h\,\sigma(t) - \sigma_h(t)\| \\
&\qquad \leqslant C\,(\|P_h\,v_0 - v_{0,h}\| + \|\Pi_h\,\sigma_0 - \sigma_{0,h}\| \\
&\qquad + \int_0^t (\|P_h\,D_\tau\,v(\tau) - D_\tau\,v(\tau)\| + \|\Pi_h\,D_\tau\,\sigma(\tau) - D_\tau\,\sigma(\tau)\|)\,d\tau) \\
&\qquad \leqslant C\,(\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + \|P_h\,v_0 - v_0\| + \|\Pi_h\,\sigma_0 - \sigma_0\| \\
&\qquad + \int_0^t (\|P_h\,D_\tau\,v(\tau) - D_\tau\,v(\tau)\| + \|\Pi_h\,D_\tau\,\sigma(\tau) - D_\tau\,\sigma(\tau)\|)\,d\tau),
\end{aligned}$$

and this, together with (1.6) and (1.8), yields

$$\begin{aligned}
&\|P_h\,v(t) - v_h(t)\| + \|\Pi_h\,\sigma(t) - \sigma_h(t)\| \\
&\qquad \leqslant C\,(\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + h^s(\|v_0\|_s + \|\sigma_0\|_s)) \\
&\qquad + Ch^s \int_0^t (\|D_\tau\,v(\tau)\|_s + \|D_\tau\,\sigma(\tau)\|_s)\,d\tau.
\end{aligned} \tag{1.22}$$

Since

$$\begin{aligned}
&\|v(t)-v_h(t)\| + \|\sigma(t)-\sigma_h(t)\| \\
&\quad \leq \|v(t)-P_h v(t)\| + \|P_h v(t)-v_h(t)\| \\
&\qquad + \|\sigma(t)-\Pi_h \sigma(t)\| + \|\Pi_h \sigma(t)-\sigma_h(t)\| \\
&\quad \leq Ch^s(\|v(t)\|_s + \|\sigma(t)\|_s) + \|P_h v(t)-v_h(t)\| + \|\Pi_h \sigma(t)-\sigma_h(t)\|,
\end{aligned}$$

by (1.6) and (1.8), and

$$\|v(t)\|_s \leq \|v_0\|_s + \int_0^t \|D_\tau v(\tau)\|_s \, d\tau ,$$

$$\|\sigma(t)\|_s \leq \|\sigma_0\|_s + \int_0^t \|D_\tau \sigma(\tau)\|_s \, d\tau ,$$

(1.22) leads to (1.9), the assertion of the theorem.

2. SOME OBSERVATIONS IN REGARD TO THE TIME-DIFFERENCING OF THE SEMIDISCRETE MODEL

(1.4_h) leads to a system of ordinary differential equations in the form

$$\begin{aligned} &M_0 D_t W - D\Sigma = F , \\ &M_1 D_t \Sigma + D^T W = 0 , \end{aligned} \tag{2.1}$$

where W corresponds to v_h, Σ corresponds to σ_h, M_0, M_1 are symmetric, positive-definite matrices, and D^T denotes the transpose of D. The application of implicit Euler time-differencing

$$\begin{aligned} &M_0 \frac{W^{n+1}-W^n}{k} - D\Sigma^{n+1} = F^{n+1} , \\ &M_1 \frac{\Sigma^{n+1}-\Sigma^n}{k} + D^T W^{n+1} = 0 , \end{aligned} \tag{2.2}$$

(k denotes the time step), necessitates the solution of

$$\begin{aligned} &M_0 W^{n+1} - kD\Sigma^{n+1} = kF^{n+1} + M_0 W^n , \\ &M_1 \Sigma^{n+1} + kD^T W^{n+1} = M_1 \Sigma^n . \end{aligned} \tag{2.3}$$

M_0 is in block-diagonal form if V_h consists of functions with no continuity requirement across inter-element boundaries, as is the case in [6], [12], [13], and the elimination of W^{n+1} in (2.3) is efficiently implementable. This leads to a system in the form

$$(M_1 + k^2 D^T M_0^{-1} D)\,\Sigma^{n+1} = G , \tag{2.4}$$

where M_1 is symmetric, positive definite and $D^T M_0^{-1} D$ is symmetric, positive-semidefinite, for the determination of Σ^{n+1}.

On the other hand, (1.3_h) leads to

$$\begin{aligned} M_0 D_t^2 U - D\Sigma &= F , \\ M_1 \Sigma + D^T U &= 0 , \end{aligned} \tag{2.5}$$

where U corresponds to u_h. (2.5) can be expressed as

$$M_0 D_t^2 U + DM_1^{-1} D^T U = F , \tag{2.6}$$

where M_0, $DM_1^{-1} D^T$ are symmetric, positive-definite [12]. If (2.6) is expressed as a system in $\{U, W\}$,

$$\begin{aligned} D_t U - W &= 0 \\ M_0 D_t W + DM_1^{-1} D^T U &= F , \end{aligned} \tag{2.7}$$

and implicit Euler time-differencing is applied to (2.7),

$$\begin{aligned} \frac{U^{n+1} - U^n}{k} - W^{n+1} &= 0 \\ M_0 \frac{W^{n+1} - W^n}{k} + DM_1^{-1} D^T U^{n+1} &= F^{n+1} , \end{aligned} \tag{2.8}$$

elimination of W^{n+1} leads to a system in the form

$$(M_0 + k^2 DM_1^{-1} D^T)\, U^{n+1} = \tilde{G} . \tag{2.9}$$

The matrix in (2.9) is symmetric, positive-definite, so that (2.9) is solvable. But M_1 is not block-diagonal, unlike M_0, so that deriving the reduced system (2.9), which includes inverting M_1, is more expensive than forming the reduced system (2.4). The time-independent counterpart of (2.5),

$$\begin{aligned} - D\Sigma &= F , \\ M_1 \Sigma + D^T U &= 0 , \end{aligned} \tag{2.10}$$

led Arnold and Brezzi [2] to relax the requirement that $\operatorname{div} \sigma_h \in L_2(\Omega)$ in order to have a block-diagonal matrix instead of M_1 and be able to eliminate Σ efficiently. This approach has to introduce a multiplier corresponding to the relaxation of the requirement $\operatorname{div} \sigma_h \in L_2(\Omega)$.

The above considerations suggest that the « velocity-stress » formulation (1.4_h) may be preferable to (1.3_h) if the approximation of the « stress » σ is of primary concern.

The application of diagonally implicit Runge-Kutta methods (see, for example, Crouzeix [8], Crouzeix and Raviart [9], Alexander [1], Burrage

[7], Dougalis and Serbin [10]) to (2.1) leads to systems similar to (2.4) so that our discussion is relevant to higher-order time differencing as well. We will not prove error estimates for such full-descrete approximation schemes based on (1.4_h). Such estimates should be obtainable by employing techniques that have been utilized in [5] or [11], for example.

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