TUNC GEVECI

On the application of mixed finite element methods to the wave equations


<http://www.numdam.org/item?id=M2AN_1988__22_2_243_0>
ON THE APPLICATION OF MIXED FINITE ELEMENT METHODS TO THE WAVE EQUATIONS (*)

by Tunc GEVECI (1)

Abstract — The convergence of certain semidiscrete approximation schemes based on the « velocity-stress » formulation of the wave equation and spaces such as those introduced by Raviart and Thomas is discussed. The discussion also applies to similar schemes for the equations of elasticity.

Résumé. — La convergence de certains schémas d'approximation semi-discrète basés sur la formulation « vitesse-contrainte » de l'équation d'onde et d'espace tel que ceux introduits par Raviart et Thomas est discuté. La discussion s'applique également pour les schémas similaires aux équations d'élasticité.

1. THE « VELOCITY-STRESS » FORMULATION OF THE WAVE EQUATION AND A SEMIDISCRETE VERSION

Let us consider the following initial-boundary value problem for the wave equation:

\[ D_t^2 u(t, x) - \Delta u(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^2, \]

\[ u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \]

\[ u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega, \]

where \( \Omega \) is a bounded domain with boundary \( \Gamma \), and \( f, u_0, v_0 \) are given functions. Introducing the « stress » \( \sigma = \nabla u \), (1.1) may be reformulated as

\[ D_t^2 u(t, x) - \text{div} \sigma(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega, \]

\[ \sigma(t, x) = \nabla u(t, x), \quad t > 0, \quad x \in \Omega, \]

\[ u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \]

\[ u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega. \]

(*) Received in October 1986.
(1) Department of Mathematical Sciences, San Diego State University San Diego, California 92182

M² AN Modélisation mathématique et Analyse numérique 0399-0516/88/02/243/8/$ 2.80
Mathematical Modelling and Numerical Analysis © AFCET Gauthier-Villars
We use the notation of Johnson and Thomée [12]:

\[ V = L^2(\Omega), \quad H = \{ \chi \in L^2(\Omega)^2 : \text{div} \chi \in L^2(\Omega) \}. \]

Using Green’s formula

\[
\int_{\Omega} u \text{div} \chi \, dx = \int_{\Gamma} u\chi \cdot n \, ds - \int_{\Omega} \nabla u \cdot \chi \, dx
\]

where \( n \) is the unit exterior normal to \( \Gamma \), a Galerkin version of (1.2) is to seek \( u(t) \in V, \sigma(t) \in H, t > 0 \), satisfying

\[
(D_t^2 u(t), w) - (\text{div} \sigma(t), w) = (f(t), w), \quad w \in V,
\]

(1.3)

\[
(\sigma(t) \chi) + (u(t), \text{div} \chi) = 0, \quad \chi \in H,
\]

\[
u(0) = u_0, \quad D_t \nu(0) = v_0,
\]

where the parentheses denote the appropriate inner products (\( L^2 \)-inner product in \( V \), \( L^2(\Omega)^2 \)-inner product in \( H \)). If \( V_h \subset V \) and \( H_h \subset H \) are finite dimensional subspaces, such as the spaces introduced by Raviart and Thomas [13], and by Brezzi, Douglas, Jr. and Marini [6], a semidiscrete version of (1.3) seeks \( u_h(t) \in V_h, \sigma_h(t) \in H_h, t > 0 \), satisfying

\[
(D_t^2 u_h(t), w_h) - (\text{div} \sigma_h(t), w_h) = (f(t), w_h), \quad w_h \in V_h,
\]

(1.3h)

\[
(\sigma_h(t), \chi_h) + (u_h(t), \text{div} \chi_h) = 0, \quad \chi_h \in H_h,
\]

\[
u_h(0) = u_{0,h}, \quad D_t \nu_h(0) = v_{0,h},
\]

where \( u_{0,h}, v_{0,h} \in V_h \) are approximations to \( u_0 \) and \( v_0 \), respectively.

Johnson and Thomée [12] have discussed the parabolic counterpart of (1.3). The analysis of convergence of (1.3h) can be carried out along similar lines, parallel to Baker and Bramble [4], for example. (1.3h) is treated essentially as a non-conforming « displacement » model for the wave equation (1.1). The purpose of this note is to discuss the convergence of the « velocity-stress » models based on pairs of spaces \( (V_h, H_h) \) such as those in [6], [12], [13]. Thus, defining \( v = D_t u, v_h = D_t u_h \), (1.3) and (1.3h) are transformed, respectively, to

\[
(D_t v(t), w) - (\text{div} \sigma(t), w) = (f(t), w), \quad w \in V,
\]

(1.4)

\[
(D_t \sigma(t), \chi) + (v(t), \text{div} \chi) = 0, \quad \chi \in H,
\]

\[
v(0) = v_0, \quad \sigma(0) = \sigma_0 = \nabla u_0,
\]

where \( v(t) \in V, \sigma(t) \in H, t \geq 0 \), and

\[
(D_t v_h(t), w_h) - (\text{div} \sigma_h(t), w_h) = (f(t), w_h), w_h \in V_h,
\]


\[(1.4_h) \quad (D, \sigma_h(t), \chi_h) + (v_h(t), \text{div} \chi_h) = 0, \quad \chi_h \in H_h, \]

\[v_h(0) = v_{0.h}, \quad \sigma_h(0) = \sigma_{0.h},\]

where \(v_h(t) \in V_h, \ \sigma_h(t) \in H_h, \ t \geq 0.\)

We now list the basic features of the space \(V_h, H_h\) which lead to a straightforward analysis of the convergence of \(v_h\) to \(v\) and \(\sigma_h\) to \(\sigma:\)

(H.1) There exists a linear operator \(\Pi_h : H \to H_h\) such that

\[(1.5) \quad (\text{div} \Pi_h \chi, w_h) = (\text{div} \chi, w_h) \quad \forall w_h \in V_h, \quad \chi \in H,\]

\[(1.6) \quad \|\Pi_h \chi - \chi\| \leq Ch^s \|\chi\| \quad \text{for} \quad 1 \leq s \leq r, \quad r \geq 2\]

(\(\|\|\) is the \(L^2(\Omega)\)-norm, and \(\|\|_s\) is the \(H^s(\Omega)\)-norm).

(H.2) There exists a linear operator \(P_h : V \to V_h\) such that

\[(1.7) \quad (P_h v, \text{div} \chi_h) = (v, \text{div} \chi_h) \quad \forall \chi_h \in H_h, \quad v \in V,\]

\[(1.8) \quad \|P_h v - v\| \leq Ch^s \|v\| \quad \text{for} \quad 1 \leq s \leq r, \quad r \geq 2\]

(\(\|\|\) is the \(L^2(\Omega)\)-norm, and \(\|\|_s\) is the \(H^s(\Omega)\)-norm, and, as usual, \(C\) denotes a generic constant which depends only on the data and on the particular discretization scheme).

If \(\Omega\) is a polygonal domain and \(V_h, H_h\) are the Raviart-Thomas spaces [12], [13], or if these spaces are the pairs introduced in the paper by Brezzi, Douglas, Jr., and Marini [6], \(\text{div} \chi_h \in V_h\), and \(P_h\) can be taken to be the \(L^2\)-projection. For an example of a pair \((V_h, H_h)\) satisfying the above hypotheses (with \(r = 2\)), where \(P_h\) is not the \(L^2\)-projection, we refer the reader to the paper by Johnson and Thomée [12]. We would also like to point out that (H.1) and (H.2) are valid for the mixed method that has been introduced by Arnold, Douglas, Jr., and Gupta [3] to approximate solution of plane elasticity problems. Our analysis is readily adapted to the corresponding (genuine) velocity-stress formulation of the time-dependent problem.

We can now state and prove our convergence result:

**Theorem:** If \(u\) is the solution of (1.1), \(v = D_t u, \ \sigma = \nabla u\), and if the pair \(\{v_h, \sigma_h\}\) is the solution of (1.4_h), under the hypotheses (H.1) and (H.2) we have, for \(1 \leq s \leq r, \ r \geq 2,\)

\[(1.9) \quad \|v_h(t) - v(t)\| + \|\sigma_h(t) - \sigma(t)\| \leq C (\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\|) + \]

\[+ Ch^s \left( \|v_0\|_s + \|\sigma_0\|_s + \int_0^t \|D_\tau v(\tau)\|_s + \|D_\tau \sigma(\tau)\|_s \ d\tau \right).\]
**Proof**: Let us denote by $X$ the space $V \times H$, the elements of which will be designated as $\xi = \{v, \sigma\}$ or $\zeta = \{w, \chi\}$ and set

$$
((\xi, \zeta)) = (v, w) + (\sigma, \chi),
$$

$$
\|\xi\| = \sqrt{((\xi, \xi))}.
$$

Let $X_h = V_h \times H_h$ be equipped with $((., .))$ and the induced norm $\|., .\|$. Elements of $X_h$ will be designated as $\xi_h = \{v_h, \sigma_h\}$ or $\zeta_h = \{w_h, \chi_h\}$. We define the bilinear form $a(., .)$ on $X$ by

$$
a(\xi, \zeta) = -(\text{div} \sigma, w) + (v, \text{div} \chi)
$$

for $\xi = \{v, \sigma\}, \zeta = \{w, \chi\}$.

We can now express (1.4) as

$$
((D, \xi(t), \zeta)) + a(\xi(t), \zeta) = (f(t), w), \quad \zeta \in X
$$

$$
(\xi(t) = \{v(t), \sigma(t)\}, \zeta = \{w, \chi\}),
$$

and we can express (1.4) as

$$
((D, \xi_h(t), \zeta_h)) + a(\xi_h(t), \zeta_h) = (f(t), w_h), \quad \zeta_h \in X_h
$$

$$
(\xi_h(t) = \{v_h(t), \sigma_h(t)\}, \zeta_h = \{w_h, \chi_h\}).
$$

Let us define $P_h \xi = \{P_h v, \Pi_h \sigma\}$ for $\xi = \{v, \sigma\} \in X$, and observe that

$$
a(P_h \xi, \zeta_h) = a(\xi, \zeta_h), \quad \zeta_h \in X_h
$$

by (H.1) ((1.5)) and (H.2) ((1.7)).

Therefore we obtain from (1.11)

$$
((D, \xi(t), \zeta_h)) + a(\xi_h(t), \zeta_h) = (f(t), w_h) + ((P_h D, \xi(t)) - D, ((\xi(t), \zeta_h)), \quad \zeta_h \in X_h.
$$

Setting $\varepsilon_h(t) = P_h \xi(t) - \xi_h(t)$, (1.11) and (1.13) yield

$$
((D, \xi_h(t), \zeta_h)) + a(\varepsilon_h(t), \zeta_h) = ((P_h D, \xi(t)) - D, ((\xi(t), \zeta_h)), \quad \zeta_h \in X_h.
$$

Let us define $\Lambda_h : X_h \to X_h$ by

$$
((\Lambda_h \xi_h, \zeta_h)) = a(\xi_h, \zeta_h), \quad \xi_h, \zeta_h \in X_h.
$$

Since

$$
a(\xi_h, \zeta_h) = -a(\xi_h, \zeta_h), \quad \xi_h, \zeta_h \in X_h,
$$

$$
M^2 AN Modélisation mathématique et Analyse numérique
Mathematical Modelling and Numerical Analysis

T GEVECI
as is readily seen \((cf. \ (1.10))\), \(\Lambda_h\) is shew-adjoint,
\[
\begin{align*}
\((\Lambda_h \xi_h, \xi_h)\) &= -\((\xi_h, \Lambda_h \xi_h)\), \quad \xi_h, \xi_h \in X_h,
\end{align*}
\]
and \(-\Lambda_h\) generates the unitary group \(e^{-t\Lambda_h}\). In particular,
\[
\begin{align*}
\|e^{-t\Lambda_h} \xi_h(0)\| &= \|\xi_h(0)\|, \quad t \in \mathbb{R}.
\end{align*}
\]
Let us denote by \(P^0_h : X \to X_h\) the projection with respect \((., .)\).

We can now express (1.14) as
\[
\begin{align*}
D_t \varepsilon_h(t) + \Lambda_h \varepsilon_h(t) &= P^0_h (P_h D_t \xi(t) - D_t \xi(t))
\end{align*}
\]
so that
\[
\begin{align*}
\varepsilon_h(t) = e^{-t\Lambda_h} \varepsilon_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h} P^0_h (P_h D_\tau \xi(\tau) - D_\tau \xi(\tau)) \, d\tau.
\end{align*}
\]
(1.18) and (1.20) yield the estimate
\[
\begin{align*}
\|\varepsilon_h(t)\| &\leq \|\varepsilon_h(0)\| + \int_0^t \|P_h D_\tau \xi(\tau) - D_\tau \xi(\tau)\| \, d\tau
\end{align*}
\]
\((P^0_h\) is the \((., .)\)-projection).
(1.21) is readily translated to
\[
\begin{align*}
\|P_h v(t) - v_h(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\|
&\leq C \left(\|P_h v_0 - v_{0,h}\| + \|\Pi_h \sigma_0 - \sigma_{0,h}\|ight. \\
&\quad + \left. \int_0^t \left(\|P_h D_\tau v(\tau) - D_\tau v(\tau)\| + \|\Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau)\|\right) \, d\tau\right)
\end{align*}
\]
\[
\begin{align*}
&\leq C \left(\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + \|P_h v_0 - v_0\| + \|\Pi_h \sigma_0 - \sigma_0\|ight. \\
&\quad + \left. \int_0^t \left(\|P_h D_\tau v(\tau) - D_\tau v(\tau)\| + \|\Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau)\|\right) \, d\tau\right),
\end{align*}
\]
and this, together with (1.6) and (1.8), yields
\[
\begin{align*}
\|P_h v(t) - v_h(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\|
&\leq C \left(\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + h^s (\|v_0\|_s + \|\sigma_0\|_s)\right) \\
&\quad + C h^s \int_0^t \left(\|D_\tau v(\tau)\|_s + \|D_\tau \sigma(\tau)\|_s\right) \, d\tau.
\end{align*}
\]
Since
\[ \| v(t) - v_h(t) \| + \| \sigma(t) - \sigma_h(t) \| \]
\[ \leq \| v(t) - P_h v(t) \| + \| P_h v(t) - v_h(t) \| 
+ \| \sigma(t) - \Pi_h \sigma(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \| \]
\[ \leq Ch^t (\| v(t) \|_s + \| \sigma(t) \|_s ) + \| P_h v(t) - v_h(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \| , \]
by (1.6) and (1.8), and
\[ \| v(t) \|_s \leq \| v_0 \|_s + \int_0^t \| D_v v(\tau) \|_s d\tau , \]
\[ \| \sigma(t) \|_s \leq \| \sigma_0 \|_s + \int_0^t \| D_\sigma \sigma(\tau) \|_s d\tau , \]
(1.22) leads to (1.9), the assertion of the theorem.

2. SOME OBSERVATIONS IN REGARD TO THE TIME-DIFFERENCING OF THE SEMIDISCRETE MODEL

(1.4,1) leads to a system of ordinary differential equations in the form
\[ M_0 D_t W - D \Sigma = F , \]
\[ M_1 D_t \Sigma + D^T W = 0 , \]
where \( W \) corresponds to \( v_h \), \( \Sigma \) corresponds to \( \sigma_h \), \( M_0, M_1 \) are symmetric, positive-definite matrices, and \( D^T \) denotes the transpose of \( D \). The application of implicit Euler time-differencing
\[ M_0 \frac{W^{n+1} - W^n}{k} - D \Sigma^{n+1} = F^{n+1} , \]
\[ M_1 \frac{\Sigma^{n+1} - \Sigma^n}{k} + D^T W^{n+1} = 0 , \]
(\( k \) denotes the time step), necessitates the solution of
\[ M_0 W^{n+1} - kD \Sigma^{n+1} = kF^{n+1} + M_0 W^n , \]
\[ M_1 \Sigma^{n+1} + kD^T W^{n+1} = M_1 \Sigma^n . \]

\( M_0 \) is in block-diagonal form if \( V_h \) consists of functions with no continuity requirement across inter-element boundaries, as is the case in [6], [12], [13], and the elimination of \( W^{n+1} \) in (2.3) is efficiently implementable. This leads to a system in the form
\[ (M_1 + k^2 D^T M_0^{-1} D) \Sigma^{n+1} = G , \]

\( M^2 \) AN Modélisation mathématique et Analyse numérique
Mathematical Modelling and Numerical Analysis
MIXED FINITE ELEMENT METHODS

where $M_1$ is symmetric, positive definite and $D^T M_0^{-1} D$ is symmetric, positive-semidefinite, for the determination of $\Sigma^n + 1$.

On the other hand, (1.3) leads to

$$
M_0 D_t^2 U - D \Sigma = F,
$$

$$
M_1 \Sigma + D^T U = 0,
$$

where $U$ corresponds to $u_h$. (2.5) can be expressed as

$$
M_0 D_t^2 U + DM_1^{-1} D^T U = F,
$$

where $M_0$, $DM_1^{-1} D^T$ are symmetric, positive-definite [12]. If (2.6) is expressed as a system in \( \{U, W\} \),

$$
D_t U - W = 0
$$

$$
M_0 D_t W + DM_1^{-1} D^T U = F,
$$

and implicit Euler time-differencing is applied to (2.7),

$$
\frac{U^{n+1} - U^n}{k} - W^{n+1} = 0
$$

$$
M_0 \frac{W^{n+1} - W^n}{k} + DM_1^{-1} D^T U^{n+1} = F^{n+1},
$$

elimination of $W^{n+1}$ leads to a system in the form

$$
(M_0 + k^2 DM_1^{-1} D^T) U^{n+1} = \tilde{G}.
$$

The matrix in (2.9) is symmetric, positive-definite, so that (2.9) is solvable. But $M_1$ is not block-diagonal, unlike $M_0$, so that deriving the reduced system (2.9), which includes inverting $M_1$, is more expensive than forming the reduced system (2.4). The time-independent counterpart of (2.5),

$$
-D \Sigma = F,
$$

$$
M_1 \Sigma + D^T U = 0,
$$

led Arnold and Brezzi [2] to relax the requirement that $\text{div} \sigma_h \in L_2(\Omega)$ in order to have a block-diagonal matrix instead of $M_1$ and be able to eliminate $\Sigma$ efficiently. This approach has to introduce a multiplier corresponding to the relaxation of the requirement $\text{div} \sigma_h \in L_2(\Omega)$.

The above considerations suggest that the « velocity-stress » formulation (1.4) may be preferable to (1.3) if the approximation of the « stress » $\sigma$ is of primary concern.

The application of diagonally implicit Runge-Kutta methods (see, for example, Crouzeix [8], Crouzeix and Raviart [9], Alexander [1], Burrage vol. 22, n° 2, 1988
(7), Dougalis and Serbin [10]) to (2.1) leads to systems similar to (2.4) so that our discussion is relevant to higher-order time differencing as well. We will not prove error estimates for such full-discrete approximation schemes based on \((1.4_h)\). Such estimates should be obtainable by employing techniques that have been utilized in [5] or [11], for example.

REFERENCES