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ASYMPTOTIC TIME-BEHAVIOR FOR WEIGHTED SCALAR CONSERVATION LAWS (*)

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Abstract. — *For weighted scalar conservation laws introduced as model equations for gas flows in axisymetric coordinates or in a nozzle, the asymptotic time-behavior of a entropy weak solution is obtained according to the behaviors of the weight and flux functions, thanks to the explicit formula previously derived in [10].*

Résumé. — *Considérant une loi de conservation scalaire avec poids qui modélise l'évolution d'un gaz en géométrie axisymétrique ou dans une tuyère, on utilise la formule explicite obtenue dans [10] pour préciser le comportement asymptotique de la solution faible entropique suivant le comportement de la fonction-flux et de la fonction-poids de l'équation.*

1. INTRODUCTION : WEIGHTED SCALAR CONSERVATION LAWS

We are interested in *weighted scalar nonlinear hyperbolic conservation laws* in the half space

$$(1.1)\quad \frac{\partial}{\partial t}(r(x)\,u(x,t)) + \frac{\partial}{\partial x}(r(x)\,f(u(x,t))) = 0, x > 0\ , t > 0$$

with a $\mathscr{C}^2$ convex flux-function $f : \mathbb{R} \to \mathbb{R}$ and a $\mathscr{C}^2$ positive weight-function $r :]0, \infty[\to]0, \infty[$. Such equations are considered as model equations for gas dynamics in axisymetric coordinates where the weight-function satisfies

$$(\text{H.1})\qquad r \in L^\infty(0,1) \quad\text{and}\quad \frac{1}{r} \in L^\infty(1,\infty)$$

(for example take $r(x) = x^\alpha$, $\alpha = 0, 1, 2$), or as model equations for gas

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flows in a nozzle where there exists two positive constant r_- and r_+ such that

(H.2) $$0 < r_- \leqslant r(.) \leqslant r_+ < +\infty .$$

For equations of the form (1.1), an uniqueness and existence result was proved in Le Floch-Nedelec [10] thanks to an explicit formula generalizing the one of Lax [7]. Moreover, a suitable formulation of boundary condition (at $x = 0$) was proposed. Refer also to previous works of Whitham [17] and Schonbek [15]. In this paper, we look for the asymptotic time-behavior of the solution of (1.1). We generalize well-known results of time-behavior for equations (1.1) without weight-function (that is $r(.) \equiv 1$)). The rate of convergence is specified according to the properties of the weight-function and the flux-function. For classical results on time-behavior of conservation laws, we refer to Lax [7], Dafermos [2], Conway [1], Liu-Pierre [13]... We present our result of time-behavior in the following section 2. Before, we detail some important properties of the mixed problem associated to the weighted scalar conservation law (1.1). The proofs can be found in [8]-[11].

We look for weak solutions (that is in the sense of distributions) of (1.1) satisfying an initial condition

(1.2) $$u(x, 0) = u_0(x) , \quad x > 0$$

with an initial data $u_0 : \mathbb{R}_+ \to \mathbb{R}$. And for the sake of uniqueness, one adds an entropy condition (Lax [7], Oleinik [14], Kruskov [6])

(1.3) $$u(x-0, t) \geqslant u(x+0, t) , \quad x > 0, t > 0 .$$

We impose a zero boundary condition at $x = 0$, since we are only interested in this paper in the rate of convergence and not in the nonlinear aspect of the boundary condition. Concerning boundary conditions, see [10] and also Le Floch [8], [9], Dubois-Le Floch [3, 4].

Our asumptions concerning the initial data are as follows

(H.3) $$r . u_0 \in L^1(\mathbb{R}_+) , \quad r . f(u_0) \in L^\infty(\mathbb{R}_+) .$$

And, the flux-function satisfies for simplifications

(H.4) $$f'' > 0 , \quad f(0) = f'(0) = 0 .$$

(for example, take $f(u) = \frac{u^2}{2}$). Then, the problem (1.1)-(1.3) admits one and only one solution $u(.,.)$ such that $r . u$ belongs to $L^\infty(\mathbb{R}_t^+ ; L^1(\mathbb{R}_x^+))$ and $rf(u)$ to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$. It satisfies also the following *stability properties* for almost every $t \geqslant 0$: the L^∞-stability

(1.4) $$\|r(.)\, f(u(., t))\|_{L^\infty(\mathbb{R}_x^+)} \leqslant \|r(.)\, f(u_0(.))\|_{L^\infty(\mathbb{R}_x^+)}$$

and the L^1-semi-group property

$$\|(u(.,t)-v(.,t)).r(.)\|_{L^1(\mathbb{R}_x^+)} \leq \|(u_0(.)-v_0(.)).r(.)\|_{L^1(\mathbb{R}_x^+)} \tag{1.5}$$

for two solutions u and v corresponding to two initial data u_0 and v_0 respectively. The property (1.5) generalizes a previous result of Keyfitz [4].

Moreover, take $a = f'$ and let $f_+^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ and $f_-^{-1} : \mathbb{R}_- \to \mathbb{R}_-$ be the two inverse functions of the convex function f. *For each* $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, we can define — thanks to the hypotheses (H.1) or (H.2) (see [10]) — the function :

$$\mathbb{R}_+ \times \{-1,+1\} \in (c,\varepsilon) \mapsto y(c,\varepsilon) \in \mathbb{R}^+$$

by the algebraic relation

$$t = \int_{y(c,\varepsilon)}^{x} \frac{d\xi}{a\left(f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right)\right)} \tag{1.6a}$$

and the function $G : \mathbb{R}_+ \times \{-1,1\} \ni (c,\varepsilon) \to G(c,\varepsilon)$ as follows :

$$G(c,\varepsilon) = \int_0^{y(c,\varepsilon)} u_0(\xi)\, r(\xi)\, d\xi - c\,.\,t + \int_{y(c,\varepsilon)}^{x} f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right) r(\xi)\, d\xi\,. \tag{1.6b}$$

Then the value $u(x,t)$ of the solution u of (1.1)-(1.3) is provided by the following *explicit formula*

$$u(x,t) = f_{\varepsilon(x,t)}^{-1}\left(\frac{c(x,t)}{r(x)}\right) \tag{1.7}$$

where $(c(x,t), \varepsilon(x,t))$ minimizes the function G.

2. ASYMPTOTIC TIME-BEHAVIOR

Using the explicit formula (1.6)-(1.7), we are able as in Lax [7] to get an uniform decay in power of t for the solution u of problem (1.1)-(1.3), according to the behavior of the function f. Assuming that

$$k_-\,.\,|v|^{p-2} \leq f''(v) \leq k_+\,.\,|v|^{p-2}\,, \quad \forall v \in \mathbb{R}\,; p \geq 2, 0 < k_- \leq k_+ \tag{2.1}$$

we have :

THEOREM *Under hypotheses* (H 1) *or* (H 2) *and* (H 3)-(H 4), *the solution* $u(.,.)$ *of* (1 1)-(1 3), *when the time t tends to infinity, decreases with the following rate*

$$|u(x,t)| \leq \left(\frac{kM}{r(x)}\right)^{\frac{1}{p}} \cdot \frac{1}{t^{1/p}} \quad t>0, x>0 \tag{2 2}$$

where the constants k and M depend only on the flux-function and the initial data respectively

$$k = 2p\frac{k_+}{k_-^2}$$

and

$$M = \|r \,.\, u_0\|_{L^1(\mathbb{R}_+)} \quad \blacksquare$$

In the case of flows in axisymetric coordinates

$$r(x) = x^{\alpha}, \quad \alpha \geq 0$$

the estimation (2 2) becomes

$$|u(x,t)| \leq \frac{c}{x^{\alpha/p}} \cdot \frac{1}{t^{1/p}} \quad (c>0) \tag{2 3}$$

Note that, when $\alpha > 0$, this inequality (2 3) provides also the behavior of u when $x \to 0+$ and $x \to +\infty$

Proof of theorem Multiplying (2 1) by $v \in \mathbb{R}$ and integrating lead to

$$k_- \cdot \frac{|v|^p}{p} \leq a(v)\,v - f(v) \leq k_+ \frac{|v|^p}{p} \tag{2 4}$$

because of

$$\frac{d}{dv}(a(v)\,v - f(v)) = a'(v)\,v = f''*(v)\,v$$

Moreover, by two successive integrations of (2 1), we have also

$$k_- \cdot \frac{|v|^p}{p(p-1)} \leq f(v) \leq k_+ \cdot \frac{|v|^p}{p(p-1)} \tag{2 5}$$

Then from (2 4)-(2 5) it results that

$$\frac{k_-}{k_+}(p-1)\,f(v) \leq a(v)\,v - f(v) \leq \frac{k_+}{k_-}(p-1)\,f(v)$$

or, for $v > 0$:

$$\text{(2.6}a\text{)} \qquad \frac{k_-}{k_+}(p-1)\frac{f(v)}{a(v)} \leqslant v - \frac{f(v)}{a(v)} \leqslant \frac{k_+}{k_-}(p-1)\frac{f(v)}{a(v)}$$

and for $v < 0$:

$$\text{(2.6b)} \qquad \frac{k_+}{k_-}(p-1)\frac{f(v)}{a(v)} \leqslant v - \frac{f(v)}{a(v)} \leqslant \frac{k_-}{k_+}(p-1)\frac{f(v)}{a(v)}.$$

Henceforth, the following minoration of the function G defined by (1.6) is an immediate consequence of (2.6) (used with the value $v = f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right)$)

$$G(c, \varepsilon) = \int_0^{y(c,\varepsilon)} u_0\, r\, d\xi + \int_{y(c,\varepsilon)}^x \left\{ f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right) - \frac{\dfrac{c}{r(\xi)}}{a f_\varepsilon^{-1}\left(\dfrac{c}{r(\xi)}\right)} \right\} r(\xi)\, d\xi$$

$$\geqslant -M + \int_{y(c,\varepsilon)}^x \frac{k_-}{k_+}(p-1)\frac{\dfrac{c}{r(\xi)}}{a f_\varepsilon^{-1}\left(\dfrac{c}{r(\xi)}\right)} \cdot r(\xi)\, d\xi$$

where $M = \|r \,.\, u_0\|_{L^1(\mathbb{R}^+)}$. Thus, the inequality

$$\text{(2.7)} \qquad G(c, \varepsilon) \geqslant -M + \frac{k_-}{k_+}(p-1)\, c \,.\, t$$

holds for all $c = 0$, $\varepsilon = \pm 1$, $t > 0$.

Furthermore, for each (x, t) in $\mathbb{R}_+ \times \mathbb{R}_+$, the minimum value of the function G — which is by notation attained at $(c(x,t), \varepsilon(x,t))$ — may be majorized by the value of G at $c = 0$:

$$\text{(2.8)} \qquad G(c(x,t), \varepsilon(x,t)) \leqslant \int_0^x r u_0\, d\xi \leqslant M.$$

Now comparing (2.7) and (2.8), il results the estimation

$$(p-1)\, t \frac{k_-}{k_+} c(x,t) \leqslant 2M$$

or

$$0 \leqslant c(x,t) \leqslant \frac{k_+}{k_-} \frac{2M}{(p-1)\,t}.$$

Thus in virtue of (1.7), we have proved that $rf(u)$ decreases uniformly in $x \in \mathbb{R}_+$ when $t \to \infty$:

$$\sup_{x \in \mathbb{R}_+} |r(x) f(u(x,t))| \leq \frac{k_+}{k_-} \frac{2M}{(p-1)t}.$$

It remains to use again (2.5) :

$$k_- \cdot \frac{|u(x,t)|^p}{p(p-1)} \leq f(u(x,t)) \leq \frac{2Mk_+}{k_- \cdot r(x)} \cdot \frac{1}{t}$$

which gives (2.2). ■

When the solution u is bounded — say the weight-function r satisfies the properties (H.2) — it suffices to assume that the inequalities (2.1) hold in the neighborhood of $v = 0$ with one $p_0 \geq 2$ and also that f'' is uniformly bounded :

$$\text{Cte} \leq f'' \leq \text{Cte}'.$$

Namely, thanks to the theorem applied with $p = 2$, we know that the solution u tends to zero uniformly in $x \in \mathbb{R}_+$ when the time goes to infinity

$$\sup_{x \in \mathbb{R}_+} |u(x,t)| \leq \left(\frac{kM}{r_-}\right)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{t}} \quad t > 0.$$

So for t sufficiently large, say $t > T_0$, $u(x,t)$ is for each x in $\mathbb{R}_+$ in the neighborhood where (2.1) holds. And the same proof as previously gives

$$|u(x,t)| \leq \left(\frac{kM}{r_-}\right)^{\frac{1}{p_0}} \cdot \frac{1}{t^{1/p_0}}, \quad \forall x \in \mathbb{R}_+, \forall t > T_0.$$

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