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ON THE APPROXIMATION OF THE SPECTRUM OF THE STOKES OPERATOR (*)

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Abstract. — *Error estimates are derived for the approximate calculation of the eigenvalues of the Stokes Operator. These estimates are valid for the regularized versions of mixed methods that satisfy the uniform Ladyzhenskaya-Babuška-Brezzi condition.*

Résumé. — *Nous obtenons ici des estimations d'erreur pour le calcul approché des valeurs propres de l'opérateur de Stokes. Ces estimations sont valables pour les versions régularisées des méthodes mixtes qu satisfont la condition uniforme de Ladyzhenskaya-Babuška-Brezzi.*

THE BACKGROUND AND THE CONVERGENCE RESULT

Let $\Omega \subset \mathbf{R}^n$ $(n = 2$ or $3)$ be a bounded domain with boundary $\partial\Omega$. The Stokes problem consists of finding u, an $\mathbf{R}^n$-valued function, and p, a scalar function, such that

$$(1) \qquad \begin{aligned} -\nu\,\Delta u + \operatorname{grad} p &= f \quad \text{in} \quad \Omega\,,\\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega\,,\\ u &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

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Here, $\nu > 0$ is the viscosity, f is a given $\mathbf{R}^n$-valued function, Δ denotes the Laplacian acting componentwise, grad denotes the gradient, and div denotes the divergence.

$H_0^1(\Omega)$, $H^r(\Omega)$, $r \geq 1$, denote the standard Sobolev spaces, $(L^2(\Omega))^n$, $(H_0^1(\Omega))^n$, $(H^r(\Omega))^n$ denote the spaces of $\mathbf{R}^n$-valued functions with components in the respective spaces. $(\cdot\,,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^n$, $\|\cdot\|_0$ denotes the induced norm. The norm in $H^r(\Omega)$ or $(H^r(\Omega))^n$ is denoted by $\|\cdot\|_r$. Ω is assumed to have sufficiently smooth boundary so that the assertions that follow are valid.

Let

$$V = \{v \in (H_0^1(\Omega))^n : \operatorname{div} v = 0 \text{ in } \Omega\} ,$$
$$H = \{v \in (L^2(\Omega))^n : \operatorname{div} v = 0 \text{ in } \Omega,\ v \,.\, n = 0 \text{ on } \partial\Omega\}$$

where n denotes the unit normal to the boundary, as in Temam [12].

It is assumed that the solution u of (1) is in $V \cap (H^2(\Omega))^n$, $p \in \{q \in L^2(\Omega) : (q, 1) = 0\}$ and

$$\|u\|_2 + \|p\|_1 \leq C(\nu, \Omega)\|f\|_0 .$$

Thus, the Stokes operator

$$A = -\nu P_H \Delta : V \cap (H^2(\Omega))^n \subset H \to H ,$$

where $P_H : (L^2(\Omega))^n \to H$ is the orthogonal $(L^2(\Omega))^n$-projection, is positive-definite, self-adjoint, has compact inverse, so that there exists a sequence of eigenvalues of A,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots ,\ \lim_{j \to \infty} \lambda_j = \infty ,$$

and a corresponding sequence $\{u_j\}_{j=1}^{\infty}$ of eigenfunctions which are orthonormal and complete in H [12], [13].

The approximate calculation of the eigenvalues of the Stokes operator A, or of an operator similar to A, is of interest in regard to the stability of incompressible fluid flow or the vibrations of an incompressible elastic medium, for example. The aim of this note is to establish the convergence of certain efficiently implementable approximation schemes which are based on the regularization of mixed methods that are used for the approximate solution of (1).

Let $W^h \subset (H_0^1(\Omega))^n$ be a finite dimensional subspace. Here h refers to the maximum diameter of the rectangular or triangular subregions (or their 3-dimensional counterparts) constituting a subdivision of Ω. We will confine the discussion to the conforming case for the sake of brevity. Let $Q^h \subset L^2(\Omega)$ be another finite dimensional space. We set

$$a(u, v) = \nu \int_{\Omega} \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx , \quad u, v \in (H_0^1(\Omega))^n .$$

An approximation to the solution (u, p) of (1) is a pair $(u^h, p^h) \in W^h \times Q^h$ which satisfies

$$(2) \qquad \begin{aligned} a(u^h, w^h) - (p^h, \operatorname{div} w^h) &= (f, w^h) , \quad & w^h \in W^h , \\ (\operatorname{div} u^h, q^h) &= 0 , & q^h \in Q^h . \end{aligned}$$

Let us define the discrete divergence operator $\operatorname{div}_h : W^h \to Q^h$ by

$$(\operatorname{div}_h w^h, q^h) = (\operatorname{div} w^h, q^h) , \quad q^h \in Q^h ,$$

and its adjoint $- \operatorname{grad}_h : Q^h \to W^h$ by

$$(- \operatorname{grad}_h q^h, w^h) = (q^h, \operatorname{div}_h w^h) , \quad w^h \in W^h .$$

We will assume

(H.1) The kernel of grad_h consists of constants, and

(H.2) The LBB (Ladyzhenskaya-Babuška-Brezzi) condition : There exists $\alpha > 0$, independent of h, such that

$$\sup_{\substack{w^h \in W^h \\ w^h \neq 0}} \frac{(q^h, \operatorname{div} w^h)}{\|w^h\|_1} \geq \alpha \|q^h\|_0$$

for $q^h \in \{q^h \in Q^h : (q^h, 1) = 0\}$.

Under the hypotheses (H.1) and (H.2), it is well known that there exists a unique solution $(u^h, p^h) \in W^h \times \{q^h \in Q^h : (q^h, 1) = 0\}$ of (2) and that

$$(3) \qquad \|u^h\|_1 + \|p^h\|_0 \leq C(\nu, \alpha, \Omega) \|f\|_0 ,$$

$$(4) \quad \|u - u^h\|_1 + \|p - p^h\|_0 \leq C(\nu, \alpha, \Omega) \left\{ \inf_{w^h \in W^h} \|u - w^h\|_1 + \inf_{q^h \in Q^h} \|p - q^h\|_0 \right\}$$

(see, for example, Girault and Raviart [5]). To be specific, let us assume that (4) and the approximation properties of W^h, Q^h lead to the estimates

$$(5) \qquad \|u - u^h\|_1 + \|p - p^h\|_0 \leq C(\nu, \alpha, \Omega, f) h^{r-1} ,$$

$$(6) \qquad \|u - u^h\|_0 \leq C(\nu, \alpha, \Omega, f) h^r ,$$

$r \geq 2$ [5].

Let us define $V^h = \{v^h \in W^h : \operatorname{div}_h v^h = 0\}$, the discrete counterpart of V, and $A_h : V^h \to V^h$ by

$$(A_h u^h, v^h) = a(u^h, v^h), \quad v^h \in V^h .$$

A_h is the discrete counterpart of the Stokes operator A. It is positive-definite, self-adjoint. Let us denote its eigenvalues by $\{\lambda_j^h\}_{j=1}^{N_h}$, $0 < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_{N_h}^h$, and let $\{u_j^h\}_{j=1}^{N_h}$ form a complete set of orthonormal eigenfunctions corresponding to $\{\lambda_j\}_{j=1}^{N_h}$, in the given order. Assuming (H.1), (H.2) and the error estimates (5), (6), error estimates such as

$$|\lambda_j^h - \lambda_j| \leq C(\nu, \alpha, \Omega, j)\, h^{2(r-1)} \tag{7}$$

follow from the results of Mercier, Osborn, Rappaz, Raviart [8] and Peterson [10].

The computation of $\{\lambda_j^h\}_{j=1}^{N_h}$ may be carried out within the self-adjoint framework if a basis for V^h is available (see, for example, [6], [7], [14]). Otherwise we may consider a regularized version of (2) which computes $(u^{\varepsilon,h}, p^{\varepsilon,h}) \in W^h \times Q^h$ such that

$$a(u^{\varepsilon,h}, w^h) - (p^{\varepsilon,h}, \operatorname{div}_h w^h) = (f, w^h),\ w^h \in W^h ,$$
$$\varepsilon(p^{\varepsilon,h}, q^h) + (\operatorname{div}_h u^{\varepsilon,h}, q^h) = 0,\ q^h \in Q^h ,$$

i.e.,

$$p^{\varepsilon,h} = -\frac{1}{\varepsilon}\operatorname{div}_h u^{\varepsilon,h} ,$$

and

$$a(u^{\varepsilon,h}, w^h) + \frac{1}{\varepsilon}(\operatorname{div}_h u^{\varepsilon,h}, \operatorname{div}_h w^h) = (f, w^h),\ w^h \in W^h . \tag{8}$$

This formulation has been quite popular in recent years. In some cases the penalty term may be evaluated by reduced (inexact) integration (hence the name, reduced integration — penalty methods), and the implementation is efficient (see, for example, [9]).

If (H.1) and (H.2) are satisfied, it is known that

$$\|u^{\varepsilon,h}\|_1 \leq C(\nu, \alpha, \Omega)\, \|f\|_0 , \tag{9}$$

$$\|u^{\varepsilon,h} - u^h\|_1 \leq C(\nu, \alpha, \Omega)\, \|f\|_0\, \varepsilon , \tag{10}$$

where C is independent of h (see [2], [5], [9]). Here and in the sequel C will designate possibly differing constants.

Let us define $A_{\varepsilon,h} : W^h \to W^h$ by

(11) $$(A_{\varepsilon,h}\, u^h, w^h) = a(u^h, w^h) + \frac{1}{\varepsilon}\,(\mathrm{div}_h\, u^h, \mathrm{div}_h\, w^h),\; w^h \in W^h .$$

$A_{\varepsilon,h}$ is positive-definite, self-adjoint. Let us denote the eigenvalues as $\{\lambda_j^{\varepsilon,h}\}_{j=1}^{M_h}$, $0 < \lambda_1^{\varepsilon,h} \leqslant \lambda_2^{\varepsilon,h} \leqslant \cdots \leqslant \lambda_{M_h}^{\varepsilon,h}$. These eigenvalues can be computed efficiently, for example, by Bathe's subspace iteration technique [1]. We will establish the following result :

THEOREM : *Under the hypotheses* (H.1) *and* (H.2) (*the uniform Ladyzhenskaya-Babuška-Brezzi condition*),

(12) $$0 \leqslant \lambda_j^h - \lambda_j^{\varepsilon,h} \leqslant C(\nu, \alpha, \Omega, j)\,\varepsilon$$

where C is independent of h, $0 < h \leqslant h_0$, *for some* $h_0 > 0$.

Proof: Our proof is based on the min-max principle, along the lines of Canuto [3], [4], and Strang and Fix [11].

We first note that

(13) $$\lambda_j^{\varepsilon,h} \leqslant \lambda_j^h .$$

Indeed, by the min-max principle

$$\begin{aligned}\lambda_j^{\varepsilon,h} &= \min_{\substack{S_j \subset W^h \\ \dim S_j = j}} \max_{w^h \in S_j} \frac{a(w^h, w^h) + \frac{1}{\varepsilon}(\mathrm{div}_h\, w^h, \mathrm{div}_h\, w^h)}{\|w^h\|_0^2} \\ &\leqslant \max_{u^h \in \langle u_1^h, u_2^h, \dots, u_j^h\rangle} \frac{a(u^h, u^h)}{\|u^h\|_0^2} \\ &= \lambda_j^h ,\end{aligned}$$

since $\mathrm{div}_h\, u^h = 0$ for $u^h \in \langle u_1^h, u_2^h, \dots, u_j^h\rangle =$ the linear span of $u_1^h, u_2^h, \dots, u_j^h$, a subspace of V^h.

Let us define $T_{\varepsilon,h} : W^h \to W^h$ as $A_{\varepsilon,h}^{-1}$. Thus the eigenvalues of $T_{\varepsilon,h}$ are $\mu_j^{\varepsilon,h} = \frac{1}{\lambda_j^{\varepsilon,h}}$, $j = 1, 2, \dots, M_h$. We define $P_{\varepsilon,h} : W^h \to V^h$ by

$$P_{\varepsilon,h}\, w^h = T_h(A_{\varepsilon,h}\, w^h) ,$$

where $T_h f = u^h \in V^h$, u^h being the solution of (2), so that $T_h : (L^2(\Omega))^n \to V^h$.

Since $w^h = T_{\varepsilon,h}(A_{\varepsilon,h}\, w^h)$ for $w^h \in W^h$,

$$w^h - P_{\varepsilon,h}\, w^h = (T_{\varepsilon,h} - T_h)(A^{\varepsilon,h}\, w^h) ,$$

so that

$$\|w^h - P_{\varepsilon,h}\, w^h\|_0 \leq C(\nu, \alpha, \Omega)\|A^{\varepsilon,h}\, w^h\|_0\, \varepsilon \tag{14}$$

by (10).

By the max-min principle,

$$\mu_j^{\varepsilon,h} = \max_{\substack{S_j \subset W^h \\ \dim S_j = j}} \min_{w^h \in S_j} \frac{(T_{\varepsilon,h}\, w^h, w^h)}{\|w^h\|_0^2} = \min_{\substack{w^h \in \langle u_1^{\varepsilon,h}, \dots, u_j^{\varepsilon,h}\rangle \\ \|w^h\|_0 = 1}} (T_{\varepsilon,h}\, w^h, w^h)\,, \tag{15}$$

where $\{u_j^{\varepsilon,h}\}_{j=1}^{M_h}$ form a complete set of orthonormal eigenfunctions corresponding to $\{\lambda_j^{\varepsilon,h}\}_{j=1}^{M_h}$, in the given order.

For $w^h \in \langle u_1^{\varepsilon,h}, \dots, u_j^{\varepsilon,h}\rangle$, $\|w^h\|_0 = 1$, let us write

$$\begin{aligned}(T_{\varepsilon,h}\, w^h, w^h) = {} & \frac{(T_h\, P_{\varepsilon,h}\, w^h, P_{\varepsilon,h}\, w^h)}{\|P_{\varepsilon,h}\, w^h\|_0^2} \\ & + \frac{(T_h, P_{\varepsilon,h}\, w^h, P_{\varepsilon,h}\, w^h)}{\|P_{\varepsilon,h}\, w^h\|_0^2} (\|P_{\varepsilon,h}\, w^h\|_0^2 - \|w^h\|_0^2) \\ & + (T_h, P_{\varepsilon,h}\, w^h, w^h - P_{\varepsilon,h}\, w^h) \\ & + (T_{\varepsilon,h}\, w^h - T_{\varepsilon,h}\, P_{\varepsilon,h}\, w^h, w^h) \\ & + (T_{\varepsilon,h}\, P_{\varepsilon,h}\, w^h - T_h\, P_{\varepsilon,h}\, w^h, w^h)\,.\end{aligned} \tag{16}$$

Thanks to the estimates (3), (7), (9), (10) and (13), the second term on the right-hand side of (16) can be estimated as follows :

$$\begin{aligned}&\frac{(T_h\, P_{\varepsilon,h}\, w^h, P_{\varepsilon,h}\, w^h)}{\|P_{\varepsilon,h}\, w^h\|_0^2} (\|P_{\varepsilon,h}\, w^h\|_0^2 - \|w^h\|_0^2) \\ &\qquad \leq C(\nu, \alpha, \Omega)\|P_{\varepsilon,h}\, w^h - w^h\|_0 (\|P_{\varepsilon,h}\, w^h\|_0 + \|w^h\|_0) \\ &\qquad \leq C(\nu, \alpha, \Omega)\ \|A_{\varepsilon,h}\, w^h\|_0^2\, \varepsilon \\ &\qquad \leq C(\nu, \alpha, \Omega)\ (\lambda_j^{\varepsilon,h})^2\, \varepsilon \\ &\qquad \leq C(\nu, \alpha, \Omega)\ (\lambda_j^h)^2\, \varepsilon \\ &\qquad \leq C(\nu, \alpha, \Omega, j)\, \lambda_j^2\, \varepsilon\,,\end{aligned}$$

say, for $0 < h \leq h_0$.

The other terms of (16) are estimated in a similar manner and we obtain from (15),

$$(17)\quad \mu_j^{\varepsilon,h} \leqslant \min_{w^h \in \langle u_1^{\varepsilon,h}, \dots, u_j^{\varepsilon,h} \rangle} \frac{(T_h P_{\varepsilon,h} w^h, P_{\varepsilon,h} w^h)}{\|P_{\varepsilon,h} w^h\|_0^2} + C(\nu, \alpha, \Omega, j)\, \lambda_j^2\, \varepsilon\,.$$

Thanks to (14), $P^{\varepsilon,h}(\langle u_1^{\varepsilon,h}, \dots, u_j^{\varepsilon,h} \rangle) \subset V^h$ is j-dimensional for sufficiently small ε (as in Strang and Fix [11], proof of Lemma 6.1, page 229), « sufficiently small » being independent of h, by (13) and (14). Therefore we obtain from (17), again by the max-min principle,

$$\mu_j^{\varepsilon,h} \leqslant \mu_j^h + C(\nu, \alpha, \Omega, j)\, \lambda_j^2\, \varepsilon\,,$$

so that

$$(18)\quad \begin{aligned} 0 \leqslant \lambda_j^h - \lambda_j^{\varepsilon,h} &\leqslant C(\nu, \alpha, \Omega, j)\, \lambda_j^2\, \lambda_j^{\varepsilon,h} \,.\, \lambda_j^h\, \varepsilon \\ &\leqslant C(\nu, \alpha, \Omega, j)\, \lambda_j^2 (\lambda_j^h)^2\, \varepsilon \\ &\leqslant C(\nu, \alpha, \Omega, j)\, \lambda_j^4\, \varepsilon \end{aligned}$$

by (13) and (17), and the theorem is established.

Remark : One can actually improve (18) to obtain an estimate in the form

$$0 \leqslant \lambda_j^h - \lambda_j^{\varepsilon,h} \leqslant C(\nu, \alpha, \Omega, j)\, \lambda_j^2\, \varepsilon$$

using a technique similar to that of Canuto's [3], [4].

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