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THE CONVERGENCE OF A GALERKIN APPROXIMATION SCHEME FOR AN EXTENSIBLE BEAM (*)

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Abstract. — *Error estimates are derived for the convergence of a semidiscrete Galerkin approximation scheme for the equation of an extensible beam A modification of the Crank-Nicolson time discretization is also discussed*

Résumé. — *Les estimations de l'erreur sont déduites de la convergence d'un schéma d'approximation semi-discret au sens de Galerkin pour une poutre extensible. On discute aussi une modification de la discrétisation du temps de Crank-Nicolson*

1. INTRODUCTION AND THE MATHEMATICAL BACKGROUND

The transverse displacement u of an extensible beam with hinged ends, assuming that the beam corresponds to the interval $[0, 1]$, is governed by the following equation that has been suggested by Woinowsky-Krieger [13] :

$$\begin{aligned}
&D_t^2 u(t,x) + \alpha D_x^4 u(t,x) - \\
(1.1)\quad &\quad - \left[\beta + \gamma \int_0^1 (D_\xi u(t,\xi))^2 \, d\xi\right] D_x^2 u(t,x) = 0, x \in (0,1), t > 0, \\
&u(t,0) = u(t,1) = 0, D_x^2 u(t,0) = D_x^2 u(t,1) = 0, t > 0, \\
&u(0,x) = u_0(x), D_t u(0,x) = \dot{u}_0(x), x \in [0,1].
\end{aligned}$$

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Here $\alpha > 0$, $\gamma > 0$ and β are constants, and u_0, $\dot{u}_0$ are given functions. As in Dickey [5] and Ball [2], β may be positive or negative corresponding to a beam under tension or compression, respectively.

Equation (1.1) and similar equations have been investigated by several authors. We refer the reader to the papers by Dickey [5] and Ball [2] concerning the existence of generalized solutions and to the paper by Holmes and Marsden [7] for the existence of smooth solutions. In this paper we will examine the stability and convergence of a semidiscrete Galerkin approximation scheme for (1.1) and a fully discrete scheme based on it.

We use the standard notation for Sobolev spaces and norms. In particular, L^2 denotes $L^2(0,1)$, $(.,.)$ denotes the L^2-inner product, $\|.\|$ denotes the L^2-norm. H^k is $H^k(0,1)$ and $\|.\|_k$ denotes the norm of H^k. $H_0^1 = \{u \in H^1 : u(0) = u(1) = 0\}$ and $\dot{H}^2$ denotes $H_0^1 \cap H^2$.

The Galerkin formulation of (1.1) that is relevant to the approximation schemes that we will consider is as follows :

$$\text{(1.2)} \quad \begin{aligned} &\text{Find } u(t) \in \dot{H}^2 \text{ such that for each } \varphi \in \dot{H}^2 \text{ and } t > 0 \\ &(D_t^2 u(t), \varphi) + \alpha (D_x^2 u(t), D_x^2 \varphi) - \\ &\qquad - (\beta + \gamma \|D_x u(t)\|^2)(D_x^2 u(t), \varphi) = 0 \\ &\text{and} \\ &u(0) = u_0, \quad D_t u(0) = \dot{u}_0 \\ &(D_x^2 u(t,0) = D_x^2 u(t,1) = 0 \text{ are natural boundary conditions}) . \end{aligned}$$

Let us define the bilinear form

$$\text{(1.3)} \qquad a(u, \varphi) = \alpha (D_x^2 u, D_x^2 \varphi) , \quad u, \varphi \in \dot{H}^2 .$$

If the domain of A is defined as

$$\text{(1.4)} \qquad D(A) = \left\{ u \in \dot{H}^2 \cap H^4 : D_x^2 u(0) = D_x^2 u(1) = 0 \right\}$$

and $A : D(A) \subset L^2 \to L^2$ is defined by

$$\text{(1.5)} \qquad Au = \alpha D_x^4 u$$

we have

$$\text{(1.6)} \qquad (Au, \varphi) = a(u, \varphi) , \quad u \in D(A) , \quad \varphi \in \dot{H}^2 .$$

$a(.,.)$ is a bounded, coercive bilinear form on $\dot{H}^2 \times \dot{H}^2$ [4, p. 273] and A is a positive-definite, self-adjoint operator. We note that $Au = f$ means that u is the solution of the elliptic boundary value problem

$$\text{(1.7)} \qquad \begin{aligned} &\alpha D_x^4 u = f \text{ in } (0,1) , \\ &u(0) = u(1) = 0 , \quad D_x^2 u(0) = D_x^2 u(1) = 0 . \end{aligned}$$

$u \in \dot{H}^2$ and $a(u, \varphi) = (f, \varphi)$, $\varphi \in \dot{H}^2$, is the Ritz-Galerkin formulation of (1.7).

Setting

$$f(u) = -(\beta + \gamma \|D_x u\|^2) D_x^2 u , \tag{1.8}$$

(1.1) can be expressed for $u(t) \in D(A)$, $t \geq 0$ as

$$D_t^2 u(t) + Au(t) + f(u(t)) = 0, t > 0, u(0) = u_0, D_t u(0) = \dot{u}_0, \tag{1.9}$$

and (1.2) can be expressed for $u(t) \in \dot{H}^2$, $t \geq 0$, as

$$(D_t^2 u(t), \varphi) + a(u(t), \varphi) + (f(u(t)), \varphi) = 0,\ t > 0,\ \varphi \in \dot{H}^2 , \quad u(0) = u_0,\ D_t u(0) = \dot{u}_0 . \tag{1.10}$$

Let $S_h \subset \dot{H}^2$ denote the space of Hermite cubics corresponding to a partition of $[0, 1]$ to subintervals of length h (see, for example, Strang and Fix [10]). Any finite dimensional subspace of $\dot{H}^2$ leads to analysis along the same lines, but we will specifically consider the semidiscrete version of (1.10) that seeks $u_h(t) \in S_h$, $t \geq 0$, which satisfies

$$\begin{aligned} &(D_t^2 u_h(t), \varphi_h) + a(u_h(t), \varphi_h) + (f(u_h(t)), \varphi_h) = 0, t > 0, \varphi_h \in S_h, \\ &u_h(0) = u_{0,h},\ D_t u_t(0) = \dot{u}_{0,h} \end{aligned} \tag{1.11}$$

where $u_{0,h}$, $\dot{u}_{0,h} \in S_h$ are approximations to u_0, $\dot{u}_0$, respectively.

Our convergence analysis and the fully discrete scheme we consider necessitate the expression of (1.9), (1.10) and (1.11) as evolution equations. We write (1.9) as

$$D_t \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(u(t)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where I denotes the identity operator, and set $U(t) = [u(t), \dot{u}(t)]^T$ (^{T} denotes the transpose),

$$\Lambda = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}, \quad F(U) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix} \tag{1.12}$$

so that

$$\begin{aligned} D_t U(t) + \Lambda U(t) + F(U(t)) &= 0 \\ U(0) &= U_0 , \end{aligned} \tag{1.13}$$

where $U_0 = [u_0, \dot{u}_0]^T$.

The evolution equation (1.13) will be considered within the framework of the Hilbert space $H = \dot{H}^2 \times L^2$ equipped with the inner product

$$(U, V)_e = a(u, v) + (\dot{u}, \dot{v}) \tag{1.14}$$

for $U = [u, \dot{u}]^T$, $V = [v, \dot{v}]^T$, and the associated norm

$$\|U\|_e = \sqrt{a(u, u) + \|\dot{u}\|^2}\,. \tag{1.15}$$

Due to the coercivity of a $(.,.)$, $\|.\|_e$ is equivalent to the usual norm of $\dot{H}^2 \times L^2$.

The domain $D(\Lambda)$ of Λ is defined as $D(A) \times \dot{H}^2$ and $\Lambda : D(\Lambda) \subset H \to H$ is skew-adjoint ($i\Lambda$, $i = \sqrt{-1}$, is self-adjoint) so that $-\Lambda$ generates the unitary group $e^{-t\Lambda}$. In particular,

$$\|e^{-t\Lambda} U_0\|_e = \|U_0\|_e\,, \quad t \in \mathbf{R}\,. \tag{1.16}$$

The map $F : H \to H$ is C^∞. Thus, as discussed by Holmes and Marsden [7], a strong solution $U(t)$ of (1.13) exists for $U_0 \in D(\Lambda)$ and $D_t^k U(t) \in D(\Lambda^{n-k})$, $k = 0, 1, \dots, n-1$, $n = 1, 2, \dots$, for $U_0 \in D(\Lambda^n)$ and all $t \geq 0$. Here $D(\Lambda^n)$, $n = 2, 3, \dots$, is defined inductively as the set of all $U \in D(\Lambda^{n-1})$ for which $\Lambda U \in D(\Lambda^{n-1})$ and is endowed with the graph norm

$$\|U\|^2_{D(\Lambda^n)} = \|U\|^2_{D(\Lambda^{n-1})} + \|\Lambda U\|^2_{D(\Lambda^{n-1})}\,.$$

It is readily seen that $U = [u, \dot{u}]^T \in D(\Lambda^n)$ iff

$$\begin{aligned} u \in H^{2n+2},\ & u(0) = u^{(2)}(0) = \cdots = u^{(2n)}(0) = 0, \\ & u(1) = u^{(2)}(1) = \cdots = u^{(2n)}(1) = 0\,, \\ \dot{u} \in H^{2n},\ & \dot{u}(0) = \dot{u}^{(2)}(0) = \cdots = \dot{u}^{(2n-2)}(0) = 0, \\ & \dot{u}(1) = \dot{u}^{(2)}(1) = \cdots = \dot{u}^{(2n-2)}(1) = 0\,, \end{aligned} \tag{1.17}$$

and that $\|.\|_{D(\Lambda^n)}$ is equivalent to the norm of $H^{2n+2} \times H^{2n}$ on $D(\Lambda^n)$.

The existence of the solution for all $t \geq 0$ follows from the conservation of energy, energy being

$$E(t) = \frac{1}{2}\left\{\|\dot{u}\|^2 + \alpha\|D_x^2 u\|^2 + \beta\|D_x u\|^2 + \frac{\gamma}{2}\|D_x u\|^4\right\}(t) \tag{1.18}$$

(see Ball [2]). Conservation of energy follows directly from the Galerkin formulation (1.10) and leads to bounds on $\|U(t)\|_e$ in terms of $\|u(0)\|_e$ [2].

We would like to emphasize the locally Lipschitz character of F:

$$\|F(U) - F(V)\|_e \leq K(\|U\|_e, \|V\|_e)\|U - V\|_e,\ U, V \in H\,, \tag{1.19}$$

where K is a continuous function [2]. (1.19), coupled with conservation of energy (1.18) leads to the well-posedness statement

$$\|U(t) - V(t)\|_e \leq e^{M(\|U_0\|_e, \|V_0\|_e)t} \cdot \|U_0 - V_0\|_e\,, \quad t \geq 0\,, \tag{1.20}$$

where $U(t)$ and $V(t)$ denote solutions corresponding to the initial conditions U_0 and V_0, respectively, and M is a continuous function [2].

In the convergence analysis we will have occasion to refer to regularity results of the form

$$\|D_t^k U(t)\|_{D(\Lambda^n)} \leqslant C(t, \|U_0\|_{D(\Lambda^{n+k})}) \tag{1.21}$$

where C is a continuous function of its arguments. Even though we will not bother to be specific about the form of C in order not to clutter the notation and distract from the main features of the analysis, the reader should be able to convince himself that such bounds do in fact exist as long as the initial data is sufficiently regular ($U_0 \in D(\Lambda^{n+k})$ with $n+k$ sufficiently large) thanks to the papers [2], [7].

We will express the evolution form of the galerkin formulation (1.10) as follows : $U(t) = [u(t), \dot{u}(t)]^T \times \dot{H}^2 \times \dot{H}^2$ is determined so that

$$a(D_t u(t), \varphi) - a(\dot{u}(t), \varphi) = 0\,, \quad \varphi \in \dot{H}^2\,, \quad t > 0\,,$$

$$(D_t \dot{u}(t), \dot{\varphi}) + a(u(t), \dot{\varphi}) + f(u(t), \dot{\varphi}) = 0\,, \quad \dot{\varphi} \in \dot{H}^2\,, t > 0\,, \tag{1.22}$$

$$u(0) = u_0\,, \quad \dot{u}(0) = \dot{u}_0\,.$$

Introducing the bilinear form $\Pi(.,.)$ on $\dot{H}^2 \times \dot{H}^2$ by

$$\Pi(U, \Phi) = -a(\dot{u}, \varphi) + a(u, \dot{\varphi}) \tag{1.23}$$

where $\Phi = [\varphi, \dot{\varphi}]^T$, (1.22) can be written as

$$(D_t U(t), \Phi)_e + \Pi(U(t), \Phi) + (F(U(t)), \Phi)_e = 0\,,$$

$$\Phi \in \dot{H}^2 \times \dot{H}^2\,, \qquad t > 0\,, \tag{1.24}$$

and

$$U(0) = U_0\,.$$

Note that Π is skew-adjoint,

$$\Pi(U, \Phi) = -\Pi(\Phi, U) \tag{1.25}$$

and, in particular

$$\Pi(U, U) = 0\,. \tag{1.26}$$

Parallel to the above expressions, the semidiscrete Galerkin formulation (1.11) can be expressed as follows : $U_h(t) \in S_h \times S_h$, $U_h(t) = [u_h(t), \dot{u}_h(t)]^T$, $t \geqslant 0$, is determined so that for $t > 0$ and each $\Phi_h = [\varphi_h, \dot{\varphi}_h]^T$

$$(D_t U_h(t), \Phi_h)_e + \Pi(U_h(t), \Phi_h) + (F(U_h(t)), \Phi_h)_e = 0 \tag{1.27}$$

and
$$U_h(0) = U_{0,h} = [u_{0,h}, \dot{u}_{0,h}]^T .$$

Introducing the positive-definite, self-adjoint operator $A_h : S_h \to S_h$ by

$$(A_h u_h, \varphi_h) = a(u_h, \varphi_h) , \quad \varphi_h \in S_h , \tag{1.28}$$

we can express (1.27) in a manner which is parallel to (1.13) :

$$\begin{gathered} D_t U_h(t) + \Lambda_h U_h(t) + P_h^e F(U_h(t)) = 0 \\ U_h(0) = U_{0,h} , \end{gathered} \tag{1.29}$$

where

$$\Lambda_h = \begin{bmatrix} 0 & -I_h \\ A_h & 0 \end{bmatrix} , \tag{1.30}$$

$I_h : S_h \to S_h$ is the identity, and $P_h^e : H \to S_h \times S_h$ denotes projection with respect to $(.,.)_e$. Just as Λ, Λ_h is skew-adjoint and generates, in $S_h \times S_h$, the unitary semigroup $e^{-t\Lambda_h}$. In particular

$$\left\| e^{-t\Lambda_h} U_{0,h} \right\|_e = \| U_{0,h} \|_e , \quad t \in \mathbf{R} . \tag{1.31}$$

Conservation of energy (1.18) for the solution $U(t)$ of (1.13) is based on the Galerkin formulation (1.24) and is also valid for the solution $U_h(t)$ of (1.29). We therefore have the *stability* result

$$\| U_h(t) - V_h(t) \|_e \leq e^{M(\| U_{0,h} \|_e, \| V_{0,h} \|_e) t} \cdot \| U_{0,h} - V_{0,h} \|_e , \; t \geq 0, \tag{1.32}$$

where M is *independent* of h, parallel to the well-posedness statement (1.20), the proof of which is exactly the same as the proof of (1.20) in [2].

Let us denote the solution u of the elliptic boundary value problem (1.7) by Tf so that $Tf \in \dot{H}^2$ and

$$a(Tf, \varphi) = (f, \varphi) , \quad \varphi \in \dot{H}^2 . \tag{1.33}$$

The approximate solution operator $T_h : L^2 \to S_h$ is defined as

$$a(T_h f, \varphi_h) = (f, \varphi_h) , \quad \varphi_h \in S_h . \tag{1.34}$$

We have the well known approximation properties

$$\| (T - T_h) f \|_2 \leq C h^2 \| f \| , \tag{1.35}$$

$$\| (T - T_h) f \| \leq C h^4 \| f \| \tag{1.36}$$

(see, for example, [10]).

The Ritz projection $P_h^2 : \dot{H}^2 \to S_h$ is defined by

$$a(P_h^2 u, \varphi_h) = a(u, \varphi_h), \quad \varphi_h \in S_h, \tag{1.37}$$

so that $P_h^2 u = T_h Au$, and by (1.35), (1.36) we have

$$\|u - P_h^2 u\|_2 \leqslant Ch^2 \|u\|_4 \tag{1.38}$$

$$\|u - P_h^2 u\| \leqslant Ch^4 \|u\|_4 . \tag{1.39}$$

In the next section we will prove that

$$\|u(t) - u_h(t)\|_2 \leqslant C(t, \|U_0\|_{D(\Lambda^3)})\, h^2, \tag{1.40}$$

$$\|u(t) - u_h(t)\| \leqslant C(t, \|U_0\|_{D(\Lambda^3)})\, h^4. \tag{1.41}$$

The third section is devoted to the discussion of a fully discrete scheme based on a Crank-Nicolson type time discretization which conserves energy. Similar schemes have been discussed by Sanz-Serna within the context of the nonlinear Schroedinger equation [9] and within the context of the extensible string equation by Sanz-Serna and Christie [3].

2. THE RATE OF CONVERGENCE OF THE SEMIDISCRETE GALERKIN APPROXIMATION

THEOREM 1 : *With the notation of section* 1,

$$\|U_h(t) - U(t)\|_e \leqslant C(T, \|U_0\|_{D(\Lambda^3)})\, h^2, \quad 0 \leqslant t \leqslant T, \tag{2.1}$$

if

$$\|U_{0,h} - U_0\|_e = 0(h^2). \tag{2.2}$$

Remark 1 : We thus have

$$\|u_h(t) - u(t)\|_2 = 0(h^2),$$
$$\|\dot{u}_h(t) - \dot{u}(t)\| = 0(h^2),$$

for $0 \leqslant t \leqslant T$ if

$$\|u_{0,h} - u_0\|_2 = 0(h^2), \quad \|\dot{u}_{0,h} - \dot{u}_0\| = 0(h^2)$$

and, according to (1.17),

$$u_0 \in H^8, \quad u_0(0) = u_0^{(2)}(0) = u_0^{(4)}(0) = u_0^{(6)}(0) = 0,$$
$$u_0(1) = u_0^{(2)}(1) = u_0^{(4)}(0) = u_0^{(6)}(0) = 0,$$

$$\dot{u}_0 \in H^6, \quad \dot{u}_0(0) = \dot{u}_0^{(2)}(0) = \dot{u}_0^{(4)}(0) = 0,$$
$$\dot{u}_0(1) = \dot{u}_0^{(2)}(1) = \dot{u}_0^{(4)}(1) = 0.$$

Such stringent hypotheses seem to be indispensable in the case of hyperbolic equations. The reader may compare with the results for the wave equation (e.g. Baker and Bramble [1], Geveci [6]) and Rauch's recent paper [8] on the necessity of such assumptions in a specific case.

Proof of Theorem 1 : We introduce $P_h : \dot{H}^2 \times \dot{H}^2 \to S_h \times S_h$ by

$$P_h U = [P_h^2 u, P_h^2 \dot{u}]^T \tag{2.3}$$

where $U = [u, \dot{u}]^T$ and P_h^2 is the Ritz projection (1.37).

Since $U(t) - U_h(t) = (U(t) - P_h U(t)) + (P_h U(t) - U_h(t))$, and

$$\| U(t) - P_h U(t) \|_e \leq Ch^2 (\| u(t) \|_4 + \| \dot{u}(t) \|_4) \tag{2.4}$$

by (1.38), so that

$$\| U(t) - P_h U(t) \|_e \leq C(t, \| U_0 \|_{D(\Lambda^2)}) h^2, \tag{2.5}$$

thanks to the regularity statement (1.21) and the description (1.17) of $D(\Lambda^k)$, all we need to show is that $E_h(t) = P_h U(t) - U_h(t)$ satisfies

$$\| E_h(t) \|_e \leq C(t, \| U_0 \|_{D(\Lambda^3)}) h^2. \tag{2.6}$$

By the definition of P_h and Π (1.23)

$$\Pi(P_h U, \Phi_h) = \Pi(U, \Phi_h), \quad \Phi_h \in S_h \times S_h. \tag{2.7}$$

We can therefore write (1.24)

$$(D_t U(t), \Phi_h)_e + \Pi(P_h U(t), \Phi_h) + (F(U(t)), \Phi_h)_e = 0, \quad \Phi_h \in S_h \times S_h,$$

and

$$(D_t P_h U(t), \Phi_h)_e + \Pi(P_h U(t), \Phi_h) + (F(P_h U(t)), \Phi_h)_e = (\rho_h(t), \Phi_h)_e, \quad \Phi_h \in S_h \times S_h, \tag{2.8}$$

where

$$\rho_h(t) = (P_h - I) D_t U(t) + (F(P_h U(t)) - F(U(t))). \tag{2.9}$$

Since

$$\Pi(P_h U(t), \Phi_h) = (\Lambda_h P_h U(t), \Phi_h)_e, \quad \Phi_h \in S_h \times S_h, \tag{2.10}$$

((1.28), (1.30)), we can express (2.8) as

$$D_t P_h U(t) + \Lambda_h P_h U(t) + P_h^e F(P_h U(t)) = P_h^e \rho_h(t) . \tag{2.11}$$

We rewrite (1.29) :

$$D_t U_h(t) + \Lambda_h U_h(t) + P_h^e F(U_h(t)) = 0 . \tag{2.12}$$

From (2.11) and (2.12) we obtain

$$D_t E_h(t) + \Lambda_h E_h(t) = P_h^e \rho_h(t) - P_h^e(F(P_h U(t)) - F(U_h(t))) \tag{2.13}$$

so that

$$E_h(t) = e^{-t\Lambda_h} E_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h} [P_h^e \rho_h(\tau) - \\ - P_h^e(F(P_h U(\tau)) - F(U_h(\tau)))] \, d\tau . \tag{2.14}$$

Thanks to (1.31) and the fact that P_h^e is the projection in $\dot{H}^2 \times L^2$, (2.14) leads to

$$\|E_h(t)\|_e \leq \|E_h(0)\|_e + \\ + \int_0^t [\|\rho_h(\tau)\|_e + \|F(P_h U(\tau)) - F(U_h(\tau))\|_e] \, d\tau . \tag{2.15}$$

Now we make use of the local Lipschitz property (1.19) of F and the boundedness of $\|U(t)\|_{D(\Lambda)}$, $\|U_h(t)\|_e$ in terms of the initial data (*cf.* (1.21), (1.32)) ;

$$\|F(P_h U(\tau)) - F(U_h(\tau))\|_e \leq C \|E_h(\tau)\|_e \tag{2.16}$$

(We shall not indicate the quantities that C depends on expliticly. C depends, in particular, on T and $\|U_0\|_{D(\Lambda)}$. In the sequel C may stand for different quantities that are bounded in terms of the data.)

Combining (2.15) and (2.16) we obtain

$$\|E_h(t)\|_e \leq \|E_h(0)\|_e + \int_0^t \|\rho_h(\tau)\|_e \, d\tau + C \int_0^t \|E_h(\tau)\|_e \, d\tau . \tag{2.17}$$

(2.17) and Gronwall's lemma lead to

$$\|E_h(t)\|_e \leq e^{Ct} \left(\|E_h(0)\|_e + \int_0^t \|\rho_h(\tau)\|_e \, d\tau \right) \tag{2.18}$$

so that the proof of Theorem 1 will be concluded once we show that

$$\|E_h(0)\|_e \leq Ch^2 \tag{2.19}$$

and

$$\|\rho_h(t)\|_e \leq Ch^2, \quad 0 \leq t \leq T. \tag{2.20}$$

We have

$$\begin{aligned} E_h(0) &= P_h U_0 - U_{0,h} \\ &= (P_h U_0 - U_0) + (U_0 - U_{0,h}) \end{aligned}$$

so that (1.38) and (2.2) yield (2.19).

From the definition (2.9) of $\rho_h(t)$, (1.38), the Lipschitz property of F, and the regularity assumption on U_0, (2.20) is also readily obtained.

We will now prove the $O(h^4)$ estimate for $\|u_h(t) - u(t)\|$. Before we state and prove the relevant theorem we will introduce some mathematical background and notation in addition to that which was presented in section 1.

As in baker and Bramble [1], Thomée [11] and Geveci [6], we will introduce another inner product on $\dot{H}^2 \times L^2$:

$$(U, V)_{-e,h} = (u, v) + (\dot{u}, T_h \dot{v}) \tag{2.21}$$

for $\quad U = [u, \dot{u}]^T, \quad V = [v, \dot{v}]^T \in \dot{H}^2 \times L^2.$

The associated seminorm is denoted as $\|.\|_{-e,h}$ (T_h is symmetric, positive semidefinite on L^2 and positive definite on S_h so that $\|.\|_{-e,h}$ is a norm on $S_h \times S_h$).

Now, Λ_h is skew adjoint when $S_h \times S_h$ is equipped with the inner product $(.,.)_{-e,h}$ since

$$\begin{aligned} (\Lambda_h U_h, V_h)_{-e,h} &= -(\dot{u}_h, v_h) + (A_h u_h, T_h \dot{v}_h) \\ &= -(\dot{u}_h, v_h) + a(u_h, T_h \dot{v}_h) \\ &= -(\dot{u}_h, v_h) + a(T_h u_h, \dot{v}_h) \\ &= -(\dot{u}_h, v_h) + (u_h, \dot{v}_h) \\ &= -(U_h, \Lambda_h V_h)_{-e,h}. \end{aligned}$$

Therefore Λ_h generates a unitary group in $S_h \times S_h$ equipped with $(.,.)_{-e,h}$ and we have

$$\left\|e^{-t\Lambda_h} U_{0,h}\right\|_{-e,h} = \|U_{0,h}\|_{-e,h}, \quad t \in \mathbf{R}. \tag{2.22}$$

Another fact that we shall appeal to is the following :
Denote

$$\|\varphi\|_{-2,h} = \sqrt{(\varphi, T_h \varphi)} \tag{2.23}$$

Then

$$\|\varphi\|_{-2,h} \leq C(\|\varphi\|_{-2} + h^2\|\varphi\|). \tag{2.24}$$

This is proved as in Thomée [11] and immediately leads to

$$\|D_x^2 \varphi\|_{-2,h} \leq C(\|\varphi\| + h^2\|\varphi\|_2). \tag{2.25}$$

(2.24) and (2.25) will be utilized in the following way :

LEMMA 1 : *We have*

$$\|f(u) - f(v)\|_{-2,h} \leq C(\|u-v\| + h^2\|u-v\|_2) \tag{2.26}$$

where $C = C(\|u\|_2, \|v\|_2)$.

Proof :

$$\begin{aligned} f(u) - f(v) &= (\beta + \gamma\|D_x v\|^2) D_x^2 v - (\beta + \gamma\|D_x u\|^2) D_x^2 u \\ &= (\beta + \gamma\|D_x v\|^2) D_x^2(v-u) + \\ &\quad + \gamma(\|D_x v\|^2 - \|D_x u\|^2) D_x^2 u \\ &= (\beta + \gamma\|D_x v\|^2) D_x^2(v-u) + \\ &\quad + \gamma(D_x(v-u), D_x(v+u)) D_x^2 u \\ &= (\beta + \gamma\|D_x v\|^2) D_x^2(v-u) - \gamma(v-u, D_x^2(v+u)) D_x^2 u \end{aligned}$$

so that

$$\begin{aligned} \|f(u) - f(v)\|_{-2,h} &\leq C(\|v\|_2^2)\|D_x^2(v-u)\|_{-2,h} \\ &\quad + C(\|v\|_2^2, \|u\|_2^2)\|v-u\| \\ &\leq C(\|v\|_2^2, \|u\|_2^2)(\|u-v\| + h^2\|u-v\|_2) \end{aligned}$$

by (2.25).

We are now ready to prove our result :

THEOREM 2 : *Under the same conditions as in Theorem* 1,

$$\|u_h(t) - u(t)\| \leq C(T, \|U_0\|_{D(\Lambda^3)}) \cdot h^4, \quad 0 \leq t \leq T, \tag{2.27}$$

if, in addition

$$\|u_0 - u_{0,h}\| = 0(h^4) \quad \textit{and} \quad \|\dot{u}_0 - \dot{u}_{0,h}\| = 0(h^4). \tag{2.28}$$

Proof: Again,

$$U(t) - U_h(t) = (U(t) - P_h U(t)) + (P_h U(t) - U_h(t)) = (U(t) - P_h U(t)) + E_h(t) , \tag{2.29}$$

and

$$\|U(t) - P_h U(t)\|_{-e,h} \leq C(\|u(t) - P_h^2 u(t)\| + \|\dot{u}(t) - P_h^2 \dot{u}(t)\|_{-2,h}) \leq C(\|u(t) - P_h^2 u(t)\| + \|\dot{u}(t) - P_h^2 \dot{u}(t)\|)$$

since $\|T_h\| \leq C$, say, for $0 < h \leq h_0$. By the approximation property (1.39),

$$\|U(t) - P_h U(t)\|_{-e,h} \leq C(\|u(t)\|_4 + \|\dot{u}(t)\|_4) h^4 . \tag{2.30}$$

In order to estimate $E_h(t)$ we proceed as in the proof of Theorem 1 :

$$E_h(t) = e^{-t\Lambda_h} E_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h} P_h^e[\rho_h(\tau) + F(P_h U(\tau)) - F(U_h(\tau))] \, d\tau ,$$

and by (2.22)

$$\|E_h(t)\|_{-e,h} \leq \|E_h(0)\|_{-e,h} + \int_0^t \|P_h^e \rho_h(\tau)\|_{-e,h} \, d\tau + \int_0^t \|P_h^e[F(P_h U(\tau)) - F(U_h(\tau))]\|_{-e,h} \, d\tau . \tag{2.31}$$

We will estimate each term on the right of (2.31) separately

$$E_h(0) = P_h U_0 - U_{0,h} = (P_h U_0 - U_0) + (U_0 - U_{0,h}) ,$$

so that

$$\|E_h(0)\|_{-e,h} \leq \|P_h U_0 - U_0\|_{-e,h} + \|U_0 - U_{0,h}\|_{-e,h} \leq C(\|P_h^2 u_0 - u_0\| + \|P_h^2 \dot{u}_0 - \dot{u}_0\| + \|u_0 - u_{0,h}\| + \|\dot{u}_0 - \dot{u}_{0,h}\|) \leq Ch^4 . \tag{2.32}$$

As for the second term :

$$P_h^e \rho_h(\tau) = P_h^e(P_h - I) D_t U(\tau) + P_h^e(F(P_h(\tau)) - F(U(\tau))) .$$

$$P_h^e(P_h - I) D_t U = [P_h^2(P_h^2 - I) D_t u, P_h^0(P_h^2 - I) D_t \dot{u}]^T = [0, P_h^0(P_h^2 - I) D_t \dot{u}]^T \tag{2.33}$$

since $P_h^2 \circ P_h^2 = P_h^2$, P_h^2 being a projection (P_h^0 denotes the L^2-projection).

By (2.33)

$$\begin{aligned}\|P_h^2(P_h - I)\, D_t\, U\|_{-e,h}^2 &= \|P_h^0(P_h^2 - I)\, D_t\, \dot u\|_{-2,h}^2 \\ &= (P_h^0(P_h^2 - I)\, D_t\, \dot u,\ T_h\, P_h^0(P_h^2 - I)\, D_t\, \dot u) \\ &= ((P_h^2 - I)\, D_t\, \dot u,\ T_h(P_h^2 - I)\, D_t\, \dot u) \\ &= \|(P_h^2 - I)\, D_t\, \dot u\|_{-2,h}^2,\end{aligned}$$

so that

$$\begin{aligned}\|P_h^e(P_h - I)\, D_t\, U\|_{-e,h} &\leq C\,\|(P_h^2 - I)\, D_t\, \dot u\| \\ &\leq Ch^4 .\end{aligned} \tag{2.34}$$

We also have

$$\begin{aligned}\|P_h^e(F(P_h\, U) - F(U))\|_{-e,h} &= \|P_h^0(f(P_h^2 u) - f(u))\|_{-2,h} \\ &= \|f(P_h^2 u) - f(u)\|_{-2,h}\end{aligned}$$

so that, by Lemma 1,

$$\begin{aligned}\|P_h^e(F(P_h\, U) - F(U))\|_{-e,h} &\leq C\,(\|u - P_h^2 u\| + h^2\|u - P_h^2 u\|_2) \\ &\leq Ch^4 .\end{aligned} \tag{2.35}$$

Combining (2.34) and (2.35) we obtain

$$\int_0^t \|P_h^e\, \rho_h(\tau)\|_{-e,h}\, d\tau \leq Ch^4, \quad 0 \leq t \leq T . \tag{2.36}$$

In the same way as we obtained (2.35),

$$\begin{aligned}\|P_h^e(F(P_h\, U) - F(U_h))\|_{-e,h} &\leq C\,(\|u_h - P_h^2 u\| + h^2\|u_h - P_h^2 u\|_2) \\ &\leq C\,(\|u_h - P_h^2 u\| + h^4)\end{aligned}$$

from Theorem 1, and we therefore have

$$\begin{aligned}\int_0^t \|P_h^e[F(P_h\, U(\tau)) - F(U_h(\tau))]\|_{-e,h}\, d\tau &\leq Ch^4 + \\ + \int_0^t \|P_h\, U(\tau) - U_h(\tau)\|_{-e,h}\, d\tau &= Ch^4 + \int_0^t \|E_h(\tau)\|_{-e,h}\, d\tau .\end{aligned} \tag{2.37}$$

By (2.31), (2.32), (2.36) and (2.37),

$$\|E_h(t)\|_{-e,h} \leq C\left(h^4 + \int_0^t \|E_h(\tau)\|_{-e,h}\, d\tau\right)$$

so that, by Gronwall's lemma

$$\|E_h(t)\|_{-e,h} \leqslant Ch^4, \quad 0 \leqslant t \leqslant T. \tag{2.38}$$

This leads to (2.27) and the proof of the theorem is concluded.

Remark : From the proof it is clear that we also have

$$\|\dot{u}_h(t) - \dot{u}(t)\|_{-2,h} = 0(h^4)$$

which, in turn, implies

$$\|\dot{u}_h(t) - \dot{u}(t)\|_{-2} = 0(h^4)$$

where $\|\cdot\|_{-2}$ denotes the norm of the dual of $\dot{H}^2$, as in [6].

3. A FULLY DISCRETE SCHEME

Let us rewrite the semidiscrete Galerkin formulation (1.27) as

$$(D_t\, U_h(t), \Phi_h)_e + \Pi(U_h(t), \Phi_h) + \beta(u_h(t), \dot{\varphi}_h)_1 + \\ + \gamma \|u_h(t)\|_1^2\, (u_h(t), \dot{\varphi}_h)_1 = 0, \tag{3.1}$$

$$\Phi_h = [\varphi_h, \dot{\varphi}_h]^T \in S_h \times S_h, \quad t > 0,$$

$$U_h(0) = U_{0,h},$$

where

$$(u, v)_1 = (D_x\, u, D_x\, v), \quad \|u\|_1^2 = (u, u)_1.$$

Denoting

$$\bar{\partial}_t U_h^n = \frac{U_h^n - U_h^{n-1}}{k}, \quad n = 1, 2, \ldots,$$

where k is the time step, and

$$\bar{U}_h^n = \frac{U_h^n + U_h^{n-1}}{2}, \quad \bar{U}_h^n = [\bar{u}_h^n, \bar{\dot{u}}_h^n]^T,$$

the application of Crank-Nicolson time discretization to (3.1) yields the scheme

$$(\bar{\partial}_t U_h^n, \Phi_h)_e + \Pi(\bar{U}_h^n, \Phi_h) + \beta(\bar{u}_h^n, \dot{\varphi}_h)_1 + \gamma\|\bar{u}_h^n\|_1^2\, (\bar{u}_h^n, \dot{\varphi}_h)_1 = 0, \tag{3.2}$$

$$\Phi_h \in S_h \times S_h, \quad n = 1, 2, \ldots,$$

$$U_h^0 = U_{0,h}.$$

We modify (3.2) as follows :

$$(3.3)\quad (\bar{\partial}_t U_h^n, \Phi_h)_e + \Pi(\bar{U}_h^n, \Phi_h) + \beta(\bar{u}_h^n, \dot{\varphi}_h)_1 + \\ + \gamma\left(\frac{\|u_h^n\|_1^2 + \|u_h^{n-1}\|_1^2}{2}\right)(\bar{u}_h^n, \dot{\varphi}_h)_1 = 0 ,$$

$$\Phi_h \in S_h \times S_h , \quad n = 1, 2, \dots , \quad U_n^0 = U_{0,h} ,$$

The reason for this modification is the following :

LEMMA 2 : *Energy, as defined by* (1.18), *is conserved by the modified Crank-Nicholson scheme* (3.3), *i.e.,*

$$E(U_h^n) = E(U_h^{n-1}) , \quad n = 1, 2, \dots .$$

Proof : Substituting $\bar{U}_h^n$ for Φ_h in (3.3),

$$(3.4)\quad (\bar{\partial}_t U_h^n, \bar{U}_h^n)_e + \Pi(\bar{U}_h^n, \bar{U}_h^n) + \beta(\bar{u}_h^n, \bar{\dot{u}}_h^n)_1 + \\ + \frac{\gamma}{2}(\|u_h^n\|_1^2 + \|u_h^{n-1}\|_1^2)(\bar{u}_h^n, \bar{\dot{u}}_h^n) = 0 .$$

Since $\Pi(\bar{U}_h^n, \bar{U}_h^n) = 0$ ((1.26)), and

$$(\bar{\partial}_t U_h^n, \bar{U}_h^n)_e = \frac{1}{2}\bar{\partial}_t \|U_h^n\|_e^2 ,$$

$$(\bar{u}_h^n, \bar{\dot{u}}_h^n)_1 = (\bar{u}_h^n, \bar{\partial}_t u_h^n)_1 = \frac{1}{2}\bar{\partial}_t \|u_h^n\|_1^2 ,$$

(3.4) yields

$$\frac{1}{2}\bar{\partial}_t \|U_h^n\|_e^2 + \frac{\beta}{2}\bar{\partial}_t \|u_h^n\|_1^2 + \frac{\gamma}{2}(\|u_h^n\|_1^2 + \|u_h^{n-1}\|_1^2)\frac{1}{2}\bar{\partial}_t \|u_h^n\|_1^2 = 0 ,$$

and this implies

$$\alpha\|D_x^2 u_h^n\|^2 + \|\dot{u}_h^n\|^2 + \beta\|u_h^n\|_1^2 + \frac{\gamma}{2}\|u_h^n\|_1^2 = \alpha\|D_x^2 u_h^{n-1}\|^2 + \\ + \|\dot{u}_h^{n-1}\|^2 + \beta\|u_h^{n-1}\|_1^2 + \frac{\gamma}{2}\|u_h^{n-1}\|_1^2$$

i.e. $$E(U_h^n) = E(U_h^{n-1}) .$$

Just as in Ball's discussion of the existence of solutions of the original equation [2], conservation of energy leads to the boundedness of

$\|D_x^2 u_h^n\|$ and $\|u_h^n\|$, $n = 1, 2,$, in terms of the initial data and the following convergence result can be established

THEOREM 3 *If the hypotheses of Theorem 1 are valid,*

$$\|U_h^n - U(kn)\|_e \leqslant C(k^2 + h^2), \quad kn \leqslant T,$$

where U_h^n, $n = 1, 2,$, *is generated by the modified Crank-Nicolson scheme* (3 3)

If the hypotheses of Theorem 2 are valid,

$$\|U_h^n - U(kn)\|_{-e\,h} \leqslant C(k^2 + h^4), \quad kn \leqslant T$$

The proof will be omitted since it is lengthly but straightforward along the lines of the proofs of Theorem 1, Theorem 2, and Thomée [12, Ch 10], thanks to Lemma 2

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