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ARNOLDI-TCHEBYCHEV PROCEDURE FOR LARGE SCALE NONSYMMETRIC MATRICES (*)

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Abstract. — *The Tchebychev acceleration technique is used in conjunction with the Arnoldi iterative method for solving the eigenvalue problems of large scale nonsymmetric matrices. The procedure is based on a newly developed algorithm to identify the optimal Tchebychev ellipse of the complex eigenspectrum. Initial results show that the procedure is fast, reliable and easy to use. Our procedure does not require a search for all possible ellipses which enclose the spectrum. Applications to nonsymmetric linear systems can also be easily done without any further modification.*

Résumé. — *La technique d'accélération de Tchébycheff est utilisée conjointement avec la méthode itérative d'Arnoldi pour résoudre les problèmes de valeurs propres des matrices non symétriques de grande taille. La procédure repose sur un algorithme récemment développé pour identifier l'ellipse optimale de Tchébycheff correspondant au spectre complexe des valeurs propres. Les résultats initiaux montrent que la procédure est rapide, fiable et facile à utiliser. Notre procédure évite une recherche exhaustive de toutes les ellipses contenant le spectre. Sans modification, notre méthode permet également de traiter le cas des systèmes linéaires non symétriques.*

I. INTRODUCTION

The eigenvalue problem of large matrices is of increasing interest because their applications are widespread in scientific and engineering computing. The stability analysis in an electronic circuit design or in structure dynamics, for instance, requires the precise knowledge of the eigenvalues near the imaginary axis. The sensitivity analysis of these eigenvalues with respect to important parameters of the system allows engineers to optimize their designs.

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The eigenvalue problem of a large symmetric matrix has been studied by a number of authors by using the subspace iteration method (e.g., Clint and Jennings, 1971 ; Stewart, 1976 ; Jennings and Stewart, 1980 ; Jennings, 1981) or Lanczos method (e.g., Nour-Omid *et al.*, 1983 ; Cullum and Willoughby, 1985). For the nonsymmetric case, the literature has been relatively limited. A number of algorithms for symmetric cases has been adapted for nonsymmetric matrices such as the Lanczos procedure (Taylor, 1983 ; Parlett *et al.*, 1985 ; Cullum and Willoughby, 1986) or the Arnoldi method (Saad, 1980, 1985). Saad (1980, 1984) has found that the Arnoldi method requires too much memory space and may have difficulties in extracting the wanted eigenvalues, in particular, when the spectrum has all its wanted eigenvalues clustered together while the unwanted ones separated favorably from one another. To avoid these shortcomings of the iterative Arnoldi method and to improve the overall performance, Saad (1984) has proposed to use it in conjunction with the Tchebychev iteration. The introduction of Tchebychev iteration is for the purpose of amplifying the components of the initial vector in the direction of the wanted eigenvalues and at the same time damping those in the remaining eigenvectors. In the corresponding problem for solving nonsymmetric linear systems, most of the works have been devoted to the use of Tchebychev polynomials for accelerating linear iterative method (Manteuffel, 1975, 1977, 1978). Manteuffel's work on the determination of the optimal ellipse containing the spectrum of the matrix of the linear system has been adapted by Saad (1984) to the nonsymmetric eigenvalue problem. However, Manteuffel's algorithm cannot be extended to the case where the reference eigenvalue is complex. To avoid this difficulty, Saad (1984) has used the real part of the reference eigenvalue as the reference point. As a result, the Tchebychev ellipse he has found is not exactly optimal, neither is its convergence factor. In addition, the procedure to look for the optimal ellipse proposed by Manteuffel and later used by Saad (1984) requires an enumeration of all possible ellipses to check for feasibility and optimality. Thus the process is combinatorial in particular in the procedure to define the convex hull of the spectrum (Saad, 1983).

In this paper, we shall also use the iterative Arnoldi with the Tchebychev acceleration technique to find a number of wanted eigenvalues of a large nonsymmetric matrix. We shall use a newly developed method, which overcomes the two above shortcomings, to identify accurately the optimal Tchebychev ellipse of the complex eigenspectrum (Ho, 1987). Section 2 will detail the Tchebychev iteration, followed by a description of the optimal ellipse and the algorithm for its identification in section 3. The Arnoldi-Tchebychev procedure and some initial results will be presented in section 4 followed by the concluding remarks in section 5.

II. TCHEBYCHEV ITERATION

The objective is to solve the eigenvalue problem of the form :

$$Au = \lambda u \,, \tag{1}$$

where A is a nonsymmetric diagonalizable real matrix of dimension N. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of A labelled in decreasing order of their real parts. Suppose that we are interested in λ_i with $i = \{1, \ldots, r\}$. We shall be looking for an iterative solution of the form

$$z_n = p_n(A)\, z_0 \,, \tag{2}$$

where p_n is a degree n polynomial and z_0 is some initial vector expressible in the eigenbasis $\{u_i\}$, or $z_0 = \sum_{i=1}^{N} \theta_i\, u_i$.

We then have

$$z_n = \sum_{i=1}^{N} \theta_i\, p_n(\lambda_i)\, u_i = \sum_{i=1}^{r} \theta_i\, p_n(\lambda_i)\, u_i + \sum_{i=r+1}^{N} \theta_i\, p_n(\lambda_i)\, u_i \,. \tag{3}$$

It is clear that we want the second part on the right hand side of (3) to be small compared to the first part. Therefore we wish to choose p_n which is small on the discrete set $S = \{\lambda_{r+1}, \ldots, \lambda_N\}$ and satisfies the normalization condition

$$p_n(\lambda_r) = 1 \,.$$

A simple procedure is to look for such a polynomial on a continuous domain E containing S and excluding λ_i for $i = \{1, \ldots, r\}$. The problem becomes

$$\min_{p \in P_n} \max_{\lambda_i \in E} |p(\lambda_i)| \tag{4}$$

where P_n is the space of all polynomials of degree not exceeding n. In our problem, we restrict E to be an ellipse, centered at d with the focal distance c, symmetric with respect to the real axis because the spectrum of A is symmetric with respect to the real axis on the complex plane.

The best polynomial of (4), called the minimax polynomial, is then

$$p_n(\lambda_i) = \frac{T_n[(\lambda_i - d)/c]}{T_n[(\lambda_r - d)/c]} \,, \tag{5}$$

where T_n is the Tchebychev polynomial of degree n of the first kind (see, for example, Manteuffel, 1977 ; Saad, 1984 ; Chatelin, 1988). And E is the Tchebychev ellipse. $T_n[(\lambda_r - d)/c]$ in (5) is in fact a scaling factor, and λ_r is the reference point.

Note that for $n = 1, 2, \ldots$

$$T_{n+1}(\lambda) = 2\,\lambda T_n(\lambda) - T_{n-1}(\lambda)\,, \tag{6}$$

with $T_0(\lambda) = 1$, $T_1(\lambda) = \lambda$.

Define $\sigma_{i+1} = \rho_i/\rho_{i+1}$ with $\rho_i = T_i[(\lambda_r - d)/c]$, the vector z_n can be computed for $i = 1, 2, \ldots$

$$z_{i+1} = 2\,\frac{\sigma_{i+1}}{c}\,(A - dI)\,z_i - \sigma_i\,\sigma_{i+1}\,z_{i-1} \tag{7}$$

and

$$\sigma_{i+1} = \frac{1}{2/\sigma_1 - \sigma_i}\,, \tag{8}$$

with $\sigma_1 = c/(\lambda_r - d)$ and $z_1 = \dfrac{\sigma_1}{c}\,(A - dI)\,z_0$, where c is the focal distance of the ellipse and d is its center.

The asymptotic convergence factor at λ_i, referred hereafter as the convergence factor $R_i(d, c^2)$ relative to the reference point λ_r is defined by

$$R_i(d, c^2) = \lim_{n \to \infty} |p_n(\lambda_i)^{1/n}| = \left| e^{\left[\cosh^{-1}\left(\frac{d-\lambda_i}{c}\right) - \cosh^{-1}\left(\frac{d-\lambda_r}{c}\right)\right]} \right|. \tag{9}$$

The subscript refers to the eigenvalue λ_i and the elements in the parentheses are for the ellipse parameters. Note that $\cosh^{-1}(w) = \ln\,[w + (w^2 - 1)^{1/2}]$, we have

$$R_i(d, c^2) = \left| \frac{(d - \lambda_i) + ((d - \lambda_i)^2 - c^2)^{1/2}}{(d - \lambda_r) + ((d - \lambda_r)^2 - c^2)^{1/2}} \right|. \tag{10}$$

Each λ_i of the set of unwanted eigenvalues is associated with a convergence factor $R_i(d, c^2)$. One way of optimizing the choice of d and c^2 is to make the maximum $R_i(d, c^2)$ as small as possible. The parameters d and c^2 will then satisfy

$$\min_{d, c^2} \max_{\lambda_i} R_i(d, c^2) = \min \max_{\lambda_i} \left| \frac{(d - \lambda_i) + ((d - \lambda_i)^2 - c^2)^{1/2}}{(d - \lambda_r) + ((d - \lambda_r)^2 - c^2)^{1/2}} \right|. \tag{11}$$

The choice of d and c^2 of an ellipse which contains all the unwanted eigenvalues and yields the smallest convergence factor is the solution to the above mini-max problem. This defines the optimal ellipse relative to λ_r (see also Manteuffel, 1977).

One of the most important characteristics of the Tchebychev ellipse is that all the eigenvalues on the ellipse have the same convergence factor relative to a reference eigenvalue. Equation (10) becomes

$$R_i(d, c^2) = \frac{a_i + b_i}{a_r + b_r}, \tag{12}$$

where a_i and b_i are half of the length of the two axes of the ellipse passing through λ_i, a_r and b_r are half of the length of the two axes of the reference ellipse.

An ellipse passing through λ_i must satisfy :

$$\frac{X_i^2}{a^2} + \frac{y_i^2}{b^2} = 1 , \tag{13}$$

where $X_i = x_i - d$.

The focal distance of the ellipse is defined as :

$$c^2 = |a^2 - b^2| = |a_r^2 - b_r^2| . \tag{14}$$

This definition allows us to treat the focal distance, the semi-major and semi-minor axes in real values. Note that the ellipse passing through the reference point λ_r has also the same focal distance.

III. THE OPTIMAL ELLIPSE AND ITS CONVERGENCE FACTOR

In this paper we shall refer to an ellipse which contains all unwanted eigenvalues of the spectrum as a feasible ellipse. An ellipse which passes through two eigenvalues and has the smallest convergence factor relative to a reference eigenvalue λ_r is called a pairwise optimal ellipse. An ellipse which passes through three eigenvalues is called a three point ellipse. Since our spectrum is symmetric with respect to the real axis, we shall refer to the set of unwanted eigenvalues whose imaginary parts are greater than or equal to 0 as S^+.

A. The Mini-Max solution

1. *One eigenvalue*

Suppose that S^+ consists of only one undesired eigenvalue, say $\lambda_p = (x_p, y_p)$, the only local minimum of R occurs at

$$d = x_p , \quad a = 0 , \quad c = b = y_p .$$

This defines the feasible as well as the optimal ellipse, and

$$R_p(x_p) = \frac{b}{a_r + b_r},$$

where a_r and b_r are determined by the ellipse passing through λ_r centered at d with the focal distance c. A theoretical proof of a similar problem was given by Manteuffel (1977).

Ho (1987) has shown that as d moves away from x_p, for each d, we may have a family of ellipses passing through λ_p. Within this family there exists one ellipse with the smallest convergence factor. This convergence factor increases as d moves away from x_p.

2. *Two eigenvalues*

Suppose that S^+ consists of two eigenvalues $\lambda_p = (x_p, y_p)$ and $\lambda_q = (x_q, y_q)$. Again with three parameters to determine, namely d, a and b, there exists a family of ellipses which passes through these two points. Among them there is one with the smallest convergence factor.

a) $x_p = x_q$

This is a degenerate case where $d = x_p = x_q$, $a = 0$, $b = \max\{y_p, y_q\}$. The rate of convergence R is defined as in the one eigenvalue case.

b) $y_p = y_q = 0$

This is also another degenerate case where $d = (x_p + x_q)/2$, $a = |x_q - x_p|/2$, and $b = 0$. The rate of convergence can then be calculated by (12)

c) $y_p = y_q \neq 0$

$$d = \frac{x_p + x_q}{2}.$$

This problem is equivalent to looking for the optimal ellipse centered at d passing through λ_p (or λ_q). Since the optimal ellipse exists, we may identify it by looking at the variation of R versus a or b. In fact, since the ellipse centered at d passes through λ_p, we are able to express a as a function of b. Similarly, c, a_r, b_r (see (13)-(14)) and R can also be expressed as functions of b. Let us denote the derivative with respect to b by $'$, we have

$$R' = 0$$

when

$$\frac{a_p' + 1}{a_p + b_p} - \frac{a_r' + b_r'}{a_r + b_r} = 0,$$

or

$$\frac{a_p + b_p}{a_r + b_r} = \frac{a'_p + 1}{a'_r + b'_r} . \tag{15}$$

The subscript p has been added to the above equation to indicate that the ellipse passes through λ_p. The ellipse is optimal when equation (15) is satisfied and $a = a_p$, $b = b_p$ (see Ho, 1987).

d) $x_p \neq x_q$, $y_p \neq y_q$

Suppose that $y_p > y_q$, from above we know that, given d, there exists an optimal ellipse which passes through λ_p. As d moves toward x_q, its convergence factor increases. For each d, the optimal ellipse, passing through λ_p, sweeps the spectrum until it passes through λ_q with a convergence factor of $R_{pq}(d)$. That means the optimal ellipse which passes through λ_p and λ_q exists with its convergence factor $R_{pq} < R_{pq}(d)$.

Since the pairwise optimal ellipse exists, (15) is also applicable to this case. Given b, other parameters in (15) can also be derived.

Let $\lambda_p = (x_p, y_p)$, $\lambda_q = (x_q, y_q)$, $\lambda_r = (x_r, y_r)$ and $X_i = x_i - d$.

An ellipse which passes through two points λ_p and λ_p must satisfies (13). From these two equations we may have

$$d = \frac{(x_p + x_q)}{2} + \left(\frac{a}{b}\right)^2 \frac{y_q^2 - y_p^2}{2(x_q - x_p)} . \tag{16}$$

where

$$\left(\frac{a}{b}\right)^2 = \frac{X_q^2}{b^2 - y_q^2} . \tag{17}$$

Note that $a_p = a_q = a$, $b_p = b_q = b$. We have X_q, or d, as a function of b (with $a_m = x_q - x_p$, $b_m = (y_q^2 - y_p^2)/a_m$)

$$X_q^2 + 2\frac{b^2 - y_q^2}{b_m} X_q - (b^2 - y_q^2)\frac{a_m}{b_m} = 0 . \tag{18}$$

From (17), a can then be expressed as a function of b. With ((13)-(14)), we can also write c^2, a_r and b_r as functions of b. The other terms in (15) can be calculated by taking the derivative of ((13)-(14)) with respect to b (see Ho, 1987). The optimal solution is reached when equation (15) is satisfied. This can easily be done by iteration.

3. *Three or more eigenvalues*

Suppose that S^+ has three or more eigenvalues. Ho (1987) has proved that the optimal ellipse relative to a reference eigenvalue is either a feasible

pairwise optimal ellipse or a feasible three point ellipse whose associated pairwise optimal ellipses do not contain the third point.

Given λ_p, λ_q, λ_t, assume that $x_p < x_t < x_q$, if a three point ellipse exists then

$$\left(\frac{a}{b}\right)^2 = \frac{(x_q - x_t)}{\zeta} \tag{19}$$

which requires

$$\zeta = \frac{y_q^2 - y_p^2}{x_p - x_q} - \frac{y_t^2 - y_p^2}{x_p - x_t} > 0 . \tag{20}$$

A similar criterion was also given by Manteuffel (1977, 1978). And d and b can be calculated from (16) and (17). The uniqueness of this three point ellipse has been proved by Manteuffel (1977).

To find the mini-max solution, Manteuffel's method (1977, 1978) requires to take systematically each pair of eigenvalues in the positive convex hull and to find its pairwise optimal ellipse and test its feasibility. If yes, it is the mini-max solution. If no, one must take each combination of the three eigenvalues and look for a feasible three point ellipse with the smallest convergence factor. This ellipse is the mini-max solution (see also Saad, 1984).

In our procedure, we may solve the problem in a much simpler way by first identifying a feasible ellipse passing through three points of the unwanted spectrum S^+. This ellipse can easily be found by taking two eigenvalues whose real parts are the maximum and minimum values and a third point whose imaginary part is the largest value of the set of unwanted eigenvalues. If no such ellipse exists, the optimal solution must be a pairwise ellipse. If such a three-point ellipse exists, we have to test its feasibility : if it is not feasible, we have to replace the third point by an eigenvalue furthermost from this three point ellipse. The process is repeated until feasibility is achieved. We then have to test the feasibility of the two related pairwise optimal ellipses, say E_{pq} and E_{tq}, assume that t and p are on one side with respect to the ellipse center. If one of these two is feasible, we have that pairwise optimal ellipse as the optimal ellipse of the unwanted spectrum. If both of them are not feasible and none of them encloses the third point then the three point ellipse is the optimal ellipse. If either one of the two non-feasible associated ellipses encloses the third point, say E_{tq} encloses λ_p, the above three point feasible ellipse is not the optimal ellipse of the spectrum. Suppose that a λ_s is outside E_{tq}, and is also the last eigenvalue that the ellipse passing through t and q sweeps through before reaching p, then E_{tsq} is the intermediate ellipse between E_{tq} and E_{ptq} and is also feasible. Its convergence factor is smaller than that of

E_{ptq}. With this new feasible ellipse the process can continue until the optimal ellipse for S^+ is found.

IV. ARNOLDI-TCHEBYCHEV AND SOME INITIAL RESULTS

We want to compute a number of r eigenvalues with largest real parts of a large scale nonsymmetric matrix. The above acceleration technique requires that we have some initial guess for the eigenspectrum. This is done be using the Arnoldi iterative method to generate an approximate spectrum from the resulting Hessenberg matrix (see Saad, 1984 ; Chatelin, 1988). Given an initial vector v_1 and a number $m > r$,

1. perform m steps of the Arnoldi algorithm starting with v_1, compute m eigenvalues of the resulting Hessenberg matrix (by QR for example) ;
2. calculate the residual norms of the first r associated eigenvectors ; if the precision criterion is satisfied then EXIT else continue ;
3. from S constituted by the set of $m - r$ unwanted eigenvalues calculated from the Hessenberg matrix, identify its optimal ellipse relative to the rth eigenvalue ;
4. generate an initial vector for Tchebychev iteration from the r approximate eigenvectors obtained above ;
5. perform m steps of Tchebychev iteration to generate the new initial vector v_1 for the Arnoldi algorithm, go to 1.

For illustration purpose, we use a simple example to compare the computed values with a set of eigenvalues known a priori. The matrix A is generated by $A = XDX^{-1}$, where D is the diagonal matrix containing 100 complex conjugate eigenvalues as shown by the $\times$ signs on figure 1. Due to the symmetry of the spectrum, the negative half is not shown. We generate the column vectors constituting the matrix X randomly, and at the same time conserve the conjugacy in accordance to the associated eigenvalues. The initial vector for the first Arnoldi iteration is also generated randomly. We have chosen $m = 30$ and $r = 7$, the number of wanted eigenvalues. The unwanted eigenvalues obtained from Hessenberg matrix of the first Arnoldi pass and the fourth pass are shown respectively by the + and $\circ$ signs on figure 1. The first r computed eigenvalues are not shown because they are not significantly distinguishable from the exact values on the graph. The residual norms (for details see Saad, 1984) of the first nine and the last five associated eigenvectors are shown on table 1 as indicated on column 1. The intermediate values are not shown because they are not sufficiently precise and do not yield further information. On the second column, after the first Arnoldi pass the first three eigenvectors have the residual norms of the order of 10^{-03} the others have the residual norms of 10^{00} and 10^{01}. Note

that the reference eigenvalue is the complex number labelled 7 on figure 1, thus its conjugate value is also considered as a reference point. We have chosen this reference point because it is close to ninth eigenvalue. This will make the Tchebychev acceleration less effective. However, on pass 2, i.e. after one Tchebychev iteration, a marked improvement can be seen on the first four eigenvalues with their associated residual norms of the order of less then 10^{-07} (table 1). The initial vector for the Tchebychev iteration is a linear combination of the product of the first r residual norms and their associated eigenvectors. This is done to lighten the effects of the highly accurate eigenvectors on the following iteration. On the third Arnoldi pass, improvement can be observed on the seventh and eighth eigenvalues. After three Tchebychev iterations, the Hessenberg matrix yields all first eight eigenvalues with their associated residual norms of the order of less than 10^{-06} as illustrated on the last column of table 1. The effect of the Tchebychev acceleration technique can be seen by comparing the residual norms associated with the seventh or the eighth eigenvalues, outside the ellipse, and with the ninth one, inside the ellipse, on the last column of table 1. Note that for the first pass, the Hessenberg matrix fails to give the ninth to the fourteenth eigenvalues as shown on figure 1. On table 1, the ninth residual norm of pass 1 is not associated to the ninth eigenvalue but to the fifteenth one. Only after the Tchebychev acceleration, the first nine

TABLE 1

The residual norms associated with eigenvalues of the Hessenberg matrix. A has dimension of 100×100, $m = 30$, $n = 35$ *and* $r = 7$ *(see text).*

Order of EV	Pass 1	Pass 2	Pass 3	Pass 4
1	0.15120D−03	0.31228D−13	0.55607D−13	0.30239D−13
2	0.60903D−03	0.12979D−11	0.11129D−11	0.82278D−12
3	0.60930D−03	0.12979D−11	0.11129D−11	0.82278D−12
4	0.47227D+00	0.96918D−07	0.65561D−07	0.82425D−08
5	0.12759D+01	0.80226D−04	0.18192D−03	0.66705D−08
6	0.12759D+01	0.80226D−04	0.18192D−03	0.66705D−08
7	0.19775D+01	0.14150D+00	0.69209D−03	0.95335D−06
8	0.19775D+01	0.14150D+00	0.69209D−03	0.95335D−06
9	0.10506D+01	0.20894D−01	0.80828D−01	0.38643D−01
	...			
26	0.15740D+01	0.52655D+00	0.38668D−01	0.16685D+00
27	0.10277D+01	0.22782D−01	0.30384D−02	0.73458D−02
28	0.10277D+01	0.22782D−01	0.30384D−02	0.73458D−02
29	0.71092D+00	0.18046D−01	0.10138D−01	0.16225D−01
30	0.23431D+00	0.61599D−04	0.13873D−04	0.12013D−04

residual norms correspond to the first nine eigenvalues. As a result, the first Tchebychev ellipse, shown by the dot-dash line on figure 1, excludes the first 12 eigenvalues. The Tchebychev ellipse of the third iteration is shown in solid line and the ellipse of the second iteration is represented by the broken line slightly outside the final one. It should also be noted that since the third ellipse, passing through the ninth and the last eigenvalues (whose associated residual norms are 10^{-01} and 10^{-04}, respectively), is also feasible for the exact unwanted spectrum, it is practically the optimal ellipse of the exact unwanted spectrum.

The above example was also run again without the Tchebychev acceleration. It took 38 Arnoldi passes to obtain the residual norms of the order of less than 10^{-05} for the first eight eigenvalues.

Last but not least, from figure 1 the eigenvalues yielded by the Hessenberg matrix generated by the Arnoldi procedure are mostly situated on the periphery of the exact spectrum of A (Chatelin, 1988). This is an inherent advantage for the Tchebychev acceleration. The experiment also shows that the Hessenberg eigenvalues seem to approximate more precisely the exact ones at the two extreme ends of the spectrum (*fig.* 1 and table 1).

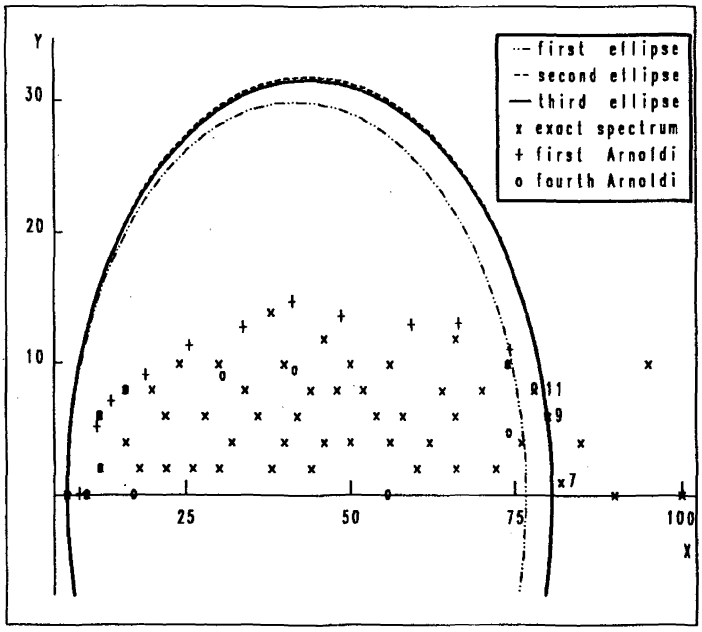

Figure 1. — The eigenspectrum of the matrix and the iterated Tchebychev ellipses.

V. CONCLUDING REMARKS

In this study, we have trated the Tchebychev ellipse in the general complex case, and used a simple and accurate procedure to calculate the optimal ellipse of the unwanted eigenspectrum. This represents the essential difference between our work and that of Saad (1984, 1985). First, the final ellipse used by Saad may not be the optimal Tchebychev ellipse for the mini-max solution, because Saad only finds the exact ellipse relative to the real part Re (λ_r) of the complex reference eigenvalue λ_r. Second, our algorithm allows us to identify precisely the solution to the mini-max problem without going through the process of enumerating all possible solutions as done by Manteuffel (1977, 1978) or Saad (1984).

As stated above, this example is only used for illustration purpose, further testing and comparison with other methods are necessary to ascertain the quality and the performance of the procedure. However, these initial results are quite encouraging, in particular, the contribution of the Tchebychev acceleration can clearly be seen on the difference of the precision of the eigenvalues inside and outside the ellipse.

When the reference point is at the origin, our procedure can also be used to solve the nonsymmetric linear system problems in a more effective way than the one described in Manteuffel (1977).

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