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## A NEW APPROACH OF TIMOSHENKO'S BEAM THEORY BY ASYMPTOTIC EXPANSION METHOD (\*)

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*Abstract — In this work we obtain a generalization of Timoshenko's beam theory by applying the asymptotic expansion method to a mixed variational formulation of the three dimensional linearized elasticity model. A classical subject of major discussion in this model is the proper definition of the so called Timoshenko's constants taking into account the fact that the shear stresses vary on each cross section. Due to the technique employed we shall be able to define these constants in a clear way and show its dependence on the geometry of the cross section and on Poisson's ratio. Finally we present several numerical examples showing the relationship between the classical and the new constants for different geometries.*

*Resume — En appliquant la methode des developpements asymptotiques a un modele variationnel mixte de l'elasticite linearisee on obtient une generalisation de la theorie de poutres de Timoshenko.*

*Associees a cette generalisation on obtient aussi une definition et une generalisation des constantes de Timoshenko tenant en compte la flexion additionnelle due a l'effort tranchant. La technique employee permet de demontrer sa dependance par rapport a la geometrie et au coefficient de Poisson.*

*Finalment differents exemples numeriques sont traites montrant la relation entre les nouvelles constantes et les constantes classiques pour differentes geometries.*

### 1 NOTATIONS

In this work the summation convention on repeated indexes is used. Latin indexes such as  $i, j, k$ , take values on the set  $\{1, 2, 3\}$  while Greek indexes such as  $\alpha, \beta, \gamma$ , take values on the set  $\{1, 2\}$ .

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Let  $\omega^\varepsilon$  be an open bounded, simply connected, subset of the plane  $Ox_1^\varepsilon x_2^\varepsilon$  with a sufficiently smooth boundary, whose area is  $A(\omega^\varepsilon) = \text{meas } \omega^\varepsilon = \varepsilon^2$ . In what follows, we shall consider a beam occupying volume  $\Omega^\varepsilon = \omega^\varepsilon \times (0, L)$ ,  $L > 0$ , and we shall write :

$$\gamma^\varepsilon = \partial\omega^\varepsilon, \quad \Gamma_0^\varepsilon = \omega^\varepsilon \times \{0, L\}, \quad \Gamma_1^\varepsilon = \gamma^\varepsilon \times (0, L). \quad (1.1)$$

We denote by  $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$ ,  $(x_1^\varepsilon, x_2^\varepsilon) \in \omega^\varepsilon$ , a generic point in  $\bar{\Omega}^\varepsilon$  and by  $\partial_\alpha^\varepsilon$  the differential operator  $\partial/\partial x_\alpha^\varepsilon$ . The outward unit normal to  $\partial\omega^\varepsilon$  will be denoted by  $n^\varepsilon = (n_\alpha^\varepsilon)$ .

From now on, we assume that the cross section  $\omega^\varepsilon$  is simply connected and the beam is made of an homogeneous, isotropic and linear elastic material of modulus of elasticity  $E$  and Poisson's ratio  $\nu$  which we suppose to be independent of  $\varepsilon$ . The method can be extended to the case of a connected cross section and anisotropic material and/or variable cross section but the notations become more complex and we shall not consider them here.

Moreover, with no loss of generality, we assume that  $Ox_1^\varepsilon x_2^\varepsilon x_3^\varepsilon$  is a principal system of inertia associated with the homogeneous body  $\Omega^\varepsilon$ . Consequently, we have in particular :

$$\int_{\omega^\varepsilon} x_\alpha^\varepsilon = \int_{\omega^\varepsilon} x_1^\varepsilon x_2^\varepsilon = 0. \quad (1.2)$$

We shall now define some functions and constants which play an important role in what follows and which characterize the geometry of the cross section  $\omega^\varepsilon$ .

1) Functions  $\Phi_{\alpha\beta}^\varepsilon$  are defined by :

$$\begin{cases} \Phi_{11}^\varepsilon(x_1^\varepsilon, x_2^\varepsilon) = -\Phi_{22}^\varepsilon(x_1^\varepsilon, x_2^\varepsilon) = \frac{1}{2} [(x_1^\varepsilon)^2 - (x_2^\varepsilon)^2], \\ \Phi_{12}^\varepsilon(x_1^\varepsilon, x_2^\varepsilon) = \Phi_{21}^\varepsilon(x_1^\varepsilon, x_2^\varepsilon) = x_1^\varepsilon x_2^\varepsilon. \end{cases} \quad (1.3)$$

2) Functions  $w^\varepsilon$  (the warping function of  $\omega^\varepsilon$ ),  $\Psi^\varepsilon$  (Saint Venant's torsion function or Prandtl's potential function),  $\eta_\beta^\varepsilon$  and  $\theta_\beta^\varepsilon$  are defined in a unique way, by the following problems :

$$\begin{cases} -\partial_{\alpha\alpha}^\varepsilon w^\varepsilon = 0 & \text{in } \omega^\varepsilon \\ \partial^\varepsilon w^\varepsilon / \partial n^\varepsilon = x_2^\varepsilon n_1^\varepsilon - x_1^\varepsilon n_2^\varepsilon & \text{on } \gamma^\varepsilon \\ \int_{\omega^\varepsilon} w^\varepsilon = 0 \end{cases} \quad (1.4)$$

$$\begin{cases} -\partial_{\alpha\alpha}^\varepsilon \Psi^\varepsilon = 2 & \text{in } \omega^\varepsilon \\ \Psi^\varepsilon = 0 & \text{on } \gamma^\varepsilon \end{cases} \quad (1.5)$$

$$\left\{ \begin{array}{ll} -\partial_{\alpha\alpha}^\varepsilon \eta_\beta^\varepsilon = -2 x_\beta^\varepsilon & \text{in } \omega^t \\ \partial^\varepsilon \eta_\beta^\varepsilon / \partial n^\varepsilon = 0 & \text{on } \gamma^\varepsilon \\ \int_{\omega^t} \eta_\beta^\varepsilon = 0 \end{array} \right. \quad (1.6)$$

$$\left\{ \begin{array}{ll} -\partial_{\alpha\alpha}^\varepsilon \theta_\beta^\varepsilon = 2 x_\beta^\varepsilon & \text{in } \omega^\varepsilon \\ \partial^\varepsilon \theta_\beta^\varepsilon / \partial n^\varepsilon = -\Phi_{\beta\alpha}^\varepsilon n_\alpha^\varepsilon & \text{on } \gamma^\varepsilon \\ \int_{\omega^t} \theta_\beta^\varepsilon = 0 \end{array} \right. \quad (1.7)$$

3) Constants  $I_\beta^{\omega^\varepsilon}$ ,  $I_\beta^{\Psi^\varepsilon}$ ,  $L_{\alpha\beta}^{\eta^\varepsilon}$ ,  $L_{\alpha\beta}^{\theta^\varepsilon}$ ,  $K_{\alpha\beta}^{\eta^\varepsilon}$  and  $K_{\alpha\beta}^{\theta^\varepsilon}$  are defined from the corresponding functions by :

$$I_\beta^{\omega^\varepsilon} = 2 \int_{\omega^\varepsilon} x_\beta^\varepsilon \omega^\varepsilon \quad (1.8)$$

$$I_1^{\Psi^\varepsilon} = 2 \int_{\omega^t} x_2^\varepsilon \Psi^\varepsilon, \quad I_2^{\Psi^\varepsilon} = -2 \int_{\omega^\varepsilon} x_1^\varepsilon \Psi^\varepsilon, \quad (1.9)$$

$$L_{\alpha\beta}^{\eta^\varepsilon} = \int_{\omega^\varepsilon} x_\alpha^\varepsilon \eta_\beta^\varepsilon, \quad L_{\alpha\beta}^{\theta^\varepsilon} = \int_{\omega^\varepsilon} x_\alpha^\varepsilon \theta_\beta^\varepsilon, \quad (1.10)$$

$$K_{\alpha\beta}^{\eta^\varepsilon} = \int_{\omega^t} \Phi_{\alpha\mu}^\varepsilon \partial_\mu^\varepsilon \eta_\beta^\varepsilon, \quad K_{\alpha\beta}^{\theta^\varepsilon} = \int_{\omega^t} \Phi_{\alpha\mu}^\varepsilon \partial_\mu^\varepsilon \theta_\beta^\varepsilon \quad (1.11)$$

4) Constants  $H_\alpha^\varepsilon$ ,  $I_\alpha^\varepsilon$  (second area moments of  $\omega^\varepsilon$ ) and  $J^\varepsilon$  (torsional constant) are defined as follows :

$$H_\alpha^\varepsilon = \frac{1}{2} \int_{\omega^\varepsilon} x_\alpha^\varepsilon [(x_1^\varepsilon)^2 + (x_2^\varepsilon)^2], \quad H_3^\varepsilon = \frac{1}{4} \int_{\omega^t} [(x_1^\varepsilon)^2 + (x_2^\varepsilon)^2] \quad (1.12)$$

$$I_\alpha^\varepsilon = \int_{\omega^\varepsilon} (x_\alpha^\varepsilon)^2 \quad (1.13)$$

$$J^\varepsilon = 2 \int_{\omega^\varepsilon} \Psi^\varepsilon = I_1^\varepsilon + I_2^\varepsilon - \int_{\omega^\varepsilon} [(\partial_1^\varepsilon \omega^\varepsilon)^2 + (\partial_2^\varepsilon \omega^\varepsilon)^2]. \quad \blacksquare \quad (1.14)$$

2. TIMOSHENKO'S CLASSICAL BEAM THEORY

We denote by  $f_i^\varepsilon(x^\varepsilon)$  (resp.  $g_i^\varepsilon(x^\varepsilon)$ ) the  $i$ -th component of the volume (resp. surface) density of the applied body forces (resp. surface tractions) at a point  $x^\varepsilon \in \Omega^\varepsilon$  (resp.  $x^\varepsilon \in \Gamma_1^\varepsilon$ ). Moreover,  $u^\varepsilon = (u_i^\varepsilon) : \Omega^\varepsilon \rightarrow \mathbb{R}^3$  denotes the displacement field due to the applied forces and  $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon) : \Omega^\varepsilon \rightarrow \mathbb{R}_s^9 = \{\tau^\varepsilon = (\tau_{ij}^\varepsilon) \in \mathbb{R}^9 : \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon\}$  its associated stress field.

Let  $F_i^\varepsilon(x_3^\varepsilon)$  and  $M_i^\varepsilon(x_3^\varepsilon)$  denote the linear force and moment densities, respectively, in the  $x_i^\varepsilon$  direction and at a section  $\omega^\varepsilon \times \{x_3^\varepsilon\}$ , that is :

$$F_i^\varepsilon = \int_{\omega^\varepsilon} f_i^\varepsilon + \int_{\gamma^k} g_i^\varepsilon, \tag{2.1}$$

$$M_\beta^\varepsilon = \int_{\omega^\varepsilon} x_\beta^\varepsilon f_3^\varepsilon + \int_{\gamma^k} x_\beta^\varepsilon g_3^\varepsilon, \tag{2.2}$$

$$M_3^\varepsilon = \int_{\omega^\varepsilon} (x_1^\varepsilon f_2^\varepsilon - x_2^\varepsilon f_1^\varepsilon) + \int_{\gamma^k} (x_1^\varepsilon g_2^\varepsilon - x_2^\varepsilon g_1^\varepsilon). \tag{2.3}$$

On each cross section  $\omega^\varepsilon \times \{x_3^\varepsilon\}$ , we denote the stress resultants along direction  $x_i^\varepsilon$  by  $q_i^\varepsilon(x_3^\varepsilon)$  and by  $m_i^\varepsilon(x_3^\varepsilon)$ , where :

$$q_i^\varepsilon = \int_{\omega^\varepsilon} \sigma_{3i}^\varepsilon, \tag{2.4}$$

$$m_\alpha^\varepsilon = \int_{\omega^\varepsilon} x_\alpha^\varepsilon \sigma_{33}^\varepsilon, \tag{2.5}$$

$$m_3^\varepsilon = \int_{\omega^\varepsilon} (x_1^\varepsilon \sigma_{32}^\varepsilon - x_2^\varepsilon \sigma_{31}^\varepsilon). \tag{2.6}$$

Stress resultants  $q_\alpha^\varepsilon(x_3^\varepsilon)$  and  $q_3^\varepsilon(x_3^\varepsilon)$  are designated by shear force along direction  $x_\alpha^\varepsilon$  and axial force (along direction  $x_3^\varepsilon$ ), respectively. Stress resultants  $m_\alpha^\varepsilon(x_3^\varepsilon)$  and  $m_3^\varepsilon(x_3^\varepsilon)$  are designated by bending moment associated to axis  $Ox_\beta^\varepsilon$ , ( $\beta \neq \alpha$ ) and by torsion moment (associated with axis  $Ox_3^\varepsilon$ ), respectively.

We consider a weakly clamped condition at both ends, as in Cimetière *et al.* [9]. If we introduce the admissible displacement and stress fields :

$$V^\varepsilon = \left\{ v^\varepsilon = (v_i^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3 : \int_{\omega^\varepsilon} v^\varepsilon = \int_{\omega^\varepsilon} x^\varepsilon \wedge v^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon \right\}, \tag{2.7}$$

$$\Sigma^\varepsilon = [L^2(\Omega^\varepsilon)]_s^9 = \{ \tau^\varepsilon = (\tau_{ij}^\varepsilon) \in [L^2(\Omega^\varepsilon)]^9 : \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon \}, \tag{2.8}$$

the equilibrium of the beam, as a three dimensional elastic body, may be described in a mixed variational form by the problem of finding the pair  $(\sigma^\varepsilon, u^\varepsilon) \in \Sigma^\varepsilon \times V^\varepsilon$  satisfying (Duvaut-Lions [12]) :

$$\int_{\Omega^\varepsilon} \left( \frac{1+\nu}{E} \sigma_{ij}^\varepsilon - \frac{\nu}{E} \sigma_{pp}^\varepsilon \delta_{ij} \right) \tau_{ij}^\varepsilon - \int_{\Omega^\varepsilon} \partial_i^\varepsilon u_j^\varepsilon \tau_{ij}^\varepsilon = 0, \quad \forall \tau^\varepsilon \in \Sigma^\varepsilon, \tag{2.9}$$

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \partial_i^\varepsilon v_j^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon + \int_{\Gamma_1^\varepsilon} g_i^\varepsilon v_i^\varepsilon, \quad \forall v^\varepsilon \in V^\varepsilon. \tag{2.10}$$

From a well-known result of Brezzi [3] for mixed formulations, and from Korn's inequality (Duvaut-Lions [12]), the existence of a unique solution to problem (2.9)-(2.10) is obtained when the applied loads satisfy, for example, the following regularity assumptions :

$$f_i^\epsilon \in L^2(\Omega^\epsilon), \quad g_i^\epsilon \in L^2(\Gamma_1^\epsilon). \tag{2.11}$$

The particular geometry of the beam as a three dimensional solid and the fact that  $\epsilon$  is very small when compared to the beam's length  $L$ , gave rise to simple models relating the displacement ( $u^\epsilon$ ) and stress ( $\sigma^\epsilon$ ) fields to the applied loads ( $f^\epsilon$  and  $g^\epsilon$ ). Invariably these models are based on a priori assumptions on the displacement field (and consequently on the stress field) leading to remarkable simplifications on the equilibrium equations. Typical examples of these models are Saint Venant's torsion theory and the bending theories of Bernoulli-Euler-Navier and of Timoshenko.

The theory of Timoshenko [23] was formulated in 1921. It provides a simple way to take into account an additional contribution to bending deformations due to the non uniform shear stress distribution along the cross section. This effect, which is not included in the classical theory of Bernoulli-Euler-Navier, cannot be neglected for relatively short beams with relatively large transversal sections. Moreover, these stresses are also involved in the main mechanism associated with delamination in multilayered structures.

We shall now summarize Timoshenko's beam theory following Dym-Shames [13] and Fung [15]. For the sake of simplicity and since Timoshenko's theory is only concerned with bending effects, we assume that the system of applied forces satisfies :

$$\begin{cases} f_3^\epsilon = g_3^\epsilon = 0, \\ M_3^\epsilon = \int_{\omega^\epsilon} (x_1^\epsilon f_2^\epsilon - x_2^\epsilon f_1^\epsilon) + \int_{\gamma^\epsilon} (x_1^\epsilon g_2^\epsilon - x_2^\epsilon g_1^\epsilon) = 0. \end{cases} \tag{2.12}$$

In this case, the kinematic a priori hypothesis associated with Timoshenko's beam theory are :

i) The transversal displacements depend only on  $x_3^\epsilon$ , that is :

$$u_\alpha^\epsilon(x_1^\epsilon, x_2^\epsilon, x_3^\epsilon) = \hat{u}_\alpha^\epsilon(x_3^\epsilon). \tag{2.13}$$

ii) The axial displacement  $u_3^\epsilon$  is of the form :

$$u_3^\epsilon(x_1^\epsilon, x_2^\epsilon, x_3^\epsilon) = -x_\alpha^\epsilon(\partial_3^\epsilon \hat{u}_\alpha^\epsilon - \hat{v}_\alpha^\epsilon) \tag{2.14}$$

where  $\hat{v}_\alpha^\epsilon$  is a function of  $x_3^\epsilon$  only which must be determined.

iii) The shear stress components  $\sigma_{3\alpha}^\epsilon$ , are given by :

$$\sigma_{3\alpha}^\epsilon(x_1^\epsilon, x_2^\epsilon, x_3^\epsilon) = \hat{\sigma}_{3\alpha}^\epsilon(x_3^\epsilon) = \frac{Ek^\epsilon}{2(1+\nu)} \hat{v}_\alpha^\epsilon \tag{2.15}$$

where  $k^\epsilon$  is Timoshenko's constant which depends on the material the beam is made of and on the shape of its cross section. From i)-iii) the Navier-Cauchy equilibrium equations for Timoshenko's beam theory become :

$$\left\{ \begin{aligned} \hat{\beta}_\alpha^\epsilon &= \partial_3^\epsilon \hat{u}_\alpha^\epsilon - \hat{v}_\alpha^\epsilon \end{aligned} \right. \tag{2.16}$$

$$\left\{ \begin{aligned} EI_\alpha^\epsilon \partial_{33}^\epsilon \hat{\beta}_\alpha^\epsilon + \frac{Ek^\epsilon A(\omega^\epsilon)}{2(1+\nu)} (\partial_3^\epsilon \hat{u}_\alpha^\epsilon - \hat{\beta}_\alpha^\epsilon) &= 0, \quad (\text{no sum on } \alpha) \end{aligned} \right. \tag{2.17}$$

$$\left\{ \begin{aligned} \frac{Ek^\epsilon A(\omega^\epsilon)}{2(1+\nu)} \partial_3^\epsilon (\partial_3^\epsilon \hat{u}_\alpha^\epsilon - \hat{\beta}_\alpha^\epsilon) &= -F_\alpha^\epsilon. \end{aligned} \right. \tag{2.18}$$

For the case of a cantilevered beam, for example, we must add the boundary conditions :

$$\hat{\beta}_\alpha^\epsilon(x_3^\epsilon) = 0, \quad \partial_3^\epsilon \hat{u}_\alpha^\epsilon(x_3^\epsilon) = 0 \quad \text{at } x_3^\epsilon = 0 \text{ and } L. \tag{2.19}$$

Differentiating with respect to  $x_3^\epsilon$  in (2.17) and (2.18) we are able to uncouple the system and obtain the classical equations of Timoshenko's beam theory :

$$\left\{ \begin{aligned} EI_\alpha^\epsilon \partial_{333}^\epsilon \hat{\beta}_\alpha^\epsilon &= F_\alpha^\epsilon, \quad (\text{no sum on } \alpha) \end{aligned} \right. \tag{2.20}$$

$$\left\{ \begin{aligned} EI_\alpha^\epsilon \partial_{333}^\epsilon \hat{u}_\alpha^\epsilon &= F_\alpha^\epsilon - \frac{2(1+\nu)}{k^\epsilon A(\omega^\epsilon)} I_\alpha^\epsilon \partial_{33}^\epsilon F_\alpha^\epsilon, \quad (\text{no sum on } \alpha) \end{aligned} \right. \tag{2.21}$$

which must be completed with the corresponding boundary conditions.

Several aspects of this theory are not very clear. For example, from (2.13)-(2.14) and using Hooke's law we obtain

$$\sigma_{3\alpha}^\epsilon = \frac{E}{2(1+\nu)} \hat{v}_\alpha^\epsilon$$

which does not agree with (2.15). Consequently, although the displacement field associated with Timoshenko's beam theory already includes the additional bending deformation due to the shear stress distribution, the stress field itself is not correctly determined. This is due to the introduction of factor  $k^\epsilon$  in order to account for the non-uniform shear stress distribution along a cross section of the beam, while still retaining the one dimensional approach. Moreover, it is not clear how this factor should be calculated. Timoshenko [23] stated that  $k^\epsilon$  depends on the shape of the cross section and proposed  $k^\epsilon = 2/3$  for the rectangular case. Mindlin [18] suggests that its

value can be selected in such a way that the solution of (2.17) agrees with certain exact solutions of the three dimensional equations. Most of the definitions make it a function of the shape of the cross section and of Poisson's ratio.

In order to illustrate this dependence we reproduce in figure 2.1 a list of values for  $k^\epsilon$  taken from Dym-Shames [13] for different shapes of the cross section.

We remark that these constants are used independently of the loading direction and do not take into account possible coupled bending effects. Moreover, for most of the cases the indicated constants are used independently of the relative dimensions of the cross section.

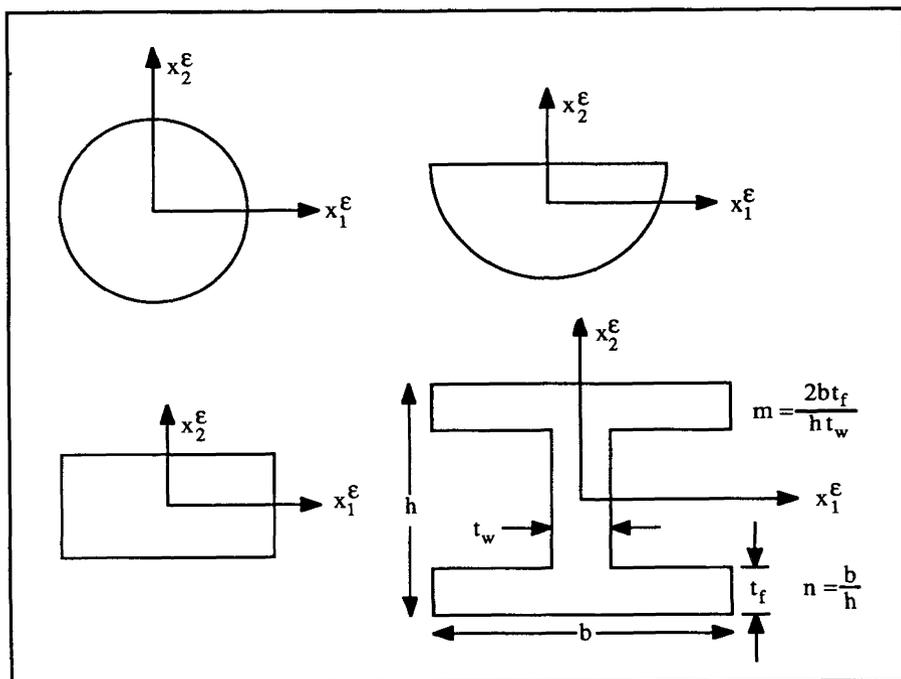


Fig. 2.1. — Timoshenko's classical constants :

- a) Circle :  $k^\epsilon = \frac{6(1 + \nu)}{7 + 6 \nu}$  . b) Semicircle :  $k^\epsilon = \frac{1 + \nu}{1.305 + 1.273 \nu}$  . c) Rectangle :  $k^\epsilon = \frac{10(1 + \nu)}{12 + 11 \nu}$  .
- d) I-shaped beam :

$$k^\epsilon = \frac{10(1 + \nu)(1 + 3 m)^2}{[(12 + 72 m + 150 m^2 + 90 m^3) + \nu(11 + 66 m + 135 m^2 + 90 m^3) + 30 n^2(m + m^2) + 5 \nu n^2(8 m + 9 m^2)]}$$

Another major drawback of Timoshenko's theory resides on the fact that even when no surface loads are applied on a portion of  $\Gamma_1^\varepsilon$  we always have a shear stress contribution given by  $\sigma_{3\alpha}^\varepsilon n_\alpha^\varepsilon$ , in contradiction with the equilibrium equations.

In the present work we obtain a generalization of Timoshenko's beam theory by using the asymptotic expansion method. The model obtained does not contain the contradictions mentioned above and allows us to justify from a mathematical point of view the classical equilibrium equations. The a priori hypotheses show up as necessary conditions for the existence of the first term of an asymptotic expansion of the stress ( $\sigma^\varepsilon$ ) and displacement ( $u^\varepsilon$ ) fields.

The governing equilibrium equation associated with the model we introduce is (compare with (2.21)):

$$EI_\alpha^\varepsilon \partial_{333}^\varepsilon \hat{u}_\alpha^\varepsilon = F_\alpha^\varepsilon - T_{\alpha\beta}^\varepsilon \partial_{33}^\varepsilon F_\beta^\varepsilon. \quad (2.22)$$

In this equation, « Timoshenko's matrix » components  $T_{\alpha\beta}^\varepsilon$  are defined in a precise way as a function of the geometry of the cross section and of Poisson's ratio. Its calculation is extremely simple (see Section 6) and just involves the calculation of functions and constants given in (1.3)-(1.14).

With the exception of some symmetrical cross sections, matrix  $T^\varepsilon = (T_{\alpha\beta}^\varepsilon)$  is not diagonal and consequently equation (2.22) includes coupled bending effects. Finally, assuming that  $\partial_{33}^\varepsilon F_\beta^\varepsilon = 0$ , comparison of  $T_{\alpha\alpha}^\varepsilon$  with  $2(1 + \nu) I_\alpha^\varepsilon / [k^\varepsilon A(\omega^\varepsilon)]$  allows us to give a precise definition of the classical Timoshenko's constant that should be used whenever the resultant of the applied loads acts along  $Ox_\alpha^\varepsilon$  (see Sections 5 and 6).

The model (2.22) that we are going to obtain may be considered as a second order approximation (in a sense to be precised) of the three-dimensional solution ( $\sigma^\varepsilon, u^\varepsilon$ ). In order for this to hold it is necessary to assume that the system of applied forces is of the following form:

$$\begin{cases} f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^{1-r} f_\alpha^0(x), & f_3^\varepsilon(x^\varepsilon) = \varepsilon^{-r} f_3^0(x), \\ g_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^{2-r} g_\alpha^0(x), & g_3^\varepsilon(x^\varepsilon) = \varepsilon^{1-r} g_3^0(x). \end{cases} \quad (2.23)$$

where  $x = (x_1, x_2, x_3) = (\varepsilon^{-1} x_1^\varepsilon, \varepsilon^{-1} x_2^\varepsilon, x_3^\varepsilon)$ ,  $r$  is an arbitrary real number and functions  $f_i^0, g_i^0$  are independent on  $\varepsilon$ .

These assumptions seem to constitute a restriction on the loading. However, this is not the case. In fact, the only restriction inherent to these equations is the one of being able to express the applied loads as the product of a power of  $\varepsilon$  by a function independent of  $\varepsilon$ . If that is the case, using a superposition principle, the linearity of the problem allows us to consider any relationship among the applied forces. Consequently, relations (2.23) are sufficient in order to analyse the most common cases in practice.

Furthermore, since Timoshenko's theory is just concerned with the bending effects we shall assume that the only significant effects of loading are those due to the linear transverse force densities. Consequently, we assume that :

$$\left\{ \begin{aligned} & f_3^\varepsilon = 0, \quad g_3^\varepsilon = 0, \\ & \int_{\omega^\varepsilon} x_\alpha^\varepsilon f_\alpha^\varepsilon = 0, \quad \int_{\gamma^\varepsilon} x_\alpha^\varepsilon g_\alpha^\varepsilon = 0, \quad \text{a. e. in } (0, L), \\ & \int_{\omega^\varepsilon} \Phi_{\alpha\beta}^\varepsilon f_\alpha^\varepsilon = 0, \quad \int_{\gamma^\varepsilon} \Phi_{\alpha\beta}^\varepsilon g_\alpha^\varepsilon = 0, \quad \text{a. e. in } (0, L), \\ & M_3^\varepsilon = \int_{\omega^\varepsilon} (x_1^\varepsilon f_2^\varepsilon - x_2^\varepsilon f_1^\varepsilon) + \int_{\gamma^\varepsilon} (x_1^\varepsilon g_2^\varepsilon - x_2^\varepsilon g_1^\varepsilon) = 0, \quad \text{a. e. in } (0, L). \end{aligned} \right. \tag{2.24}$$

If these conditions do not hold the asymptotic method may be used in the same way. However, the final model is more complex because it will also include other effects : bending, extension, torsion and Poisson's effects (see Trabucho-Viaño [24, 25, 26]). ■

### 3. THE ASYMPTOTIC EXPANSION METHOD

The asymptotic expansion method whose foundations can be studied in Lions [17] allows us not only to justify from the mathematical point of view, the hypothesis showing up in Timoshenko's beam theory, but also to derive them. Applications of this method in solid mechanics were done, for example, by Caillerie [4], Ciarlet-Destuynder [7, 8], Destuynder [10, 11], Raoult [19] and Viaño [30] in order to justify the classical models in plate bending. The study of the linearized theory of beam bending by this method, was already introduced in Rigolot [20, 21] and it was continued in Bermudez-Viaño [2], Aganovic-Tutek [1] and Viaño [31] using a mixed variational formulation. In these works the classical Bernoulli-Euler-Navier theory is justified but the stress components  $\sigma_{i\beta}^0$ , in the first term of the asymptotic expansion (3.11) are not uniquely determined.

Using the same method and a weakly clamped condition at both ends, the geometrically nonlinear case was analyzed by Cimetière *et al.* [9]. They proved unicity of  $\sigma^0$  by imposing the compatibility with higher order terms.

This method is used in Trabucho-Viaño [24, 25, 26] in order to calculate the higher order terms in the asymptotic expansion introduced in Bermudez-Viaño [2] for the linear case. This allows us to derive and justify the most well known beam theories both in bending and torsion including Bernoulli-Euler-Navier (see Bermudez-Viaño [2]), Saint Venant, Timoshenko and Vlasov (see Trabucho-Viaño [24, 26, 27, 28]).

As a matter of fact, this work is devoted to derive a generalized Timoshenko’s model (including the classical one described in Section 2 as a particular case) together with its a priori hypothesis, directly from the three dimensional linearized elasticity model (2.9)-(2.10). The general results of Trabucho-Viaño [26] will be fundamental here and we shall reference them for the proofs.

The main idea in all these works is to consider the problem of finding the pair  $(\sigma^\varepsilon, u^\varepsilon)$  of the stress and displacement fields which solves (2.9)-(2.10) as a problem depending on the small parameter  $\varepsilon$  which tends to zero. In order to study the behaviour of the solution when  $\varepsilon$  becomes small, we make a change of variable from  $\bar{\Omega}^\varepsilon$  to a fixed domain  $\bar{\Omega} = \bar{\omega} \times [0, L]$ ,  $\omega = \varepsilon^{-1} \omega^\varepsilon$ , through the transformation

$$x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \bar{\Omega}^\varepsilon \rightarrow x = (x_1, x_2, x_3) = (\varepsilon^{-1} x_1^\varepsilon, \varepsilon^{-1} x_2^\varepsilon, x_3^\varepsilon) \in \bar{\Omega},$$

already introduced in Bermudez-Viaño [2]. This leads to a problem posed in the fixed open set  $\Omega = \omega \times (0, L)$ , which does not depend on  $\varepsilon$  and in such a way that this parameter appears in an explicit and suitable manner that makes it possible to apply the techniques of Lions [17]. Specifically, let us introduce the following notation

$$\gamma = \partial\omega, \quad \Gamma_0 = \omega \times \{0, L\}, \quad \Gamma_1 = \gamma \times (0, L), \tag{3.1}$$

and the following function spaces

$$V = \left\{ v = (v_i) \in [H^1(\Omega)]^3 \mid \int_\omega v = \int_\omega x \wedge v = 0 \text{ on } \Gamma_0 \right\}, \tag{3.2}$$

$$\Sigma = [L^2(\Omega)]_S^9 = \{ \tau = (\tau_{ij}) \in [L^2(\Omega)]^9 \mid \tau_{ij} = \tau_{ji} \} \tag{3.3}$$

equipped with the usual norms

Given  $(\sigma^\varepsilon, u^\varepsilon) \in \Sigma^\varepsilon \times V^\varepsilon$  we define the element  $(\sigma(\varepsilon), u(\varepsilon)) \in \Sigma \times V$  through the following transformations, where  $r$  is the fixed real number appearing in (2.23) (see Bermudez-Viaño [2])

$$\begin{cases} u_\alpha(\varepsilon)(x) = \varepsilon^{r+1} u_\alpha^\varepsilon(x^\varepsilon), & u_3(\varepsilon)(x) = \varepsilon^r u_3^\varepsilon(x^\varepsilon) \\ \sigma_{\alpha\beta}(\varepsilon)(x) = \varepsilon^{r-2} \sigma_{\alpha\beta}^\varepsilon(x^\varepsilon), & \sigma_{3\beta}(\varepsilon)(x) = \varepsilon^{r-1} \sigma_{3\beta}^\varepsilon(x^\varepsilon), \\ \sigma_{33}(\varepsilon)(x) = \varepsilon^r \sigma_{33}^\varepsilon(x^\varepsilon) \end{cases} \tag{3.4}$$

Then, the following result is a very simple consequence of the integral change of variable in problem (2.9)-(2.10)

**PROPOSITION 3.1** *Let  $(\sigma(\varepsilon), u(\varepsilon)) \in \Sigma \times V$  be the element obtained from the solution  $(\sigma^\varepsilon, u^\varepsilon) \in \Sigma^\varepsilon \times V^\varepsilon$  of (2.9), (2.10) through the use of*



In Trabucho-Viaño [26] it is proved that equations (3.12)-(3.14) determine in a unique way the element

$$(\sigma^0, u^0, \sigma_{33}^2, u^2, q^2) \in \Sigma \times V \times L^2(\Omega) \times V \times [L^2(0, L)]^3, \quad q_i^2 = \int_{\omega} \sigma_{3i}^2,$$

when certain regularity on the applied loads is assumed. ■

*Remark 3.1.*

In Bermudez-Viaño [2] existence of  $(u^0, \sigma_{33}^0)$  and existence, but not uniqueness, of  $\sigma_{\alpha i}^0$  are shown solving (3.12), with a clamped condition at both ends. By working with the equivalent of (3.13)-(3.14), for the geometrically nonlinear case, and considering a weakly clamped condition at both ends, unicity of  $\sigma_{\alpha i}^0$  is shown in Cimetière *et al.* [9]. ■

**4. APPROXIMATION ON THE ORIGINAL BEAM  $\Omega^\epsilon$**

From (3.11) we may suppose in an heuristic way that  $(\sigma(\epsilon), u(\epsilon))$  is approximated in  $\Omega$  by  $(\sigma^0, u^0)$  or by  $(\sigma^0, u^0) + \epsilon^2(\sigma^2, u^2)$  as  $\epsilon$  becomes small. Consequently by transforming these quantities back to  $\bar{\Omega}^\epsilon$  we obtain quantities  $(\sigma^{0\epsilon}, u^{0\epsilon})$  and  $(\sigma^{0\epsilon}, u^{0\epsilon}) + (\sigma^{2\epsilon}, u^{2\epsilon})$ , which may be considered as the first and second order approximations, respectively, of  $(\sigma^\epsilon, u^\epsilon)$  solution of (2.9)-(2.10) in  $\Omega^\epsilon$ . Specifically, for  $p = 0, 2, 4$  elements  $(\sigma^{p\epsilon}, u^{p\epsilon})$  are defined by :

$$\begin{cases} u_{\alpha}^{p\epsilon}(x^\epsilon) = \epsilon^{-1-r+p} u_{\alpha}^p(x), & u_{\beta}^{p\epsilon}(x^\epsilon) = \epsilon^{-r+p} u_{\beta}^p(x), \\ \sigma_{\alpha\beta}^{p\epsilon}(x^\epsilon) = \epsilon^{2-r+p} \sigma_{\alpha\beta}^p(x), & \sigma_{3\beta}^{p\epsilon}(x^\epsilon) = \epsilon^{1-r+p} \sigma_{3\beta}^p(x), \\ \sigma_{33}^{p\epsilon}(x^\epsilon) = \epsilon^{-r+p} \sigma_{33}^p(x). \end{cases} \quad (4.1)$$

We characterize elements  $(\sigma^{0\epsilon}, u^{0\epsilon})$  (partially contained in Bermudez-Viaño [2]) and  $(\sigma_{33}^{2\epsilon}, u^{2\epsilon})$  through the following result which is an immediate consequence of (4.1) and from a more general result contained in Trabucho-Viaño [26] where the particular case (2.24) is not assumed.

**PROPOSITION 4.1 :** *Let the system of applied forces be such that (2.23) and (2.24) hold. Then, elements  $(\sigma^{0\epsilon}, u^{0\epsilon}, \sigma_{33}^{2\epsilon}, u^{2\epsilon}) \in \Sigma^\epsilon \times V^\epsilon \times L^2(\Omega^\epsilon) \times V^\epsilon$ , defined in (4.1), together with  $q_{\beta}^{2\epsilon} = \int_{\omega^\epsilon} \sigma_{3\beta}^{2\epsilon}$  are uniquely determined in the following way :*

i) Displacements  $u_{\beta}^{0\epsilon}$  depend only on  $x_3^{\epsilon}$  and are the unique solution to the following variational problem :

$$\begin{cases} u_{\beta}^{0\epsilon} \in H_0^2(0, L) \\ EI_{\beta}^{\epsilon} \int_0^L \partial_{33}^{\epsilon} u_{\beta}^{0\epsilon} \partial_{33}^{\epsilon} v = \int_0^L F_{\beta}^{\epsilon} v, \quad \forall v \in H_0^2(0, L), \quad (\text{no sum on } \beta) \end{cases} \quad (4.2)$$

ii) Displacement  $u_3^{0\epsilon}$  and stress component  $\sigma_{33}^{0\epsilon}$  are obtained from  $u_{\beta}^{0\epsilon}$  by :

$$u_3^{0\epsilon} = -x_{\alpha}^{\epsilon} \partial_3^{\epsilon} u_{\alpha}^{0\epsilon} \quad (4.3)$$

$$\sigma_{33}^{0\epsilon} = E \partial_3^{\epsilon} u_3^{0\epsilon} = -E x_{\alpha}^{\epsilon} \partial_{33}^{\epsilon} u_{\alpha}^{0\epsilon}. \quad (4.4)$$

iii) Displacements  $u_i^{2\epsilon}$  are of the following form :

$$u_1^{2\epsilon} = z_1^{2\epsilon} + x_2^{\epsilon} z^{2\epsilon} + \nu \Phi_{1\beta}^{\epsilon} \partial_{33}^{\epsilon} u_{\beta}^{0\epsilon} \quad (4.5)$$

$$u_2^{2\epsilon} = z_2^{2\epsilon} - x_1^{\epsilon} z^{2\epsilon} + \nu \Phi_{2\beta}^{\epsilon} \partial_{33}^{\epsilon} u_{\beta}^{0\epsilon} \quad (4.6)$$

$$u_3^{2\epsilon} = \underline{u}_3^{2\epsilon} - x_{\alpha}^{\epsilon} \partial_3^{\epsilon} z^{2\epsilon} - w^{\epsilon} \partial_3^{\epsilon} z^{2\epsilon} + [(1 + \nu) \eta_{\alpha}^{\epsilon} + \nu \theta_{\alpha}^{\epsilon}] \partial_{333}^{\epsilon} u_{\alpha}^{0\epsilon}, \quad (4.7)$$

where  $z^{2\epsilon}$ ,  $\underline{u}_3^{2\epsilon}$  and  $z_{\alpha}^{2\epsilon}$  depend only on variable  $x_3^{\epsilon}$  and are characterized in the following way from the data and from the components already known :

a) Function  $z^{2\epsilon}$  represents the angle of twist and it solves the problem :

$$\begin{cases} z^{2\epsilon} \in H^1(0, L) \\ \frac{EJ^{\epsilon}}{2(1 + \nu)} \int_0^L \partial_3^{\epsilon} z^{2\epsilon} \partial_3^{\epsilon} v = \int_0^L M_3^{0\epsilon} v, \quad \forall v \in H_0^1(0, L) \\ z^{2\epsilon}(x_3^{\epsilon}) = \frac{\nu}{I_1^{\epsilon} + I_2^{\epsilon}} [H_2^{\epsilon} \partial_{33}^{\epsilon} u_1^{0\epsilon}(x_3^{\epsilon}) - H_1^{\epsilon} \partial_{33}^{\epsilon} u_2^{0\epsilon}(x_3^{\epsilon})], \quad \text{at } x_3^{\epsilon} = 0 \text{ and } L \end{cases} \quad (4.8)$$

where :

$$M_3^{0\epsilon} = -\frac{E}{2(1 + \nu)} [(1 + \nu) I_{\alpha}^{w^{\epsilon}} + \nu I_{\alpha}^{\Psi^{\epsilon}}] \partial_{3333}^{\epsilon} u_{\alpha}^{0\epsilon}. \quad (4.9)$$

b) The stretching component  $\underline{u}_3^{2\epsilon}$  is obtained solving the following variational problem :

$$\begin{cases} \underline{u}_3^{2\epsilon} \in H_0^1(0, L) \\ EA(\omega^{\epsilon}) \int_0^L \partial_3^{\epsilon} \underline{u}_3^{2\epsilon} \partial_3^{\epsilon} v = \nu \int_0^L G_3^{2\epsilon} \partial_3^{\epsilon} v, \quad \forall v \in H_0^1(0, L) \end{cases} \quad (4.10)$$

where :

$$G_3^{2\varepsilon} = EH_\alpha^\varepsilon \partial_{333}^\varepsilon u_\alpha^{0\varepsilon}. \quad (4.11)$$

c) The bending component  $z_\alpha^{2\varepsilon}$  is the unique solution of the following problem (no sum on  $\alpha$ ):

$$\left\{ \begin{array}{l} z_\alpha^{2\varepsilon} \in H^2(0, L) \\ EI_\alpha^\varepsilon \int_0^L \partial_{33}^\varepsilon z_\alpha^{2\varepsilon} \partial_{33}^\varepsilon v = \int_0^L M_\alpha^{2\varepsilon} \partial_{33}^\varepsilon v, \quad \forall v \in H_0^2(0, L); \\ z_\alpha^{2\varepsilon}(x_3^\varepsilon) = \frac{\nu(I_\beta^\varepsilon - I_\alpha^\varepsilon)}{2A(\omega^\varepsilon)} \partial_{33}^\varepsilon u_\alpha^{0\varepsilon}(x_3^\varepsilon) \quad \text{at } x_3^\varepsilon = 0 \text{ and } L, \quad (\beta \neq \alpha); \\ \partial_{33}^\varepsilon z_\alpha^{2\varepsilon}(x_3^\varepsilon) = \frac{1}{I_\alpha^\varepsilon} \left\{ -\frac{I_\alpha^w}{2} \partial_3^\varepsilon z^{2\varepsilon} + \right. \\ \left. + [(1 + \nu) L_{\alpha\beta}^{\eta^\varepsilon} + \nu L_{\alpha\beta}^{\theta^\varepsilon}] \partial_{333}^\varepsilon u_\beta^{0\varepsilon} \right\} \quad \text{at } x_3^\varepsilon = 0 \text{ and } L, \end{array} \right. \quad (4.12)$$

where

$$M_\alpha^{2\varepsilon} = E \left\{ (1 + \nu) L_{\alpha\beta}^{\eta^\varepsilon} + \nu L_{\alpha\beta}^{\theta^\varepsilon} + \frac{\nu}{2} K_{\alpha\beta}^{\eta^\varepsilon} + \frac{\nu^2}{2(1 + \nu)} (K_{\alpha\beta}^{\theta^\varepsilon} + H_3^\varepsilon \delta_{\alpha\beta}) \right\} \partial_{3333}^\varepsilon u_\beta^{0\varepsilon} - \frac{E}{2(1 + \nu)} [(1 + \nu) I_\alpha^w + \nu I_\alpha^{\Psi^\varepsilon}] \partial_{33}^\varepsilon z^{2\varepsilon}. \quad (4.13)$$

iv) The shear stress  $\sigma_{3\beta}^{0\varepsilon}$ , the bending moment  $m_\beta^{0\varepsilon} = \int_{\omega^\varepsilon} x_\beta^\varepsilon \sigma_{33}^{0\varepsilon}$  and the shear force  $q_\beta^{0\varepsilon} = \int_{\omega^\varepsilon} \sigma_{3\beta}^{0\varepsilon}$  components are uniquely determined by :

$$\sigma_{31}^{0\varepsilon} = \frac{E}{2(1 + \nu)} \left\{ -\partial_2^\varepsilon \Psi^\varepsilon \partial_3^\varepsilon z^{2\varepsilon} + [(1 + \nu) \partial_1^\varepsilon \eta_\beta^\varepsilon + \nu(\partial_1^\varepsilon \theta_\beta^\varepsilon + \Phi_{1\beta}^\varepsilon)] \partial_{333}^\varepsilon u_\beta^{0\varepsilon} \right\} \quad (4.14)$$

$$\sigma_{32}^{0\varepsilon} = \frac{E}{2(1 + \nu)} \left\{ \partial_1^\varepsilon \Psi^\varepsilon \partial_3^\varepsilon z^{2\varepsilon} + [(1 + \nu) \partial_2^\varepsilon \eta_\beta^\varepsilon + \nu(\partial_2^\varepsilon \theta_\beta^\varepsilon + \Phi_{2\beta}^\varepsilon)] \partial_{333}^\varepsilon u_\beta^{0\varepsilon} \right\} \quad (4.15)$$

$$m_\beta^{0\varepsilon} = -EI_\beta^\varepsilon \partial_{33}^\varepsilon u_\beta^{0\varepsilon}, \quad (\text{no sum on } \beta) \quad (4.16)$$

$$q_\beta^{0\varepsilon} = \partial_3^\varepsilon m_\beta^{0\varepsilon} = -EI_\beta^\varepsilon \partial_{333}^\varepsilon u_\beta^{0\varepsilon}, \quad (\text{no sum on } \beta). \quad (4.17)$$

v) The plane stress components  $\sigma_{\alpha\beta}^{0\epsilon}$  are obtained solving the following plane elasticity problem :

$$\sigma_{\alpha\beta}^{0\epsilon} = S_{\alpha\beta}^{\epsilon}(\underline{u}^{4\epsilon}) = \frac{E}{1+\nu} \gamma_{\alpha\beta}^{\epsilon}(\underline{u}^{4\epsilon}) + \frac{\nu E}{(1+\nu)(1-2\nu)} \gamma_{\mu\mu}^{\epsilon}(\underline{u}^{4\epsilon}) \delta_{\alpha\beta} \quad (4.18)$$

where, for all  $v^{\epsilon} \in [H^1(\omega^{\epsilon})]^2$ ,  $\gamma_{\alpha\beta}^{\epsilon}(v^{\epsilon}) = (\partial_{\alpha}^{\epsilon} v_{\beta}^{\epsilon} + \partial_{\beta}^{\epsilon} v_{\alpha}^{\epsilon})/2$  and where  $\underline{u}^{4\epsilon} = (\underline{u}_{\alpha}^{4\epsilon})$  is the unique solution of :

$$\left\{ \begin{array}{l} \underline{u}^{4\epsilon} \in (L^2[0, L; H^1(\omega^{\epsilon})])^2 \text{ and } \forall \zeta^{\epsilon} \in [H^1(\omega^{\epsilon})]^2: \\ \int_{\omega^{\epsilon}} S_{\alpha\beta}^{\epsilon}(\underline{u}^{4\epsilon}) \gamma_{\alpha\beta}^{\epsilon}(\zeta^{\epsilon}) = \int_{\omega^{\epsilon}} f_{\beta}^{\epsilon} \zeta_{\beta}^{\epsilon} + \int_{\gamma^{\epsilon}} g_{\beta}^{\epsilon} \zeta_{\beta}^{\epsilon} + \int_{\omega^{\epsilon}} \partial_3^{\epsilon} \sigma_{3\beta}^{0\epsilon} \zeta_{\beta}^{\epsilon}, \\ a. e. \text{ in } (0, L), \\ \int_{\omega^{\epsilon}} \underline{u}_{\alpha}^{4\epsilon} = \int_{\omega^{\epsilon}} (x_2^{\epsilon} \underline{u}_1^{4\epsilon} - x_1^{\epsilon} \underline{u}_2^{4\epsilon}) = 0, \text{ a. e. in } (0, L). \end{array} \right. \quad (4.19)$$

vi) The axial stress  $\sigma_{33}^{2\epsilon}$ , bending moment  $m_{\beta}^{2\epsilon} = \int_{\omega^{\epsilon}} x_{\beta}^{\epsilon} \sigma_{33}^{2\epsilon}$  and shear force  $q_{\beta}^{2\epsilon} = \int_{\omega^{\epsilon}} \sigma_{3\beta}^{2\epsilon}$  components are given by :

$$\sigma_{33}^{2\epsilon} = E \partial_3^{\epsilon} u_3^{2\epsilon} + \nu \sigma_{\mu\mu}^{0\epsilon} = E \{ \partial_3^{\epsilon} u_3^{2\epsilon} - x_{\alpha}^{\epsilon} \partial_{33}^{\epsilon} z_{\alpha}^{2\epsilon} - w^{\epsilon} \partial_{33}^{\epsilon} z^{2\epsilon} + [(1+\nu) \eta_{\alpha}^{\epsilon} + \nu \theta_{\alpha}^{\epsilon}] \partial_{333}^{\epsilon} u_{\alpha}^{0\epsilon} \} + \nu S_{\mu\mu}^{\epsilon}(\underline{u}^{4\epsilon}) \quad (4.20)$$

$$m_{\beta}^{2\epsilon} = -EI_{\beta}^{\epsilon} \partial_{33}^{\epsilon} z_{\beta}^{2\epsilon} + M_{\beta}^{2\epsilon}, \quad (\text{no sum on } \beta) \quad (4.21)$$

$$q_{\beta}^{2\epsilon} = \partial_3^{\epsilon} m_{\beta}^{2\epsilon}. \quad \blacksquare \quad (4.22)$$

From this result we obtain very important consequences from the physical point of view. In fact, equations (4.2)-(4.4), (4.16) and (4.17) are the classical equations associated with Bernoulli-Euler-Navier's beam theory (see Bermudez-Viaño [2]). Equations (4.5)-(4.7) represent second order displacement components (with respect to Bernoulli-Euler-Navier's theory). In these terms not only are the classical effects of Saint Venant's torsion theory with Poisson's effects (Fraejis de Veubeke [14]) exhibited in a general form but the bending terms  $z_{\alpha}^{2\epsilon}$  which are connected with Timoshenko's theory are also found, as we shall show in the next section.

Firstly, equations (4.8), (4.9) for  $z^{2\epsilon}$  represent a torsion problem; although the total moment about  $Ox_3^{\epsilon}$  due to the applied body forces and surface tractions is zero, there may be a nonzero angle of twist  $z^{2\epsilon}$  due to the geometry of the cross section. In fact, if the resultant of the applied loads does not pass through the shear centre of the cross section, a moment  $M_3^{0\epsilon}$  about  $Ox_3^{\epsilon}$  is created originating a rotation on each cross section. We remark that if the cross section possesses two axes of symmetry,

or if it only has one axis and if the resultant of the applied loads acts along that axis, then  $z^{2\epsilon}$  is zero

Secondly, equations (4 10), (4 11) for  $\underline{u}_3^\epsilon$  represent a contribution to the axial displacement mainly due to Poisson's effects and to the geometry of the cross section. Once again this term is zero if the cross section possesses two axes of symmetry, or if it only has one axis and if the resultant of the applied loads acts perpendicularly to that axis.

Moreover, equations (4 12), (4 13) for  $z_\alpha^{2\epsilon}$  represent an additional bending due to two different types of effects. The first one results from the fact that plane sections perpendicular to the centroidal axis, before deformation, do not remain neither plane nor perpendicular with respect to that axis when bending is present. This is exactly the effect considered in Timoshenko's beam theory that we shall study in detail in the next section. The second effect given by the term involving  $z^{2\epsilon}$  in (4 13) represents an additional contribution to bending due to the fact that the total resultant of the applied forces may not necessarily pass through the shear centre and consequently besides the additional torsional effects already mentioned there is also an additional bending effect. This term is not present in a explicit way in the classical torsion-bending theories. Associated with this additional bending displacement we obtain the bending moment and the shear force components given by (4 21) and (4 22), respectively.

The last terms in (4 5), (4 6) and in (4 7) represent a deformation due to Poisson's effect associated with the bending moments and with the shear force components of the Bernoulli-Euler-Navier displacement field, respectively. The classical torsion theory of Saint Venant with Poisson's effects includes the terms in (4 5) and (4 6) but the last term in (4 7) seems to have never been presented in this explicit way.

The first term in equations (4 14) and (4 15) represents a classical contribution to the shear stresses from torsion while the other terms represent a contribution mainly due to shear force components associated with the Bernoulli-Euler-Navier displacement field. It constitutes a generalization of the corresponding classical form in torsion theory (see Hlavacek-Necas [16] and Trabucho-Viaño [27]). One of the purposes of Timoshenko's classical beam theory is to give an approximation of this quantity when the torsion effects are neglected (see (2 15)).

Equations (4 18)-(4 19) are simply a plane elasticity problem on each cross section and represent the fact that a cross section does not necessarily behave like a rigid body on its own plane. It is interesting to observe that this phenomenon is obtained as a higher order effect.

The general form obtained from the former terms is transferred to the axial stress component given by (4 20) where the third term represents an effect due to the fact that the angle of twist per unit length is not necessarily constant as in the classical Saint Venant's torsion theory. ■

5. A GENERALIZATION OF TIMOSHENKO'S BEAM THEORY

In the previous section we observed that the first order terms  $u^{0\epsilon}$ ,  $\sigma_{33}^{0\epsilon}$ ,  $m_{\beta}^{0\epsilon}$ ,  $q_{\beta}^{0\epsilon}$  represent the classical bending theory of Bernoulli-Euler-Navier. Moreover, we also observe that the terms giving the shear stress components are associated with the second order approximation  $(\tilde{\sigma}^{\epsilon}, \tilde{u}^{\epsilon}) = (\sigma^{0\epsilon}, u^{0\epsilon}) + (\sigma^{2\epsilon}, u^{2\epsilon})$  of  $(\sigma^{\epsilon}, u^{\epsilon})$ . We remark that the correct order of approximation should be given by an appropriate error estimation and convergence. Partial results in this direction were obtained by Aganovic-Tutek [1] and Bermudez-Viaño [2].

From Proposition 4.1 we conclude that the transverse displacements are approximated by :

$$\tilde{u}_1^{\epsilon} = (u_1^{0\epsilon} + z_1^{2\epsilon}) + x_2^{\epsilon} z^{\epsilon} + \nu \Phi_{1\beta}^{\epsilon} \partial_{33}^{\epsilon} u_{\beta}^{0\epsilon} \tag{5.1}$$

$$\tilde{u}_2^{\epsilon} = (u_2^{0\epsilon} + z_2^{2\epsilon}) - x_1^{\epsilon} z^{\epsilon} + \nu \Phi_{2\beta}^{\epsilon} \partial_{33}^{\epsilon} u_{\beta}^{0\epsilon} . \tag{5.2}$$

From these expressions we conclude that displacements  $\tilde{u}_{\beta}^{\epsilon}$  include a bending effect given by term  $u_{\beta}^{0\epsilon} + z_{\beta}^{2\epsilon}$ , a torsion effect due to the presence of  $z^{\epsilon}$  and a last term associated with a Poisson's effect. As a consequence equations (5.1) and (5.2) constitute a generalization of the displacement field associated with the bending-torsion theory with Poisson's effects which does not include term  $z_{\beta}^{2\epsilon}$  (see Fraejis de Veubeke [14]).

Since Timoshenko's beam theory does not take into account torsion or Poisson's effects, a possible model generalizing Timoshenko's classical theory may be obtained by neglecting these effects in  $\tilde{\sigma}_{33}^{\epsilon}$ ,  $\tilde{u}^{\epsilon}$  and in  $\tilde{\sigma}_{\beta i}^{0\epsilon}$  which in fact amounts to approximate  $(\sigma^{\epsilon}, u^{\epsilon})$  by  $(\hat{\sigma}^{\epsilon}, \hat{u}^{\epsilon})$  given by :

$$\hat{u}_{\beta}^{\epsilon} = u_{\beta}^{0\epsilon} + z_{\beta}^{2\epsilon} \tag{5.3}$$

$$\hat{u}_3^{\epsilon} = \underline{u}_3^{2\epsilon} - x_{\alpha}^{\epsilon} \partial_3^{\epsilon} \hat{u}_{\alpha}^{\epsilon} + [(1 + \nu) \eta_{\alpha}^{\epsilon} + \nu \theta_{\alpha}^{\epsilon}] \partial_{333}^{\epsilon} u_{\alpha}^{0\epsilon} \tag{5.4}$$

$$\hat{\sigma}_{33}^{\epsilon} = E \partial_3^{\epsilon} \{ \underline{u}_3^{2\epsilon} - x_{\alpha}^{\epsilon} \partial_3^{\epsilon} \hat{u}_{\alpha}^{\epsilon} + [(1 + \nu) \eta_{\alpha}^{\epsilon} + \nu \theta_{\alpha}^{\epsilon}] \partial_{333}^{\epsilon} u_{\alpha}^{0\epsilon} \} \tag{5.5}$$

$$\hat{\sigma}_{3\alpha}^{\epsilon} = \frac{E}{2(1 + \nu)} \{ [(1 + \nu) \partial_{\alpha}^{\epsilon} \eta_{\beta}^{\epsilon} + \nu \partial_{\alpha}^{\epsilon} \theta_{\beta}^{\epsilon}] \partial_{333}^{\epsilon} u_{\beta}^{0\epsilon} \} \tag{5.6}$$

$$\hat{\sigma}_{\alpha\beta}^{\epsilon} = 0 . \tag{5.7}$$

We remark that in (5.3) we obtained a priori hypothesis (2.13). Expression (5.4) gives us a generalization of (2.14) because besides including quantity  $\underline{u}_3^{2\epsilon}$  it consists mainly in replacing the classical term  $x_{\alpha}^{\epsilon} \hat{v}_{\alpha}^{\epsilon}$  by  $[(1 + \nu) \eta_{\beta}^{\epsilon} + \nu \theta_{\beta}^{\epsilon}] \partial_{333}^{\epsilon} u_{\beta}^{0\epsilon}$ .

It is also clear that (5.6) generalizes the classical expression (2.15) and takes into account the variation of the shear stress component  $\hat{\sigma}_{3\alpha}^{\epsilon}$  through the cross section. We observe that (5.6) may also be obtained directly from

the displacement field  $\hat{u}_i^\varepsilon$  through Hooke's law eliminating one of the contradictions pointed out in the classical theory.

We shall now study these equations in more detail and show how they include Timoshenko's equation (2.21), for sufficiently smooth data. In fact, the differential equation associated with (4.2) is :

$$EI_\alpha^\varepsilon \partial_{3333}^\varepsilon u_\alpha^{0\varepsilon} = F_\alpha^\varepsilon \text{ in } (0, L), \quad (\text{no sum on } \alpha), \quad (5.8)$$

and the one associated with (4.8)-(4.9) is :

$$\frac{EJ^\varepsilon}{2(1+\nu)} \partial_{33}^\varepsilon z^{2\varepsilon} = -M_3^{0\varepsilon}. \quad (5.9)$$

From (4.9) and (5.8) we obtain :

$$\partial_{33}^\varepsilon z^{2\varepsilon} = \frac{(1+\nu) I_\alpha^{w^\varepsilon} + \nu I_\alpha^{\Psi^\varepsilon}}{EJ^\varepsilon I_\alpha^\varepsilon} F_\alpha^\varepsilon, \quad (\text{no sum on } \alpha). \quad (5.10)$$

Now, the differential equation associated with (4.12)-(4.13) may be written as :

$$EI_\alpha^\varepsilon \partial_{3333}^\varepsilon z_\alpha^{2\varepsilon} = -T_{\alpha\beta}^\varepsilon \partial_{33}^\varepsilon F_\beta^\varepsilon, \quad (\text{no sum on } \alpha) \quad (5.11)$$

where, with no sum on  $\beta$  :

$$T_{\alpha\beta}^\varepsilon = -\frac{1}{I_\beta^\varepsilon} \left\{ (1+\nu) L_{\alpha\beta}^{\eta^\varepsilon} + \nu L_{\alpha\beta}^{\theta^\varepsilon} + \frac{\nu}{2} K_{\alpha\beta}^{\eta^\varepsilon} + \frac{\nu^2}{2(1+\nu)} (K_{\alpha\beta}^{\theta^\varepsilon} + H_3^\varepsilon \delta_{\alpha\beta}) \right. \\ \left. - \frac{1}{2(1+\nu)J^\varepsilon} [(1+\nu) I_\alpha^{w^\varepsilon} + \nu I_\alpha^{\Psi^\varepsilon}] [(1+\nu) I_\beta^{w^\varepsilon} + \nu I_\beta^{\Psi^\varepsilon}] \right\} \quad (5.12)$$

From (5.8) and (5.11) we conclude that  $\hat{u}_\alpha^\varepsilon$  is the unique solution of the following differential equation (no sum on  $\alpha$ ) :

$$EI_\alpha^\varepsilon \partial_{3333}^\varepsilon \hat{u}_\alpha^\varepsilon = F_\alpha^\varepsilon - T_{\alpha\beta}^\varepsilon \partial_{33}^\varepsilon F_\beta^\varepsilon, \quad (5.13)$$

subjected to the following boundary conditions at  $x_3^\varepsilon = 0$  and  $L$  (no sum on  $\alpha$ ) :

$$\hat{u}_\alpha^\varepsilon(x_3^\varepsilon) = \frac{\nu(I_\beta^\varepsilon - I_\alpha^\varepsilon)}{2A(\omega^\varepsilon)} \partial_{33}^\varepsilon u_\alpha^{0\varepsilon}(x_3^\varepsilon), \quad (\beta \neq \alpha), \quad (5.14)$$

$$\partial_3^\varepsilon \hat{u}_\alpha^\varepsilon(x_3^\varepsilon) = \frac{1}{I_\alpha^\varepsilon} \left\{ -\frac{I_\alpha^{w^\varepsilon}}{2} \partial_{33}^\varepsilon z^{2\varepsilon} + [(1+\nu) L_{\alpha\beta}^{\eta^\varepsilon} + \nu L_{\alpha\beta}^{\theta^\varepsilon}] \partial_{333}^\varepsilon u_\beta^{0\varepsilon} \right\}. \quad (5.15)$$

By comparing equation (5.13) with (2.21), we see that the total bending displacement components  $\hat{u}_\alpha^\varepsilon = u_\alpha^{0\varepsilon} + z_\alpha^{2\varepsilon}$  are the solution of a boundary

value problem generalizing Timoshenko's equation (2.21). Boundary conditions (5.14) and (5.15) are not the same as those given in (2.19) because now torsion effects are also included. Neglecting, as in the classical theory, torsion and Poisson's effects, it is logical not to consider the first term on the right-hand side of (5.15). We remark that this term is zero if  $Ox_\alpha^\epsilon$  is an axis of symmetry.

In summary, the generalized Timoshenko model proposed is given by (5.3)-(5.7) where  $\hat{u}_\alpha^\epsilon$  is the solution of (5.13)-(5.15) with  $u_\alpha^{0\epsilon}$  solution of (4.2) and with « Timoshenko's matrix »  $T^\epsilon = (T_{\alpha\beta}^\epsilon)$  given by (5.12). ■

*Remark 5.1.*

The major idea followed in order to obtain model (5.3)-(5.7) consists in neglecting torsion and Poisson's effects from the general equations, obtained via the asymptotic expansion method, and to obey Hooke's law in order to obtain a model as close as possible to the classical one.

However, if we do not require Hooke's law to hold, other models similar to the classical one are also possible. As an example we point out that if one wishes condition  $\hat{\sigma}_{3\alpha}^\epsilon n_\alpha^\epsilon = 0$  on  $\Gamma_1^\epsilon$  to hold (which is coherent with the equilibrium equations) then one just needs to substitute (5.6) by

$$\hat{\sigma}_{3\alpha}^\epsilon = \frac{E}{2(1 + \nu)} [(1 + \nu) \partial_\alpha^\epsilon \eta_\beta^\epsilon + \nu(\partial_\alpha^\epsilon \theta_\beta^\epsilon + \Phi_{\alpha\beta}^\epsilon)] \partial_{333}^\epsilon u_\beta^{0\epsilon}$$

obtained from (4.14), (4.15) by neglecting torsion effects only.

These considerations indicate that the correct model one should always consider is the one given directly by asymptotic expansion method. ■

**6. NEW TIMOSHENKO'S CONSTANTS**

Equation (5.13) for the transversal displacement  $\hat{u}_\alpha^\epsilon$  takes into account the coupled bending effects not included in the classical theory, through matrix  $T^\epsilon = (T_{\alpha\beta}^\epsilon)$ , which is not diagonal, in general. Consequently, in order to be able to compare (5.13) with (2.21) we assume that simple bending takes place, that is,  $\partial_{33}^\epsilon F_\beta^{0\epsilon} = 0$  ( $\beta \neq \alpha$ ). In this case, the following expression :

$$\tilde{k}_\alpha^\epsilon = \frac{2(1 + \nu) I_\alpha^\epsilon}{T_{\alpha\alpha}^\epsilon A(\omega^\epsilon)} \quad (\text{no sum on } \alpha) . \tag{6.1}$$

provides a precise definition for the constant that should be considered for calculating the bending deformations along direction  $Ox_\alpha^\epsilon$  when the coupling effect due to loads acting along direction  $Ox_\beta^\epsilon$  is to be neglected.

Even though, in the general case, one has  $T_{11}^\epsilon \neq T_{22}^\epsilon$ . Consequently (6.1) represents an improvement with respect to the classical theory which

assumes the same constant for any direction (see the rectangular cross section example presented next).

From definition (5.12) we see that the calculation of Timoshenko's matrix  $T^e = (T_{\alpha\beta}^e)$ , for a specific cross section and a specific material, can be done using any numerical method in order to solve problems (1.4)-(1.7) and evaluate constants (1.8)-(1.14). We shall now illustrate the calculation of this matrix for the most common cross sections. For the circular case an analytical solution is available. For the other cases, we use the finite element method with linear triangular elements. The results presented next were obtained using a large number of elements. However, extremely accurate results are also possible using just a few elements. ■

### 6.1. Circular cross section of radius $R$

This is the simplest case since analytical solutions for problems (1.4)-(1.7) are available. In fact, we have :

$$a) w^e = 0$$

$$b) \Psi^e = \frac{1}{2} \{R^2 - [(x_1^e)^2 + (x_2^e)^2]\}$$

$$c) \eta_\alpha^e = \frac{1}{4} [(x_1^e)^2 + (x_2^e)^2 - 3R^2] x_\alpha^e$$

$$d) \theta_\alpha^e = -\frac{1}{4} [(x_1^e)^2 + (x_2^e)^2 - R^2] x_\alpha^e$$

$$I_\alpha^e = \pi R^4/4 \quad J^e = \pi R^4/2$$

$$I_\alpha^{w^e} = 0.0000 \quad I_\alpha^{\Psi^e} = 0.0000$$

$$L_{11}^{\eta^e} = -7\pi R^6/48 \quad L_{22}^{\eta^e} = 7\pi R^6/48 \quad L_{\alpha\beta}^{\eta^e} = 0 \quad (\alpha \neq \beta)$$

$$L_{11}^{\theta^e} = \pi R^6/48 \quad L_{22}^{\theta^e} = \pi R^6/48 \quad L_{\alpha\beta}^{\theta^e} = 0 \quad (\alpha \neq \beta)$$

$$K_{11}^{\eta^e} = \pi R^6/24 \quad K_{22}^{\eta^e} = \pi R^6/24 \quad K_{\alpha\beta}^{\eta^e} = 0 \quad (\alpha \neq \beta)$$

$$K_{11}^{\theta^e} = -\pi R^6/24 \quad K_{22}^{\theta^e} = -\pi R^6/24 \quad K_{\alpha\beta}^{\theta^e} = 0 \quad (\alpha \neq \beta)$$

$$H_\alpha^e = 0.0000 \quad H_3^e = \pi R^6/12 . .$$

Consequently :

$$T_{12}^e = T_{21}^e = 0.0000$$

$$T_{11}^e = T_{22}^e = \frac{R^2(7 + 12\nu + 4\nu^2)}{12(1 + \nu)} .$$

Substituting in (6.1) we obtain that the new Timoshenko's constants for the circle are given by :

$$\tilde{k}_1^e = \tilde{k}_2^e = \frac{6 + 12\nu + 6\nu^2}{7 + 12\nu + 4\nu^2} . \quad (6.2)$$

Both this new and the classical constants are represented as a function of Poisson's ratio  $\nu$ , in figure 6.1. They both coincide for  $\nu = 0$  and the maximum difference is obtained for  $\nu = 0.5$ .

On the next subsections we shall give the values for the constants showing up in the definition of  $T_{\alpha\beta}^e$ , for the most common cross sections and using the finite element method. ■

## 6.2. Semicircular cross section of unitary radius

With a mesh of 1 350 elements and 724 nodes we obtain the following results :

$$\begin{array}{ll}
 I_1^{w^t} = 0.06635056 & I_2^{w^t} = 0.00000000 \\
 I_1^{\Psi^e} = 0.00703070 & I_2^{\Psi^e} = 0.00000000 \\
 I_1^e = 0.39219778 & I_2^e = 0.10964360 \\
 L_{11}^{\eta^e} = -0.22880905 & L_{22}^{\eta^e} = -0.02005027 \\
 L_{12}^k = 0.00000000 & L_{21}^k = 0.00000000 \\
 L_{11}^{\theta^k} = 0.01650639 & L_{22}^{\theta^e} = 0.01047391 \\
 L_{12}^{\theta^e} = 0.00000000 & L_{21}^{\theta^e} = 0.00000000 \\
 K_{11}^{\eta^e} = 0.03376054 & K_{22}^{\eta^e} = 0.02084043 \\
 K_{12}^k = 0.00000000 & K_{21}^{\eta^e} = 0.00000000 \\
 K_{11}^{\theta^k} = -0.05813010 & K_{22}^{\theta^k} = -0.03056944 \\
 K_{12}^{\theta^k} = 0.00000000 & K_{21}^{\theta^k} = 0.00000000 \\
 H_1^e = 0.00000000 & H_2^e = -0.01326206 \\
 H_3^e = 0.06426122 & J^e = 0.29631603
 \end{array}$$

$$T_{11}^e = \frac{14 + 30\nu + 15\nu^2}{23(1 + \nu)}$$

$$T_{22}^e = \frac{1 + \nu - \nu^2}{6(1 + \nu)}$$

$$T_{12}^e = T_{21}^e = 0.00000000$$

$$\tilde{k}_1^e = \frac{1 + 2\nu + \nu^2}{1.205 + 2.581\nu + 1.360\nu^2}$$

$$\tilde{k}_2^e = \frac{1 + 2\nu + \nu^2}{1.309 + 1.254\nu - 1.149\nu^2}$$

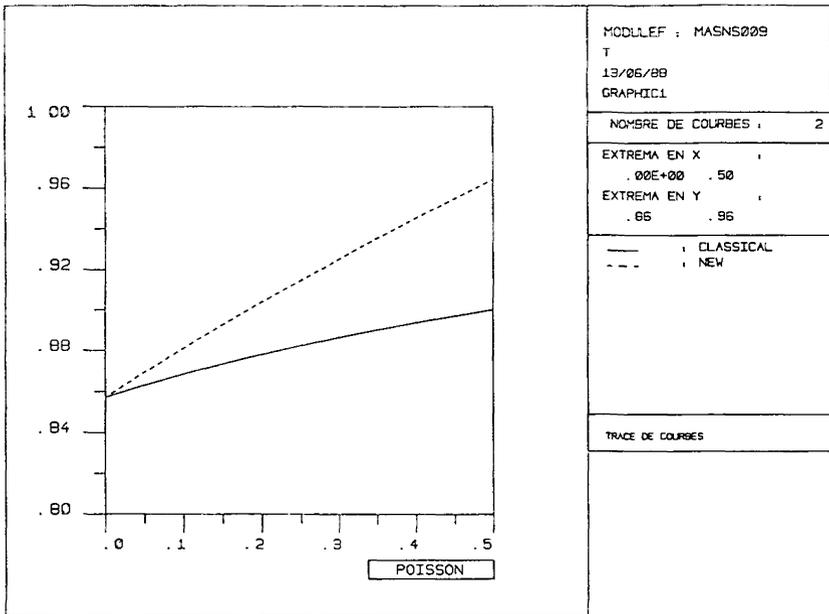


Figure 6.1. — Circular cross section.

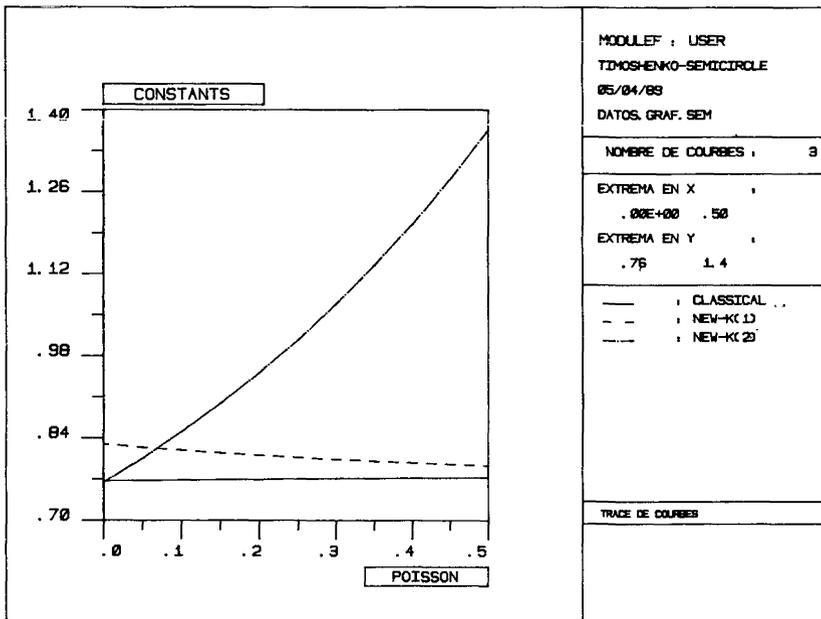


Figure 6.2. — Semicircular cross section.

In figure 6.2 both constants are plotted as functions of  $\nu$  and they are compared with the classical one. ■

**6.3. I-shaped cross section**

With the same notation as in figure 2.1 the relative dimensions of the cross section are  $h = 5$ ,  $b = 4$ ,  $t_w = 2$ ,  $t_f = 0.5$  and the mesh used is constituted by 384 triangles with 237 nodes. The results are the following :

$I_1^{w^b} =$	0.00000000	$I_2^{w^e} =$	0.00000000
$I_1^{w^e} =$	0.00000000	$I_2^{w^e} =$	0.00000000
$I_1^e =$	8.00000000	$I_2^e =$	31.00000000
$L_{11}^b =$	- 14.88317100	$L_{22}^b =$	- 2 27.90495000
$L_{12}^b =$	0.00000000	$L_{21}^b =$	0.00000000
$L_{11}^{\theta^e} =$	9.17661890	$L_{22}^{\theta^e} =$	- 25.50602800
$L_{12}^{\theta^e} =$	0.00000000	$L_{21}^{\theta^e} =$	0.00000000
$K_{11}^{\eta^e} =$	18.31755100	$K_{22}^{\eta^e} =$	- 52.47956700
$K_{12}^{\eta^e} =$	0.00000000	$K_{21}^{\eta^e} =$	0.00000000
$K_{11}^{\theta^e} =$	- 37.81388500	$K_{22}^{\theta^e} =$	- 49.81746700
$K_{12}^{\theta^e} =$	0.00000000	$K_{21}^{\theta^e} =$	0.00000000
$H_1^e =$	0.00000000	$H_2^e =$	0.00000000
$H_3^e =$	51.59578400	$J^e =$	9.84146960

$$T_{11}^e = \frac{1.86 + 1.4290 \nu - 1.293 \nu^2}{1 + \nu}$$

$$T_{22}^e = \frac{7.35 + 16.373 \nu + 8.992 \nu^2}{1 + \nu}$$

$$T_{12}^e = T_{21}^e = 0.00000000$$

$$\tilde{k}_1^e = \frac{1 + 2 \nu + \nu^2}{1.395 + 1.071 \nu - 0.970 \nu^2}$$

$$\tilde{k}_2^e = \frac{1 + 2 \nu + \nu^2}{1.423 + 3.169 \nu + 1.740 \nu^2}$$

Two constants are compared with the classical one in figure 6.3. ■

#### 6.4. Triangular cross section

As an example, we consider an equilateral triangular cross section of side  $l = 6$  and a regular mesh with 1 296 triangles and 703 nodes. The results are as follows :

$$\begin{array}{ll}
 I_1^{w^e} = 0.00000000 & I_2^{w^e} = 0.00000000 \\
 I_1^{\Psi^e} = 0.00000000 & I_2^{\Psi^e} = 0.00000000 \\
 I_1^e = 23.38259000 & I_2^e = 23.38259000 \\
 L_{11}^{\eta^e} = -92.81725600 & L_{22}^{\eta^e} = -92.81725600 \\
 L_{12}^{\eta^e} = 0.00000000 & L_{21}^{\eta^e} = 0.00000000 \\
 L_{11}^{\theta^e} = -2.22166050 & L_{22}^{\theta^e} = -2.22166050 \\
 L_{12}^{\theta^e} = 0.00000000 & L_{21}^{\theta^e} = 0.00000000 \\
 K_{11}^{\eta^e} = -4.82221040 & K_{22}^{\eta^e} = -4.82221040 \\
 K_{12}^{\eta^e} = 0.00000000 & K_{21}^{\eta^e} = 0.00000000 \\
 K_{11}^{\theta^e} = -41.65178400 & K_{22}^{\theta^e} = -41.65178400 \\
 K_{12}^{\theta^e} = 0.00000000 & K_{21}^{\theta^e} = 0.00000000 \\
 H_1^e = 0.00000000 & H_2^e = 0.00000000 \\
 H_3^e = 56.11812300 & J^e = 27.95093700
 \end{array}$$

$$T_{11}^e = T_{22}^e = \frac{3.97 + 8.137 \nu + 3.858 \nu^2}{1 + \nu}$$

$$T_{12}^e = T_{21}^e = 0.00000000$$

$$\tilde{k}_1^e = \tilde{k}_2^e = \frac{1 + 2 \nu + \nu^2}{1.323 + 2.712 \nu + 1.286 \nu^2}. \quad \blacksquare$$

#### 6.5. Unitary square cross section

For this case we use a mesh with 1 352 triangles and 729 nodes and we obtain :

$$\begin{array}{ll}
 I_1^{w^e} = 0.00000000 & I_2^{w^e} = 0.00000000 \\
 I_1^{\Psi^e} = 0.00000000 & I_2^{\Psi^e} = 0.00000000 \\
 I_1^e = 0.08333301 & I_2^e = 0.08333301 \\
 L_{11}^{\eta^e} = -0.16687270 & L_{22}^{\eta^e} = -0.16687270
 \end{array}$$

$$\begin{aligned}
 L_{12}^{\eta^s} &= 0.00000000 & L_{21}^{\eta^s} &= 0.00000000 \\
 L_{11}^{\theta^e} &= 0.00142491 & L_{22}^{\theta^e} &= 0.00142491 \\
 L_{12}^{\theta^e} &= 0.00000000 & L_{21}^{\theta^e} &= 0.00000000 \\
 K_{11}^{\eta^s} &= 0.00275738 & K_{22}^{\eta^s} &= 0.00275738 \\
 K_{12}^{\eta^e} &= 0.00000000 & K_{21}^{\eta^e} &= 0.00000000 \\
 K_{11}^{\theta^s} &= -0.00580318 & K_{22}^{\theta^s} &= -0.00580318 \\
 K_{12}^{\theta^e} &= 0.00000000 & K_{21}^{\theta^e} &= 0.00000000 \\
 H_1^e &= 0.00000000 & H_2^e &= 0.00000000 \\
 H_3^e &= 0.00972218 & J^e &= 0.13990354
 \end{aligned}$$

$$\begin{aligned}
 T_{11}^e &= T_{22}^e = \frac{1 + 1.832 \nu + 0.714 \nu^2}{5(1 + \nu)} \\
 T_{12}^e &= T_{21}^e = 0.00000000 \\
 \tilde{k}_1^e &= \tilde{k}_2^e = \frac{10 + 20 \nu + 10 \nu^2}{12 + 22 \nu + 8.585 \nu^2}
 \end{aligned}$$

The relationship between the new and the classical constant is illustrated in figure 6.4. As before the two constants agree for the limiting case  $\nu = 0$ . ■

### 6.6. Rectangular cross section

For this case Timoshenko's constants just depend on the relative dimensions of the cross section. They can be written on the form :

$$\tilde{k}_1^e = 10(1 + \nu)^2 / N_1, \quad \tilde{k}_2^e = 10(1 + \nu)^2 / N_2.$$

Considering that the smallest side is parallel to direction  $Ox_2^e$  the results are as follows :

Dimensions	$N_1$	$N_2$
$1 \times 1$	$12.014 + 22.010 \nu + 08.585 \nu^2$	$12.014 + 22.010 \nu + 08.585 \nu^2$
$2 \times 1$	$12.009 + 25.761 \nu + 13.634 \nu^2$	$12.039 + 06.970 \nu - 19.167 \nu^2$
$2.4 \times 1$	$12.010 + 26.143 \nu + 14.074 \nu^2$	$12.041 - 01.841 \nu - 38.153 \nu^2$
$2.5 \times 1$	$12.006 + 26.206 \nu + 14.148 \nu^2$	$12.039 - 04.323 \nu - 43.619 \nu^2$
$3 \times 1$	$12.003 + 26.448 \nu + 14.417 \nu^2$	$12.039 - 18.126 \nu - 75.511 \nu^2$
$5 \times 1$	$12.004 + 26.804 \nu + 14.794 \nu^2$	$12.101 - 98.791 \nu - 279.230 \nu^2$
$10 \times 1$	$12.001 + 26.951 \nu + 14.949 \nu^2$	$12.101 - 476.980 \nu - 1\ 315.87 \nu^2$

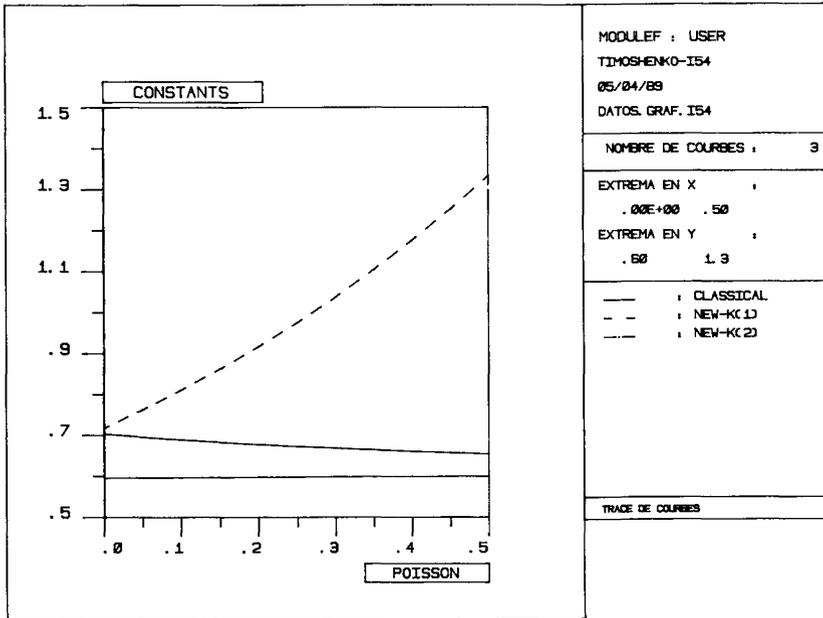


Figure 6.3. — I-shaped cross section.

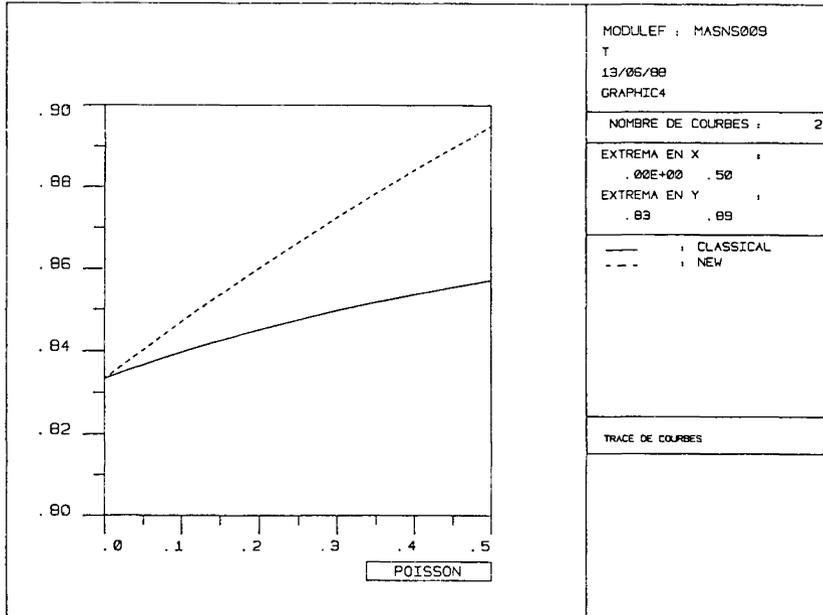


Figure 6.4. — Square cross section.

For the various rectangular cross sections functions  $\tilde{k}_\beta^e(\nu)$  are represented in figures 6 5 ( $\beta = 1$ ) and 6 6 ( $\beta = 2$ ) From the above results two major facts, which deserve a deeper study, show up The first one is related to constant  $\tilde{k}_1^e$  In fact, it looks as if as the ratio between the sides of the rectangle  $r = l_2/l_1$  goes to zero, constant  $\tilde{k}_1^e = \tilde{k}_1^r(\nu)$  converges to

$$\tilde{k}_1^0(\nu) = \frac{10(1 + \nu)^2}{12 + 27 \nu + 15 \nu^2}$$

The second fact is referred to constant  $\tilde{k}_2^e$  The graphical representation on figure 6 6 shows that when  $l_2$  is greater or equal to 2 5, there is a critical value of Poisson's ratio ( $\nu_r$ ), for which  $\tilde{k}_2^e = \tilde{k}_2^r(\nu)$  presents a singularity in the sense that

$$\lim_{\nu \rightarrow \nu_r} \tilde{k}_2^r(\nu) = + \infty \quad \text{and} \quad \lim_{\nu \rightarrow \nu_r^+} \tilde{k}_2^r(\nu) = - \infty$$

We also observe that when  $r \rightarrow 0$ ,  $\nu_r$  also goes to zero Consequently, for a given material with Poisson's ratio  $\nu$ , it looks as if there exists a relationship between the relative dimensions of the cross section ( $r = l_2/l_1$ ) for which Timoshenko's and Bernoulli-Euler-Navier's beam models coincide We may then ask what is this relationship between  $r$  and  $\nu_r$  The answer may well give a domain of validity for both theories as a function of the geometry and of Poisson's ratio

These two questions are currently under study by the authors using once again the asymptotic expansion method on the functions that appear in the definition of Timoshenko's matrix  $T^e = (T_{\alpha\beta}^e)$  (see Trabuco-Viaño [29]) It is possible to prove that replacing  $\tilde{k}_\alpha^r(\nu)$  by their first order asymptotic approximations  $\tilde{k}_\alpha^{0,r}(\nu)$ , then  $\tilde{k}_1^{0,r}(\nu)$  converges asymptotically to  $\tilde{k}_1^0(\nu)$  and  $\tilde{k}_2^{0,r}(\nu)$  presents a singularity in  $\nu = \nu_r$  satisfying the following relation which is in agreement with the numerical results obtained in figure 6 6

$$r = \left[ \frac{3(8 + 10 \nu_r + \nu_r^2)}{5(2 \nu_r + 3 \nu_r^2)} \right]^{-1/2} \quad \blacksquare$$

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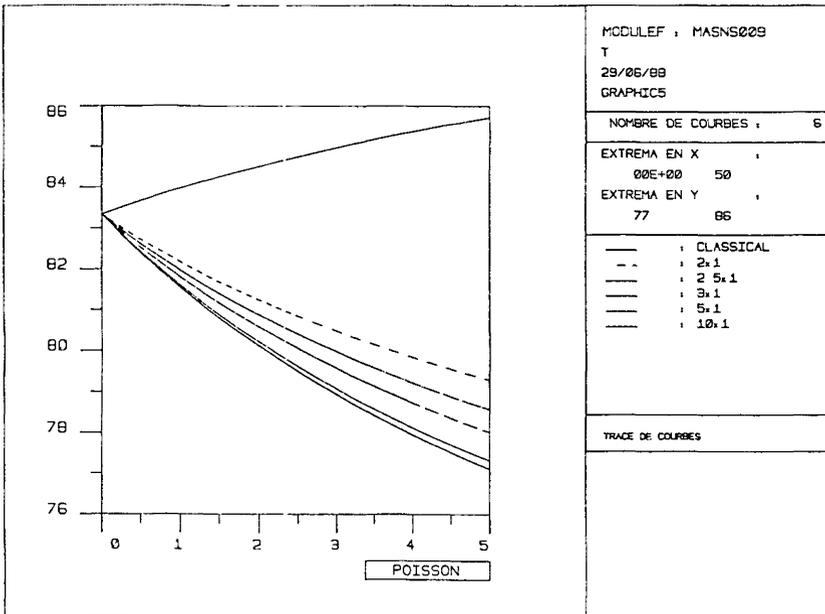


Figure 6.5. — Rectangular cross section. Constant  $k_1^0$ .

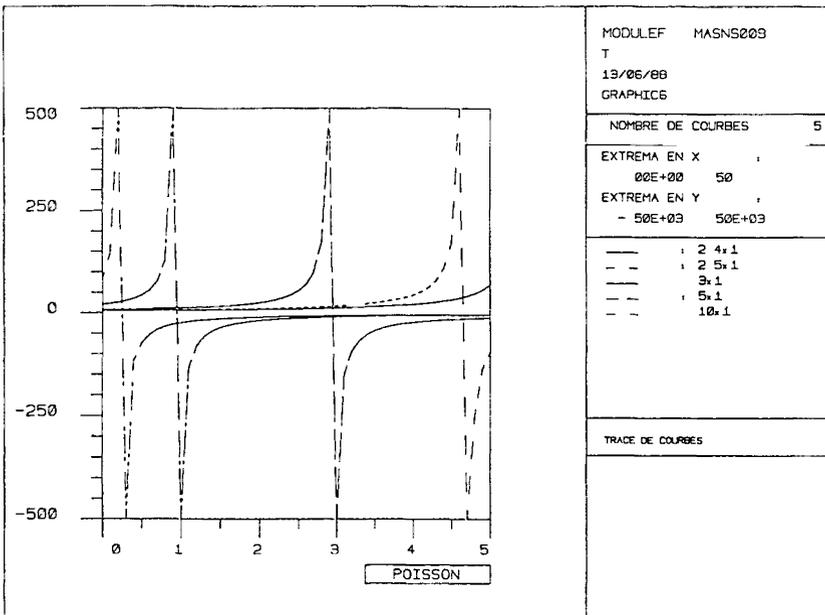


Figure 6.6. — Rectangular cross section. Constant  $k_2^1$ .

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