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EXISTENCE AND CONVERGENCE OF THE EXPANSION IN THE ASYMPTOTIC THEORY OF ELASTIC THIN PLATES (*)

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Abstract — *The asymptotic expansion method is applied to a periodic linear elastic thick plate problem with the thickness as the small parameter The purpose of this paper is to prove the existence and the convergence of the expansion If the applied forces are smooth and periodic functions, it is shown that 1°) all the terms of this expansion exist and 2°) the convergence holds if the product of the frequency of the applied forces (assumed be trigonometric polynomials) with the thickness of the plate is small enough*

Résumé — *On applique la méthode des développements aymptotiques a un problème de plaque élastique épaisse en utilisant l'épaisseur comme petit paramètre L'objet de ce travail est de démontrer l'existence et la convergence du développement en série ainsi obtenu Ceci est possible pour une plaque rectangulaire avec des conditions aux limites de type périodique sur le déplacement Si les forces appliquées sont régulieres et périodiques (par rapport aux variables « horizontales »), on montre que 1°) tous les termes du développement existent, 2°) la série converge si on suppose que les forces appliquées sont des polynômes trigonometriques tel que le produit de leur fréquence maximum par l'epaisseur de la plaque soit assez petit*

1. INTRODUCTION

Elastic bodies exist in the usual three dimensional Euclidian space. In this way, the general equations for the static equilibrium of an elastic body are partial differential equations with the variables in a *three dimensional open set* (namely : three dimensional model).

However, when the body is for example a *thin* plate (where the thickness is very small with respect to the other two dimensions), *two dimensional* models are preferred to the three dimensional models.

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In this paper we consider a three dimensional rectangular elastic plate with *periodic boundary conditions* on the lateral surface. We carry out an *asymptotic expansion* of the three dimensional solution engendering the *two dimensional* model of periodic thin plates. The aim of this paper is to study the existence and the convergence of this expansion.

The idea of using an asymptotic expansion for obtaining two dimensional models of elastic thin plates was developed about thirty years ago [12], [13]. In the first half of this century these two dimensional models were only performed according to *a priori* assumptions regarding the variation of the unknowns across the thickness of the plate (for example the *Love-Kirchhoff* assumption). Afterwards, the *asymptotic expansion method* (developed by Lions [8] for partial differential equations) was applied to plates by Ciarlet and Destuynder [3] to justify the usual linear model of thin plates. Then, Ciarlet [2] justified von Kármán equation which is the most popular non linear model of thin plates. More recently, Ciarlet and the author [5] have justified the Marguerre von Kármán equation for shallow shells.

The above method employs an asymptotic expansion of the three dimensional solution using the thickness of the plate as the small parameter. Then it can be shown that the two dimensional model sought may be identified with the leading term of this expansion. One other merit of this method, *in the linear case,* is that we can show how the three dimensional solution does indeed converge to the leading term of the expansion when the thickness vanishes. This proof was given by Destuynder [6], [7], Ciarlet and Kesavan [4] and Raoult [11].

In the *non linear case,* the problem is *the absence of an efficient theorem for the existence of the solution to the three dimensional problem* (although, in [10], there is a theorem which, unfortunately, is not used for this application). However, if the asymptotic expansion exists and converges for a fixed value of the thickness, the limit will be a three dimensional solution. But, *does the asymptotic expansion converge ?* We think this is the first question we must study to have any chance of proving the convergence of the three dimensional non linear solution to the leading term of the expansion when the thickness vanishes.

In the first place, this problem must be solved in the *simplest case which is the linear one.* In this paper, we can see how such a convergence holds for *linear periodic rectangular plates.* It is shown that every term of the expansion exists if the applied forces are smooth and periodic. Moreover the expansion converges if the product of the maximum frequency of the applied forces (assumed to be trigonometric polynomials) and the half-thickness of the plate is small enough.

Let us briefly note the contents of this paper.

In Section 2, we recall the basic equations of a thick periodic rectangular elastic plate with half-thickness $\varepsilon > 0$. These equations consist of linear

P.D.E. with boundary conditions on the three dimensional open set:

$$\Omega^{\varepsilon} :=]-1/2, 1/2[\times]-1/2, 1/2[\times]-\varepsilon, \varepsilon[.$$

The unknowns of the problem are the stress and the displacement which are vector fields. Using the classical Brezzi lemma, we see that this problem has an unique solution in a given Hilbert space if the applied forces are smooth enough.

In Section 3, we use the method given in [3] to study the dependence on ε of the solution. We transform the basic problem into a problem over a domain which is independent of ε. Hence, the parameter ε appears *explicitely* in the equations of this new problem.

In this way, the asymptotic expansion is formally written in Section 4. Each term of the expansion verifies given equations and the leading term may be identified with the solution of the *two dimensional periodic thin plates model.* The first result of this paper is that, under periodicity and smoothness (C^{∞}) assumptions, every term of this expansion exists and is also smooth (Theorem 1).

Next, the convergence of the expansion is studied in Section 5 where the second result of this paper is stated in Theorem 2. It is shown that the expansion converges in a convenient Sobolev space if the applied forces are trigonometric polynomials of frequency less than or equal to K (a positive integer) and if the half-thickness ε of the plate satisfies the inequality $\varepsilon K < Q$ (where $Q > 0$ is a constant).

In Section 6, it is concluded that, when the applied forces can be expanded in Fourier series, for a fixed value of the half-thickness ε, there is an order of frequency $K(\varepsilon)$ such that the expansion does converge if (and only if) we eliminate the term with frequency greater than $K(\varepsilon)$ in the Fourier expansion of the applied forces.

Let us first review some of the notations used in this paper:

- A is a set, ∂A its boundary, A^- its closure.
- $L^2(\Omega ; X)$ and $H^1(\Omega ; X)$ are spaces of functions with values in a finite dimensional space X, whose components lie respectively in $L^2(\Omega)$ and the usual Sobolev space $H^1(\Omega)$.
- $H^m(]-1, +1[; H^q(\omega))$ is a Sobolev space of functions $\phi :]-1, +1[\ni x_3 \to \phi(x_3) \in H^q(\omega)$ such that for $n = 1, \ldots, m$, the derivative $\phi^{(n)}(x_3)$ belongs to the space $H^q(\omega)$.
- $\mathbf{R}^n$ is the space of n dimensional real vectors.
- $\mathbf{S}^n$ is the space of $n \times n$ real symmetric matrices.
- $\mathbf{S}'^n$ is the space of $n \times n$ complex symmetric matrices.
- The partial derivative $\dfrac{\partial}{\partial x_i}$ is denoted ∂_i.
- As a rule, Greek indices $\alpha, \beta, \mu, \ldots,$ belong to the set $\{1, 2\}$, while Latin indices $i, j, k, \ldots,$ belong to the set $\{1, 2, 3\}$. The repeated index

convention is systematically used in conjunction with the above rule. For instance :

$$\int_\Omega \left\{ \frac{1+\nu}{E} \sigma_{ij} \tau_{ij} - \frac{\nu}{E} \sigma_{ii} \tau_{jj} \right\} dx \,, \quad \int_\Omega 2 \frac{1+\nu}{E} \delta_{\alpha 3} \tau_{\alpha 3} \, dx \,,$$

mean respectively,

$$\int_\Omega \left\{ \frac{1+\nu}{E} \sum_{i,j=1}^{3} \sigma_{ij} \tau_{ij} - \frac{\nu}{E} \left(\sum_{i=1}^{3} \sigma_{ii} \right) \left(\sum_{j=1}^{3} \tau_{jj} \right) \right\} dx \,,$$

$$\int_\Omega 2 \frac{1+\nu}{E} \sum_{\alpha=1}^{2} \sigma_{\alpha 3} \tau_{\alpha 3} \, dx \,.$$

• Finally, in Section 4, we will let :
for $k = (k_1, k_2) \in \mathbf{Z}^2$, $|k| := |k_1| + |k_2|$ and $k \,.\, x' := k_1 x_1 + k_2 x_2$.

2. THE THREE DIMENSIONAL PROBLEM

Let (e_i) denote the usual basis of the Euclidian space $\mathbf{R}^3$.

Given a parameter $\varepsilon > 0$ (half-thickness), let $\Omega^\varepsilon = \omega \times \left]-\varepsilon, \varepsilon\right[$, where ω is the unit square $\left]-1/2, 1/2\right[\times \left]-1/2, 1/2\right[$ of the « horizontal » plane spanned by the vectors (e_1, e_2). The closure $\Omega^{\varepsilon -}$ is called a *rectangular thick plate*.

At each point x of the boundary Γ^ε of the plate, we denote by $n^\varepsilon = n^\varepsilon(x)$ the unit outer normal vector.

Let

$$\begin{cases} \Gamma_1^\varepsilon &= \left\{ x \in \Gamma^\varepsilon \,;\, n^\varepsilon(x) \in \{-e_3, +e_3\} \right\} \,, \\ D_{\alpha \pm}^\varepsilon &= \left\{ x \in \Gamma^\varepsilon \,;\, n^\varepsilon(x) = \pm e_\alpha \right\} \,, \quad \alpha = 1, 2 \,, \end{cases}$$

be the faces of the plate.

For each $x \in \Omega^{\varepsilon -}$, the *unknowns* of the problem are

1°) $\sigma(x) = (\sigma_{ij}(x)) \in \mathbf{S}^3$,

2°) $u(x) = (u_i(x)) \in \mathbf{R}^3$,

the *stress* and the *displacement* at the point x, respectively.

The displacement vector field is said to be *admissible* if it verifies the following periodic boundary conditions for $\alpha = 1, 2$ and $i = 1, 2, 3$:

$$u_i \big|_{D_{\alpha+}^\varepsilon} = u_i \big|_{D_{\alpha-}^\varepsilon} \,.$$

Otherwise, the plate is subjected to applied forces of density :

$$f^\varepsilon = (f_i^\varepsilon) \in L^2(\Omega^\varepsilon \,;\, \mathbf{R}^3) \,,$$

$$g^\varepsilon = (g_i^\varepsilon) \in L^2(\Gamma_1^\varepsilon \,;\, \mathbf{R}^3) \,,$$

with the compatibility condition :

$$\int_{\Omega^\varepsilon} f_i^\varepsilon\, dx + \int_{\Gamma_1^\varepsilon} g_i^\varepsilon\, d\Gamma = 0\,, \quad i = 1, 2, 3\,.$$

Then, the classical problem of *linear elasticity* is the following *boundary value problem*

$$(1) \qquad \begin{cases} \dfrac{1+\nu}{E}\,\sigma_{ij} - \dfrac{\nu}{E}\,\sigma_{kk}\,\delta_{ij} = \dfrac{1}{2}\,(\partial_i u_j + \partial_j u_i)\,, \text{ in } \Omega^\varepsilon\,, \\ -\,\partial_j \sigma_{ij} = f_i^\varepsilon\,, \text{ in } \Omega^\varepsilon\,, \\ n_3^\varepsilon\,\sigma_{i3} = g_i^\varepsilon\,, \text{ on } \Gamma_1^\varepsilon\,, \\ \sigma_{\alpha i}\big|_{D^\varepsilon_{\alpha +}} = \sigma_{\alpha i}\big|_{D^\varepsilon_{\alpha -}}\ (*)\,, \\ u_i\big|_{D^\varepsilon_{\alpha +}} = u_i\big|_{D^\varepsilon_{\alpha -}}\,, \end{cases}$$

where the constants E and ν are, respectively, the Young Modulus and the Poisson coefficient of the homogeneous elastic material constituting the plate ($E > 0$, $0 < \nu < 1/2$) and (*) denotes that summation on α is not taken.

To obtain the variational formulation of equations (1), it is convenient to introduce the following displacement space :

$$\mathbf{V}^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon\,; \mathbf{R}^3)\,;\, v\big|_{D^\varepsilon_{\alpha +}} = v\big|_{D^\varepsilon_{\alpha}}\,, \alpha = 1, 2 \right\}$$

and the stress space

$$\Sigma^\varepsilon = L^2(\Omega^\varepsilon\,; \mathbf{S}^3)\,.$$

Then the boundary value problem (1) is formally equivalent to the variational problem

$$(2) \qquad \begin{cases} \text{Find } (\sigma, u) \in \Sigma^\varepsilon \times \mathbf{V}^\varepsilon \text{ such that} \\ \forall \tau \in \Sigma^\varepsilon, \ \displaystyle\int_{\Omega^\varepsilon} \left(\frac{1+\nu}{E}\,\sigma_{ij}\,\tau_{ij} - \frac{\nu}{E}\,\sigma_{ii}\,\tau_{jj} \right) dx = \\ \qquad\qquad = \dfrac{1}{2} \displaystyle\int_{\Omega^\varepsilon} \tau_{ij}(\partial_i u_j + \partial_j u_i)\, dx\,, \\ \forall v \in \mathbf{V}^\varepsilon, \ \displaystyle\int_{\Omega^\varepsilon} \sigma_{ij}\,\partial_i v_j\, dx = \int_{\Omega^\varepsilon} f_i^\varepsilon\, v_i\, dx + \int_{\Gamma_1^\varepsilon} g_i^\varepsilon\, v_i\, d\Gamma\,. \end{cases}$$

As a consequence of the Korn's inequality, and the Brezzi's lemma [1], the solution of this problem exists and is unique in the space $\Sigma^\varepsilon \times \mathbf{W}^\varepsilon$, where :

$$\mathbf{W}^\varepsilon = \left\{ v \in \mathbf{V}^\varepsilon\,;\, \int_{\Omega^\varepsilon} v_i\, dx = 0, i = 1, 2, 3 \right\}.$$

Of course, the solution depends on ε. This dependence will be noted in the next section.

3. TRANSFORMATION INTO A PROBLEM OVER A DOMAIN INDEPENDENT OF ε

Using the function F:

$$\omega^{-} \times [-1, +1] \ni x = (x_1, x_2, x_3) \rightarrow F(x) = (x_1, x_2, \varepsilon x_3) \in \Omega^{\varepsilon -} ,$$

we transform the problem (2) into a problem on the fixed set :

$$\Omega = \omega \times \left]-1, +1\right[.$$

Let

$$\left\{ \begin{array}{l} \Gamma = \partial\Omega , \\ \Gamma_1 = \{x \in \Gamma ; n(x) \in \{-e_3, +e_3\}\} , \\ D_{\alpha \pm} = \{x \in \Gamma ; n(x) = \pm e_\alpha\} , \quad \alpha = 1, 2 , \end{array} \right.$$

be the faces of Ω, where $n(x)$ is the unit outer normal vector to the boundary Γ at the point x.

Following Ciarlet and Destuynder [3], we assume that :

$$\begin{array}{l} f_\alpha^\varepsilon \circ F = f_\alpha , \\ f_3^\varepsilon \circ F = \varepsilon f_3 , \\ g_\alpha^\varepsilon \circ F - \varepsilon g_\alpha , \\ g_3^\varepsilon \circ F = \varepsilon^2 g_3 , \end{array}$$

where f_i and g_i are independent on ε.

Similary, using [3], we also change the scale of the unknowns by letting

$$\left\{ \begin{array}{l} \sigma_{33} \circ F = \varepsilon^2 \sigma_{33}^\varepsilon , \\ \sigma_{\alpha 3} \circ F = \varepsilon \sigma_{\alpha 3}^\varepsilon , \\ \sigma_{\alpha\beta} \circ F = \sigma_{\alpha\beta}^\varepsilon , \\ u_\alpha \circ F = u_\alpha^\varepsilon , \\ u_3 \circ F = \dfrac{1}{\varepsilon} u_3^\varepsilon , \end{array} \right.$$

where the solution $((\sigma_{ij}), (u_i))$ of problem (2) depends on ε. The dependence on ε of the new unknown $((\sigma_{ij}^\varepsilon), (u_i^\varepsilon))$ is denoted by the exponent ε.

Now, we introduce the displacement and the stress spaces :

$$\mathbf{V} = \left\{ v \in H^1(\Omega ; \mathbf{R}^3) ; v|_{D_{\alpha+}} = v|_{D_{\alpha-}}, \alpha = 1, 2 \right\},$$

$$\Sigma = L^2(\Omega ; \mathbf{S}^3) .$$

Then the problem (2) is equivalent to the following one

$$(3) \quad \begin{cases} \text{Find } (\sigma^\varepsilon, u^\varepsilon) \in \Sigma \times \mathbf{V} \text{ such that :} \\ \forall \tau \in \Sigma , \ \mathscr{A}_0(\sigma^\varepsilon, \tau) + \varepsilon^2 \mathscr{A}_2(\sigma^\varepsilon, \tau) + \varepsilon^4 \mathscr{A}_4(\sigma^\varepsilon, \tau) + \mathscr{B}(\tau, u^\varepsilon) = 0 , \\ \forall v \in \mathbf{V} , \quad \mathscr{B}(\sigma^\varepsilon, v) = \mathscr{F}(v) , \end{cases}$$

where the forms $\mathscr{A}_0$, $\mathscr{A}_2$, $\mathscr{A}_4$, $\mathscr{B}$ and $\mathscr{F}$ are defined as follows :

$$\mathscr{A}_0(\sigma, \tau) = \int_\Omega \left(\frac{1+\nu}{E} \sigma_{\alpha\beta} \tau_{\alpha\beta} - \frac{\nu}{E} \sigma_{\alpha\alpha} \tau_{\beta\beta} \right) dx ,$$

$$\mathscr{A}_2(\sigma, \tau) = \int_\Omega \left\{ 2 \frac{1+\nu}{E} \sigma_{\alpha 3} \tau_{\alpha 3} - \frac{\nu}{E} (\sigma_{\mu\mu} \tau_{33} + \sigma_{33} \tau_{\mu\mu}) \right\} dx ,$$

$$\mathscr{A}_4(\sigma, \tau) = \int_\Omega \frac{1}{E} \sigma_{33} \tau_{33} \, dx ,$$

$$\mathscr{B}(\tau, u) = \frac{1}{2} \int_\Omega \tau_{ij} (\partial_i u_j + \partial_j u_i) \, dx ,$$

$$\mathscr{F}(v) = - \int_\Omega f_i \, v_i \, dx - \int_{\Gamma_1} g_i \, v_i \, d\Gamma .$$

4. EXISTENCE OF THE ASYMPTOTIC EXPANSION

We formally write :

$$(4) \quad \begin{cases} \sigma^\varepsilon = \sigma^0 + \varepsilon^2 \sigma^1 + \varepsilon^4 \sigma^2 + \cdots + \varepsilon^{2p} \sigma^p + \cdots \\ u^\varepsilon = u^0 + \varepsilon^2 u^1 + \varepsilon^4 u^2 + \cdots + \varepsilon^{2p} u^p + \cdots . \end{cases}$$

Then, by equalizing the powers of ε in the problem (3), it is easily seen that the sequence $\{(\sigma^p, u^p)\}_{p \in \mathbf{N}}$ of the expansion must satisfy the equations :

$$(5) \quad \begin{cases} \mathscr{A}_0(\sigma^0, \tau) + \mathscr{B}(\tau, u^0) = 0 , \quad \forall \tau \in \Sigma , \\ \mathscr{B}(\sigma^0, v) = \mathscr{F}(v) , \quad \forall v \in \mathbf{V} , \end{cases}$$

$$(6) \quad \begin{cases} \mathscr{A}_0(\sigma^p, \tau) + \mathscr{B}(\tau, u^p) = - \mathscr{A}_2(\sigma^{p-1}, \tau) - \mathscr{A}_4(\sigma^{p-2}, \tau), \ \forall \tau \in \Sigma , \\ \mathscr{B}(\sigma^p, v) = 0 , \quad \forall v \in \mathbf{V} , \\ \text{for } p = 1, 2, 3, \ldots \text{ with } \sigma^{-1} = 0 . \end{cases}$$

Let $d_{\alpha\pm} = \{x' \in \gamma \,;\, n(x') = \pm e_\alpha\}$ be the sides of ω, where $n = (n_\alpha)$ is the normal unit vector to the boundary γ of the square ω.

In the following, h^+ (resp. h^-) denotes the restriction of a function $h : \Gamma_1 \to \mathbf{R}$ to the upper (resp. the lower) face of the set Ω.

According to the results of Ciarlet and Destuynder [3], we will see in Theorem 1 that there exists an unique leading term (σ^0, u^0) satisfying :

$$\int_\Omega u_i^0 \, dx = 0 \,, \quad i = 1, 2, 3 \,,$$

and that the displacement components u_i^0 verify :

$$\begin{cases} u_\alpha^0 = \zeta_\alpha - x_3 \, \partial_\alpha \zeta_3 \,, & \alpha = 1, 2 \,, \\ u_3^0 = \zeta_3 \,, \end{cases}$$

where $\zeta = (\zeta_i)$ only depends on $x' = (x_1, x_2) \in \omega$ and verifies the two dimensional equations of *periodic thin plates* (no summation on α in equations (*)) :

$$\text{(7)} \qquad \begin{cases} -\dfrac{E}{2(1+\nu)} \Delta \zeta_\alpha - \dfrac{E}{2(1-\nu)} \partial_{\alpha\beta} \zeta_\beta = F_\alpha \,, \text{ in } \omega \,, \\ \zeta_\beta |_{d_{\alpha+}} = \zeta_\beta |_{d_{\alpha-}} \,, \\ r_{\alpha\beta} |_{d_{\alpha+}} = r_{\alpha\beta} |_{d_{\alpha-}} \ (*) \,, \end{cases}$$

$$\text{(8)} \qquad \begin{cases} \dfrac{2E}{3(1-\nu^2)} \Delta^2 \zeta_3 = F_3 \,, \text{ in } \omega \,, \\ \zeta_3 |_{d_{\alpha+}} = \zeta_3 |_{d_{\alpha-}} \,, \\ \partial_\beta \zeta_3 |_{d_{\alpha+}} = \partial_\beta \zeta_3 |_{d_{\alpha-}} \,, \\ s_{\alpha\beta} |_{d_{\alpha+}} = s_{\alpha\beta} |_{d_{\alpha-}} \ (*) \,, \\ h_\alpha |_{d_{\alpha+}} = h_\alpha |_{d_{\alpha-}} \ (*) \,, \end{cases}$$

where :

$$\begin{aligned} r_{\alpha\beta} &= \frac{1-\nu}{2} (\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha) + \nu (\partial_\mu \zeta_\mu) \, \delta_{\alpha\beta} \,, \\ s_{\alpha\beta} &= (1-\nu) \, \partial_{\alpha\beta} \zeta_3 + \nu \, \Delta \zeta_3 \, \delta_{\alpha\beta} \,, \\ F_\alpha &= g_\alpha^+ + g_\alpha^- + \int_{-1}^{+1} f_\alpha \, dx_3 \,, \\ F_3 &= g_3^+ + g_3^- + \partial_\alpha (g_\alpha^+ - g_\alpha^-) + \int_{-1}^{+1} (f_3 + x_3 \, \partial_\alpha f_\alpha) \, dx_3 \,, \\ h_\alpha &= \frac{2E}{3(1-\nu^2)} \partial_\alpha \Delta \zeta_3 - g_\alpha^+ + g_\alpha^- - \int_{-1}^{+1} (x_3 f_\alpha) \, dx_3 \,. \end{aligned}$$

Since the boundary conditions are periodic it will be shown that the boundary layer phenomenon can be deleted when the applied forces verify the following assumptions.

A « periodic function » φ means a function φ which is 1-periodic with respect to the variables x_1 and x_2, that is to say,

$$\forall k = (k_1, k_2) \in \mathbf{Z}^2 , \quad \forall x = (x_1, x_2, x_3) \in \mathbf{R}^2 \times E ,$$

$$\varphi(x_1, x_2, x_3) = \varphi(x_1 + k_1, x_2 + k_2, x_3) ,$$

where $E = [-1, +1]$ or $\{-1, +1\}$.

THEOREM 1 : *Assume that the function f is the restriction to Ω^- of a* « periodic function » $F \in C^{\infty}(\mathbf{R}^2 \times [-1, +1] ; \mathbf{R}^3)$ *and assume that the function g is the restriction to $\Gamma_1^- = \omega^- \times \{-1, +1\}$ of a* « periodic function » *G such that $G(.,., \pm 1)$ belong to the space $C^{\infty}(\mathbf{R}^2 ; \mathbf{R}^3)$.*

Then the unique solution $(\sigma^{\varepsilon}, u^{\varepsilon})$ of the problem (3) *satisfying*

$$\int_{\Omega} u_i^{\varepsilon} \, dx = 0 , \quad i = 1, 2, 3 ,$$

is the restriction to Ω^- of a « periodic function » $X^{\varepsilon} \in C^{\infty}(\mathbf{R}^2 \times [-1, +1] ; \mathbf{S}^3 \times \mathbf{R}^3)$.

Furthermore, for every $p = 0, 1, 2, \ldots$ the term (σ^p, u^p) of the expansion (4) *satisfying*

$$\int_{\Omega} u_i^p \, dx = 0 , \quad i = 1, 2, 3 ,$$

exists, is unique and is the restriction to Ω^- of a « periodic function » $X^p \in C^{\infty}(\mathbf{R}^2 \times [-1, +1] ; \mathbf{S}^3 \times \mathbf{R}^3)$.

Proof : For the first part of the theorem, the proof of the regularity of the solution $(\sigma^{\varepsilon}, u^{\varepsilon})$ is given in [10, Theorem 4.1].

Next, equations (5) and (6) may be written as follows :

$$(9) \qquad \begin{cases} \mathscr{A}_0(\sigma, \tau) + \mathscr{B}(\tau, u) = \displaystyle\int_{\Omega} \theta_{ij} \, \tau_{ij} \, dx , & \forall \tau \in \Sigma , \\ \mathscr{B}(\sigma, v) = \displaystyle\int_{\Omega} p_i \, v_i \, dx + \int_{\Gamma_1} q_i \, v_i \, d\Gamma , & \forall v \in \mathbf{V} , \end{cases}$$

where $\theta = (\theta_{ij})$ and $p = (p_i)$ are functions on Ω with value in $\mathbf{S}^3$ and $\mathbf{R}^3$, respectively, and $q = (q_i)$ is a function on Γ_1 into $\mathbf{R}^3$.

Assume that p is the restriction to Ω^- of a « periodic function » $P \in C^{\infty}(\mathbf{R}^2 \times [-1, +1] ; \mathbf{R}^3)$, and that q is the restriction to $\Gamma_1^- =$

$\omega^- \times \{-1, +1\}$ of a « periodic function » Q such that $Q(.,.,\pm 1)$ are in the space $C^\infty(\mathbf{R}^2 ; \mathbf{R}^3)$ and assume that θ is the restriction to Ω^- of a « periodic function » $T \in C^\infty(\mathbf{R}^2 \times [-1, +1] ; \mathbf{S}^3)$.

In the first step we show that the solution $(\sigma, u) \in \Sigma \times \mathbf{V}$ of the problem (9) satisfying :

$$\int_\Omega u_i \, dx = 0 \,, \quad i = 1, 2, 3 \,, \tag{10}$$

i) is unique, ii) exists and is the restriction to Ω^- of a « periodic function » $X \in C^\infty(\mathbf{R}^2 \times [-1, +1] ; \mathbf{S}^3 \times \mathbf{R}^3)$.

In the second step we will use the result of the first step to prove the second part of the theorem.

First step :

i) *Assume that* $\theta = 0$, $p = 0$ *and* $q = 0$.

If there exists a corresponding solution $(\sigma, u) \in \Sigma \times \mathbf{V}$ of problem (7) satisfying (8), then, letting $v = u$ and $\sigma = \tau$ in (7), we have $\mathscr{A}_0(\sigma, \sigma) = 0$. Therefore we have $\sigma_{\alpha\beta} = 0$ and then

$$\begin{cases} \mathscr{B}(\tau, u) = 0 \,, & \forall \tau \in \Sigma \,, \\ \mathscr{B}(\sigma, v) = 0 \,, & \forall v \in \mathbf{V} \,. \end{cases}$$

Using the Korn's inequality (with $\tau_{ij} = \partial_i u_j + \partial_j u_i$) and (10), the first variational equality gives $u = 0$. The second one merely gives the partial differential equations $(\sigma_{\alpha 3} = \sigma_{3\alpha})$:

$$\partial_3 \sigma_{3\alpha} = 0 \,,$$

$$\partial_\beta \sigma_{\beta 3} + \partial_3 \sigma_{33} = 0 \,,$$

with the boundary conditions $(\alpha, \beta = 1, 2)$:

$$\sigma_{\beta 3}|_{D_{\alpha +}} = \sigma_{\beta 3}|_{D_{\alpha -}} \,,$$

$$\sigma_{\alpha 3} = 0 \,, \text{ on } \Gamma_1 \,, \quad \sigma_{33} = 0 \,, \text{ on } \Gamma_1 \,.$$

Obviously, the solution $\sigma_{\alpha 3} = \sigma_{33} = 0$ of the above system is unique.

ii) We will prove the existence of the vector-valued function $X = (U, S)$ with $U \in C^\infty(\mathbf{R}^2 \times [-1, +1] ; \mathbf{R}^3)$ and $S \in C^\infty(\mathbf{R}^2 \times [-1, +1] ; \mathbf{S}^3)$.

Firstly, it is convenient to introduce the space $\mathbf{V}_{KL}$ (where the index KL means « Kirchhoff-Love ») defined by :

$$\mathbf{V}_{KL} = \{ v \in \mathbf{V} \,;\, \partial_3 v_\alpha + \partial_\alpha v_3 = \partial_3 v_3 = 0, \text{ a.e. in } \Omega \} \,.$$

Obviously, this space is a closed subspace of the Hilbert space $\mathbf{V}$. Also, it is easily checked that it is *isomorphic* to the product space $\mathbf{V}_2 \times \mathbf{V}_3$ via the isomorphism $\mathbf{KL}$:

$$\begin{aligned}\mathbf{V}_2 \times \mathbf{V}_3 \ni \eta &= (\eta_1, \eta_2, \eta_3) \mapsto \mathbf{KL}(\eta) \\ &= (\eta_1 - x_3\, \partial_1 \eta_3, \eta_2 - x_3\, \partial_2 \eta_3, \eta_3) \in \mathbf{V}_{KL}\,,\end{aligned}$$

where the spaces $\mathbf{V}_2$ and $\mathbf{V}_3$ are defined by :

$$\mathbf{V}_2 = \left\{ (\eta_1, \eta_2) \in H^1(\omega, \mathbf{R}^2)\,;\ \eta_\beta \,|_{d_{\alpha+}} = \eta_\beta \,|_{d_{\alpha-}}\,,\ \alpha, \beta = 1, 2 \right\}\,,$$

$$\mathbf{V}_3 = \left\{ \begin{array}{l} \eta_3 \in H^2(\omega)\,;\ \eta_3 \,|_{d_{\alpha+}} = \eta_3 \,|_{d_{\alpha-}}\,, \\ \partial_\beta \eta_3 \,|_{d_{\alpha+}} = \partial_\beta \eta_3 \,|_{d_{\alpha-}}\,,\ \alpha, \beta = 1, 2 \end{array} \right\}.$$

Now, we consider the system (9). Its first equation may be written as follows :

$$\left\{ \begin{aligned} \frac{1+\nu}{E}\, \sigma_{\alpha\beta} - \frac{\nu}{E}\, \sigma_{\mu\mu}\, \delta_{\alpha\beta} - \frac{1}{2}\, (\partial_\alpha u_\beta + \partial_\beta u_\alpha) &= \theta_{\alpha\beta}\,, \\ - \frac{1}{2}\, (\partial_\alpha u_3 + \partial_3 u_\alpha) &= \theta_{\alpha 3}\,, \\ - \partial_3 u_3 &= \theta_{33}\,, \end{aligned} \right.$$

and, consequently, if we let $u = u^1 + w$ with :

$$\text{(11)} \quad \left\{ \begin{aligned} u_\alpha^1(x) &= \int_{-1}^{x_3} \left(\int_{-1}^{t} \partial_\alpha \theta_{33}(x_1, x_2, s)\, ds - 2\, \theta_{\alpha 3}(x_1, x_2, t) \right) dt + C_\alpha\,, \\ u_3^1(x) &= - \int_{-1}^{x_3} \theta_{33}(x_1, x_2, t)\, dt + C_3\,, \end{aligned} \right.$$

the vector field w is in the space $\mathbf{V}_{KL}$ since $\partial_3 u_3^1 = -\,\theta_{33}$ and $\partial_\alpha u_3^1 + \partial_3 u_\alpha^1 = -\,2\,\theta_{\alpha 3}$. In the above formulas, the constants C_i are computed such that the integrales on Ω of functions $u_i^1(x)$ vanish. Then, a *restrictive* form of the problem (9) may be written as follows :

$$\text{(12)} \quad \left\{ \begin{array}{l} \text{Find } w \in \mathbf{V}_{KL} \text{ and } (\sigma_{\alpha\beta}) \in L^2(\Omega\,; \mathbf{S}^2) \text{ such that, } \forall \tau \in L^2(\Omega\,; \mathbf{S}^2) : \\ \displaystyle\int_\Omega \left(\frac{1+\nu}{E}\, \sigma_{\alpha\beta}\, \tau_{\alpha\beta} - \frac{\nu}{E}\, \sigma_{\alpha\alpha}\, \tau_{\beta\beta} \right) dx - \frac{1}{2} \int_\Omega (\partial_\alpha w_\beta + \partial_\beta w_\alpha)\, \tau_{\alpha\beta}\, dx = \\ \displaystyle\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_\Omega k_{\alpha\beta}\, \tau_{\alpha\beta}\, dx\,, \\ \displaystyle\int_\Omega \sigma_{\alpha\beta} (\partial_\alpha v_\beta + \partial_\beta v_\alpha)\, dx = \int_\Omega p_i\, v_i\, dx + \int_{\Gamma_1} q_i\, v_i\, d\Gamma,\ \forall v \in \mathbf{V}_{KL}\,, \end{array} \right.$$

where the tensor $k = (k_{\alpha\beta}) \in L^2(\Omega ; \mathbf{S}^2)$ is given by the formula :

$$k_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}(\partial_\alpha u^1_\beta + \partial_\beta u^1_\alpha) . \tag{13}$$

Since the space $\mathbf{V}_{KL}$ is a closed subspace of $\mathbf{V}$, we can use the Brezzi lemma [1] and the above problem has a solution which is unique if we prescribe the condition :

$$\int_\Omega w_i \, dx = 0 , \quad i = 1, 2, 3 , \tag{14}$$

so that the condition (10) is fulfilled. On the other hand, since $w \in \mathbf{V}_{KL}$, there exists $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbf{V}_2 \times \mathbf{V}_3$ such that $w = \mathbf{KL}(\zeta)$ and the condition (14) implies the following conditions on ζ_i :

$$\int_\omega \zeta_i \, dx' = 0 , \quad i = 1, 2, 3 . \tag{15}$$

We notice that

1°) the function θ is the restriction to Ω^- of a « periodic function » $T \in C^\infty(\mathbf{R}^2 \times [-1, +1] ; \mathbf{S}^3)$, therefore, with (11) and (13), it is easily seen that k is the restriction to Ω^- of a « periodic function » $K \in C^\infty(\mathbf{R}^2 \times [-1, +1] ; \mathbf{S}^2)$;

2°) the function p is the restriction to Ω^- of a « periodic function » $P \in C^\infty(\mathbf{R}^2 \times [-1, +1] ; \mathbf{R}^3)$, and the function q is the restriction to $\Gamma_1^- = \omega^- \times \{-1, +1\}$ of a « periodic function » Q such that $Q(., ., \pm 1)$ are in the space $C^\infty(\mathbf{R}^2 ; \mathbf{R}^3)$.

Using these two remarks and by eliminating the tensor $(\sigma_{\alpha\beta})$ in the system (12), a simple computation shows that the functions ζ_i are solutions of the equations

$$\begin{cases} (\zeta_1, \zeta_2) \in \mathbf{V}_2 \quad \text{and} \quad \forall (\eta_\alpha) \in \mathbf{V}_2 : \\ \displaystyle\int_\omega \left\{ \frac{1-\nu}{2} (\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha) \, \partial_\beta \eta_\alpha + \nu (\partial_\alpha \zeta_\alpha)(\partial_\beta \eta_\beta) \right\} dx' = \int_\omega H_\alpha \, \eta_\alpha \, dx', \end{cases} \tag{16}$$

$$\begin{cases} \zeta_3 \in \mathbf{V}_3 , \\ \displaystyle\int_\omega \{ (1-\nu) \, \partial_{\alpha\beta} \zeta_3 \, \partial_{\alpha\beta} \eta_3 + \nu \, \Delta \zeta_3 \, \Delta \eta_3 \} \, dx' = \int_\omega H_3 \, \eta_3 \, dx', \ \forall \eta_3 \in \mathbf{V}_3 , \end{cases} \tag{17}$$

where H_i are C^∞ and 1-periodic functions on $\mathbf{R}^2$ given by the following formulas :

$$H_\alpha(x') = \int_{-1}^{+1} \{(1-\nu)\,\partial_\beta K_{\alpha\beta}(x', x_3) + \nu\,\partial_\alpha K_{\beta\beta}(x', x_3)\}\,dx_3$$
$$+ \frac{2(1-\nu^2)}{E}\left\{\int_{-1}^{+1} P_\alpha(x', x_3)\,dx_3 + Q_\alpha^+(x') + Q_\alpha^-(x')\right\},$$

$$H_3(x') = \int_{-1}^{+1} \{(1-\nu)\,\partial_{\alpha\beta} K_{\alpha\beta}(x', x_3) + \nu\,\Delta K_{\beta\beta}(x', x_3)\}\,x_3\,dx_3$$
$$+ \frac{2(1-\nu^2)}{3E}\left\{\int_{-1}^{+1} (P_3(x', x_3) + x_3\,\partial_\alpha P_\alpha(x', x_3))\,dx_3\right.$$
$$\left. + (Q_3^+ + Q_3^-)(x') + \partial_\alpha(Q_\alpha^+ - Q_\alpha^-)(x')\right\}.$$

We notice that, if $\theta = 0$, $p = f$ and $q = g$, the equations (16) and (17) may be explained by the equations (7) and (8), which are called the *two dimensional equations of periodic thin plates.*

Following the method introduced in [10], the functions ζ_i, $i = 1, 2, 3$ (a.e. defined on the square $\omega =]-1/2, 1/2[\times]-1/2, 1/2[$) may be extended to the square $\omega^+ =]-3/2, 3/2[\times]-3/2, 3/2[$ using a convenient translation operator. Let ζ_i^+ be the extension of the function ζ_i $(i = 1, 2, 3)$. Then we may use the classical regularity results on the bidimensional elasticity operator and on the operator Δ^2 *inside* the square ω^+. Since the functions H_i are C^∞ on ω^+, we obtain that the functions ζ_1^+, ζ_2^+ and ζ_3^+ are C^∞ on the open set ω^+ and, therefore, the functions ζ_1, ζ_2, ζ_3 are the restriction to ω^- of 1-periodic functions on $\mathbf{R}^2$.

On the other hand, since $u = w + u^1$ (where $w = \mathbf{KL}(\zeta)$) and :

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\left\{\frac{1-\nu}{2}(\partial_\alpha w_\beta + \partial_\beta w_\alpha + 2\,k_{\alpha\beta}) + \nu(\partial_\mu w_\mu + k_{\mu\mu})\,\delta_{\alpha\beta}\right\},$$

we can conclude that there exists « periodic functions » $(U_i) \in C^\infty(\mathbf{R}^2 \times [-1, +1]; \mathbf{R}^3)$ and $(S_{\alpha\beta}) \in C^\infty(\mathbf{R}^2 \times [-1, +1]; \mathbf{S}^2)$ such that the functions u_i and $\sigma_{\alpha\beta}$ are the restrictions to ω^- of U_i and $S_{\alpha\beta}$, respectively.

To complete this step, we must compute the functions $\sigma_{\alpha 3}$ and σ_{33}. Taking the second equation in (9) into consideration, they are solutions of the equations :

$$\partial_3\sigma_{\alpha 3} = -\,\partial_\beta\sigma_{\alpha\beta} - p_\alpha\,, \quad \text{in } \omega\,,$$
$$n_3\,\sigma_{\alpha 3} = q_\alpha\,, \quad \text{on } \Gamma_1\,,$$
$$\sigma_{\beta 3}|_{D_{\alpha+}} = \sigma_{\beta 3}|_{D_{\alpha-}}\,,$$
$$\partial_3\sigma_{33} = -\,\partial_\alpha\sigma_{\alpha 3} - p_3\,, \text{ in } \omega\,,$$
$$n_3\,\sigma_{33} = q_3\,, \text{ on } \Gamma_1\,.$$

Clearly, we begin by computing $\sigma_{\alpha 3}$ and then σ_{33} with $\partial_\alpha\sigma_{\alpha 3}$ as data. Because every function $x_3 \mapsto \sigma_{i3}(.,., x_3)$ must be a solution of a *first order* differential equation with *two* boundary conditions ($x_3 \in \{-1, +1\}$) we must verify the compatibility conditions :

$$q_\alpha^+ + q_\alpha^- + \int_{-1}^{+1} (\partial_\beta \sigma_{\alpha\beta} + p_\alpha)\, dx_3 = 0$$

and

$$q_3^+ + q_3^- + \int_{-1}^{+1} (\partial_\alpha \sigma_{\alpha 3} + p_3)\, dx_3 = 0\,.$$

The first one is obtained from the second variational equation in (12) with $v_\alpha = \eta_\alpha$, $(\eta_\alpha) \in \mathbf{V}_2$ and $v_3 = 0$. The second one also follows from the second variational equation in (12) with $v_\alpha = -x_3\, \partial_\alpha \eta_3$, $\eta_3 \in \mathbf{V}_3$. (Indeed, the expression of the solution $\sigma_{\alpha 3}$ in (18) may be used to obtain the second condition with $\partial_{\alpha\beta}\sigma_{\alpha\beta}$ instead of $\partial_\alpha \sigma_{\alpha 3}$.)

Finally, since $\sigma_{\alpha\beta}$ are the restrictions to Ω^- of the « periodic function » $S_{\alpha\beta}$, the unique functions $\sigma_{\alpha 3}$ and σ_{33} are the restrictions to Ω^- of the functions

$$(18) \qquad \begin{cases} S_{\alpha 3}(x', x_3) = -\displaystyle\int_{-1}^{x_3} (\partial_\beta S_{\alpha\beta} + P_\alpha)(x', t)\, dt - Q_\alpha^-(x')\,, \\ S_{33}(x', x_3) = -\displaystyle\int_{-1}^{x_3} (\partial_\alpha S_{\alpha 3} + P_3)(x', t)\, dt - Q_3^-(x')\,, \end{cases}$$

respectively. Obviously these functions are C^∞ and « periodic » and hence, the function σ is the restriction to Ω^- of the « periodic function » $S = (S_{ij}) \in C^\infty(\mathbf{R}^2 \times [-1, +1]\,; \mathbf{S}^3)$.

Second step :

We use the assumption : *f is the restriction to Ω^- of a « periodic function » $F \in C^\infty(\mathbf{R}^2 \times [-1, +1]\,; \mathbf{R}^3)$ and g is the restriction to $\Gamma_1^- = \omega^- \times \{-1, +1\}$ of a « periodic function » G such that $G(.,.,\pm 1)$ are in the space $C^\infty(\mathbf{R}^2\,; \mathbf{R}^3)$.* With the first step the term $(\sigma^0, u^0) \in \Sigma \times \mathbf{V}$ of the expansion (4) satisfying

$$\int_\Omega u_i^0\, dx = 0\,, \quad i = 1, 2, 3\,,$$

exists, is unique and is the restriction to Ω^- of a « periodic function » $X^0 \in C^\infty(\mathbf{R}^2 \times [-1, +1]\,; \mathbf{S}^3 \times \mathbf{R}^3)$. Then the function $\theta = (\theta_{ij}) \in \Sigma$ satisfying

$$\int_\Omega \theta_{ij}\, \tau_{ij}\, dx = -\mathscr{A}_2(\sigma^0, \tau)\,, \quad \forall \tau \in \Sigma\,,$$

is the restriction to Ω^- of a « periodic function » $T \in C^\infty(\mathbf{R}^2 \times [-1, +1]; \mathbf{S}^3)$. According to the first step, the term $(\sigma^1, u^1) \in \Sigma \times \mathbf{V}$ of the expansion (4) satisfying

$$\int_\Omega u_i^1 \, dx = 0 , \quad i = 1, 2, 3 ,$$

exists, is unique and is the restriction to Ω^- of a « periodic function » $X^1 \in C^\infty(\mathbf{R}^2 \times [-1, +1]; \mathbf{S}^3 \times \mathbf{R}^3)$.

Next, let $p \geq 1$ be an integer. Assume that the terms (σ^{p-1}, u^{p-1}) and (σ^p, u^p) of the expansion (4) satisfying

$$\int_\Omega u_i^{p-1} \, dx = \int_\Omega u_i^p \, dx = 0 , \quad i = 1, 2, 3 ,$$

exist, are unique and are the restrictions to Ω^- of « periodic functions » X^{p-1} and X^p belonging to the space $C^\infty(\mathbf{R}^2 \times [-1, +1]; \mathbf{S}^3 \times \mathbf{R}^3)$ (that is true if $p = 1$). Then the function $\theta = (\theta_{ij})$ satisfying

$$\int_\Omega \theta_{ij} \, \tau_{ij} \, dx = -\mathscr{A}_2(\sigma^p, \tau) - \mathscr{A}_4(\sigma^{p-1}, \tau) , \quad \forall \tau \in \Sigma ,$$

is the restriction to Ω^- of a « periodic function » $T \in C^\infty(\mathbf{R}^2 \times [-1, +1]; \mathbf{S}^3)$. With the first step the term $(\sigma^{p+1}, u^{p+1}) \in \Sigma \times \mathbf{V}$ of the expansion (4) satisfying

$$\int_\Omega u_i^{p+1} \, dx = 0 , \quad i = 1, 2, 3 ,$$

exists, is unique and is the restriction to Ω^- of a « periodic function » $X^{p+1} \in C^\infty(\mathbf{R}^2 \times [-1, +1]; \mathbf{S}^3 \times \mathbf{R}^3)$. ■

4. CONVERGENCE OF THE EXPANSION

THEOREM 2 : *Assume that the applied forces f and g are trigonometric polynomials in* $\cos(2\pi k \cdot x')$ *and* $\sin(2\pi k \cdot x')$, $k = (k_1, k_2) \in \mathbb{N}^2$, $|k| \leq K$, *and the coefficients of f are in* $C^\infty([-1, +1])$ *and let m and q be non-negative integers.*

Then there exists a constant $Q > 0$ *such that the expansion* (4) *converges to* $(\sigma^\varepsilon, u^\varepsilon)$ *in the space* $H^m(]-1, +1[; H^q(\omega))$ *if the thickness* ε *is small enough, that is to say* $\varepsilon K < Q$. *Furthermore, the convergence of the expansion is normal.*

Proof : We introduce the sequence $\{(\tau^n, v^n)\}_{n \in \mathbb{N}}$:

$$(19) \qquad \begin{cases} \tau^n = \sigma^\varepsilon - \sum_{p=0}^{n} \varepsilon^{2p} \sigma^p , \\ v^n = u^\varepsilon - \sum_{p=0}^{n} \varepsilon^{2p} u^p . \end{cases}$$

Using the definition (3) of the solution $(\sigma^\varepsilon, u^\varepsilon)$ and the definitions (5)-(6) of the terms (σ^p, u^p) the sequence $\{(\tau^n, v^n)\}_{n \in \mathbb{N}}$ must verify the following variational equalities :

$$(20) \quad \begin{cases} \mathcal{A}_0(\tau^n, \sigma) + \mathcal{B}(\sigma, v^n) = -\varepsilon^2 \mathcal{A}_2(\tau^{n-1}, \sigma) - \varepsilon^4 \mathcal{A}_4(\tau^{n-2}, \sigma) , \\ \forall \sigma \in \Sigma , \\ \mathcal{B}(\tau^n, u) = 0 , \quad \forall u \in \mathbf{V} , \\ \text{for } n = 0, 1, 2, \ldots \text{ with } \tau^{-1} = \tau^{-2} = \sigma^\varepsilon . \end{cases}$$

Following the method introduced in the proof of Theorem 1 (see (11), (12) and (18), with $\tau^n_{i3} = S_{i3}$, $Q = P = 0$), a simple computation gives

$$(21) \quad \begin{cases} w^n \in \mathbf{V}_{KL} \text{ and } t^n = (\tau^n_{\alpha\beta}) \in L^2(\Omega ; \mathbf{S}^2) : \\ \dfrac{1+\nu}{E} \tau^n_{\alpha\beta} - \dfrac{\nu}{E} \tau^n_{\mu\mu} \delta_{\alpha\beta} - \dfrac{1}{2} (\partial_\alpha w^n_\beta + \partial_\beta w^n_\alpha) = \varepsilon^2 \mathfrak{J}^2 D^2_{\alpha\beta}(t^{n-1}) + \\ \quad + \varepsilon^4 \mathfrak{J}^4 D^4_{\alpha\beta}(t^{n-2}) , \\ \displaystyle\int_\Omega \tau^n_{\alpha\beta}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) \, dx = 0 , \quad \forall v \in \mathbf{V}_{KL} , \end{cases}$$

where

- $(\mathfrak{J}h)(x_1, x_2, x_3) = \displaystyle\int_{-1}^{x_3} h(x_1, x_2, t) \, dt,$
- $\mathrm{D}^2_{\alpha\beta}$ and $D^4_{\alpha\beta}$ are, respectively, *second* and *fourth* order partial differential operators only with respect to the variables x_1 and x_2 (with constant coefficients).

Assume that the applied forces verify the following assumptions :

$$f(x_1, x_2, x_3) = \sum_{|k| \leq K} F_k(x_3) \, e^{2 i\pi k . x'} ,$$
$$g(x_1, x_2, \pm 1) = \sum_{|k| \leq K} G^\pm_k \, e^{2 i\pi k . x'} ,$$

where, $G^\pm_k \in \mathbf{C}^3$, $F_k \in C^\infty([-1, +1] ; \mathbf{C}^3)$ and $K > 0$ is a fixed integer.

Since the problem is *linear* and *periodic*, we use Theorem 1 and we look for the terms τ^n and v^n as sums in which, for every $k \in \mathbf{Z}^2$, such that

$|k| \leq K$, the sequence $\{T^{n,k}\}_{n \in \mathbb{N}}$ is in the space $C^\infty([-1, +1]; \mathbf{S}'^2)$ and the sequence $\{W^{n,k}\}_{n \in \mathbb{N}}$ in the space $\mathbf{C}^3$:

$$\left\{ \begin{aligned} \tau^n_{\alpha\beta}(x_1, x_2, x_3) &= \sum_{|k| \leq K} T^{n,k}_{\alpha\beta}(x_3)\, e^{2 i\pi k . x'} , \\ v^n_3(x_1, x_2, x_3) &= \sum_{|k| \leq K} W^{k,n}_3\, e^{2 i\pi k . x'} , \\ v^n_\alpha(x_1, x_2, x_3) &= \sum_{|k| \leq K} (W^{n,k}_\alpha - 2\, i\, \pi k_\alpha\, x_3\, W^{k,n}_3)\, e^{2 i\pi k . x'} , \end{aligned} \right.$$

Then, it is easily checked that we have :

$$(22) \quad \left\{ \begin{aligned} &\frac{1+\nu}{E} T^{n,k}_{\alpha\beta} - \frac{\nu}{E} T^{n,k}_{\mu\mu}\, \delta_{\alpha\beta} - i\pi\,(k_\alpha\, W^{n,k}_\beta + k_\beta\, W^{n,k}_\alpha) - \\ &\quad - 4\,\pi^2\, x_3\, k_\alpha\, k_\beta\, W^{n,k}_3 = \varepsilon^2\, \mathfrak{I}^2\, M^2_{\alpha\beta\lambda\mu}\, T^{n-1,k}_{\lambda\mu} + \varepsilon^4\, \mathfrak{I}^4\, M^4_{\alpha\beta\lambda\mu}\, T^{n-2,k}_{\lambda\mu} , \\ &k_\beta \int_{-1}^{1} T^{n,k}_{\alpha\beta}\, dx_3 = 0 \text{ and } k_\alpha\, k_\beta \int_{-1}^{1} x_3\, T^{n,k}_{\alpha\beta}\, dx_3 = 0 , \end{aligned} \right.$$

where $M^2 = m^2_{\alpha\beta}\, k_\alpha\, k_\beta$ and $M^4 = m^4_{\alpha\beta\lambda\mu}\, k_\alpha\, k_\beta\, k_\lambda\, k_\mu$ are tensors with constant coefficients $m^2_{\alpha\beta}$ and $m^4_{\alpha\beta\lambda\mu}$ which do not depend on k. In equation (22), we may eliminate the complex vector $W^{n,k}$. (See the formulas (26).) In this way, we obtain the equation :

$$(23) \qquad T^{n,k} = \mathcal{G}_{\alpha\beta}\, \left\{\varepsilon^2\, \mathfrak{I}^2\, M^2_{\alpha\beta\lambda\mu}\, T^{n-1,k}_{\lambda\mu} + \varepsilon^4\, \mathfrak{I}^4\, M^4_{\alpha\beta\lambda\mu}\, T^{n-1,k}_{\lambda\mu}\right\} ,$$

where the operator $\mathcal{G} = (\mathcal{G}_{\alpha\beta}) : C^\infty(\Omega^- ; \mathbf{S}'^2) \to C^\infty(\Omega^- ; \mathbf{S}'^2)$ is given by the formula :

$$(24) \quad (\mathcal{G}_{\alpha\beta}\, \tau_{\alpha\beta})\, (x) = \\ = A_{\lambda\mu} \int_{-1}^{+1} \tau_{\lambda\mu}(x', t)\, dt + x_3\, B_{\lambda\mu} \int_{-1}^{+1} \tau_{\lambda\mu}(x'\, t)\, t\, dt + C_{\lambda\mu}\, \tau_{\lambda\mu}(x) ,$$

with constant coefficients $A_{\alpha\beta}$, $B_{\alpha\beta}$ and $C_{\alpha\beta} \in \mathbf{S}'^2$. The coefficients $C_{\alpha\beta}$ do not depend on k. The coefficients $A_{\alpha\beta}$ and $B_{\alpha\beta}$ depend on k and verify the inequalities :

$$(25) \qquad \forall \mathrm{k} \in \mathbf{Z}^2 ,\ |k| \leq K , \quad \left\{ \begin{aligned} &|A_{\alpha\beta}| \leq c , \\ &|B_{\alpha\beta}| \leq c , \end{aligned} \right.$$

where c is a constant independent on k and $|t|$ denotes the norm $[t_{\lambda\mu}(t_{\lambda\mu})^c]^{1/2}$.

Let $\mu = (\mu_1, \mu_2) \in \mathbb{N}^2$ be a multi-index such that $\mu_1 + \mu_2 \leqslant m$ and let θ an integer such that $\theta \leqslant q$. Next, let $\partial^{\mu, \theta}$ denote the partial differential operator :

$$\partial^{\mu, \theta} = \frac{\partial^{\mu_1 + \mu_2 + \theta}}{\partial x_1^{\mu_1} \partial x_2^{\mu_2} \partial x_3^{\theta}}, \text{ if } \theta \geqslant 0 \text{ , and } \partial^{\mu, \theta} = \mathfrak{I}^{-\theta} \frac{\partial^{\mu_1 + \mu_2}}{\partial x_1^{\mu_1} \partial x_2^{\mu_2}}, \text{ if } \theta < 0 .$$

If $\tau(x) = e^{2 i \pi k . x'} T(x_3)$ denotes a tensor with $T \in C^{\infty}([-1, +1] ; \mathbf{S}'^2)$, the following seminorm :

$$\|\tau\|_{\mu, \theta} = \left(\int_{\Omega} (\partial^{\mu, \theta} \tau_{\alpha\beta})(x) (\partial^{\mu, \theta} \tau_{\alpha\beta}^c)(x) \, dx \right)^{1/2}$$

will verify :

$$\|\tau\|_{\mu, \theta} = (2 \pi)^{\mu_1 + \mu_2} k_1^{\mu_1} k_2^{\mu_2} \left(\int_{-1}^{+1} \frac{d^{\theta} T_{\alpha\beta}}{dx_3^{\theta}} (x_3) \frac{d^{\theta} T_{\alpha\beta}^c}{dx_3^{\theta}} (x_3) \, dx_3 \right)^{1/2},$$

if $\theta \geqslant 0$, and

$$\|\tau\|_{\mu, \theta} = (2 \pi)^{\mu_1 + \mu_2} k_1^{\mu_1} k_2^{\mu_2} \left(\int_{-1}^{+1} (\mathfrak{I}^{-\theta} T_{\alpha\beta})(x_3) (\mathfrak{I}^{-\theta} T_{\alpha\beta}^c)(x_3) \, dx_3 \right)^{1/2},$$

if $\theta < 0$.

Then, with equation (23) and inequalities (25), we see that there exists a constant $C > 0$ such that :

$$\begin{cases} \|\tau^{n,k}\|_{\mu, \theta} \leqslant C \ \{ \varepsilon^2 |k|^2 \|\tau^{n-1,k}\|_{\mu, \theta - 2} + \varepsilon^4 |k|^4 \|\tau^{n-2,k}\|_{\mu, \theta - 4} \} , \\ \text{for } n = 1, 2, \ldots, \text{ and } \ |k| \leqslant K . \end{cases}$$

Finally, we denote the norm of the tensor $\tau^{n,k}$ in the space $H^q(]-1, 1[; H^m(\omega))$ by $\|\tau^{n,k}\|$ which is defined by the formula

$$\|\tau^{n,k}\| = \sum_{\theta = 0}^{q} \sum_{\mu_1 + \mu_2 \leqslant m} \|\tau^{n,k}\|_{\mu, \theta} .$$

Since we have for $j = 2, 4$:

$$\sum_{\theta = -j}^{q - j} \sum_{\mu_1 + \mu_2 \leqslant m} \|\tau^{n,k}\|_{\mu, \theta} \leqslant c \|\tau^{n,k}\| ,$$

where c denotes a constant independent on n and k, we see that there exists an other constant $C > 0$ such that the following inequality holds :

$$\begin{cases} \|\tau^{n,k}\| \leqslant C \ \{ \varepsilon^2 |k|^2 \|\tau^{n-1,k}\| + \varepsilon^4 |k|^4 \|\tau^{n-2,k}\| \} , \\ \text{for } n = 1, 2, \ldots, \text{ and } \ |k| \leqslant K . \end{cases}$$

Hence, there exist a constant $Q > 0$, independent on k and ε, such that the sequence $\{\|\tau^{n,k}\|\}_{n \in \mathbb{N}}$ does converge to zero if $\varepsilon|k| < Q$.

Now, it is easy to complete the proof of the convergence. Indeed, we have the following expression of the vector $W^{n,k}$:

$$(26) \quad \left\{ \begin{aligned} W_\delta^{n,k} &= R_{\alpha\beta\delta} \int_{-1}^{+1} \{\varepsilon^2 \mathfrak{I}^2 M_{\alpha\beta\lambda\mu}^2 T_{\lambda\mu}^{n-1,k} + \varepsilon^4 \mathfrak{I}^4 M_{\alpha\beta\lambda\mu}^4 T_{\lambda\mu}^{n-1,k}\}(t)\, dt\,, \\ W_3^{n,k} &= S_{\alpha\beta} \int_{-1}^{+1} \{\varepsilon^2 \mathfrak{I}^2 M_{\alpha\beta\lambda\mu}^2 T_{\lambda\mu}^{n-1,k} + \varepsilon^4 \mathfrak{I}^4 M_{\alpha\beta\lambda\mu}^4 T_{\lambda\mu}^{n-1,k}\}(t)\, dt\,, \end{aligned} \right.$$

where $R_{\alpha\beta\delta}$ and $S_{\alpha\beta}$ are constants which are bounded with respect to k and we also have :

$$\tau_{\alpha 3}^n = -\mathfrak{I}\, \partial_\beta \tau_{\alpha\beta}^n \quad \text{and} \quad \tau_{33}^n - \mathfrak{I}^2\, \partial_{\alpha\beta} \tau_{\alpha\beta}^n\,.$$

The normal convergence is obtained in a similar way. ■

5. CONCLUSION

The main result of Section 4 says that the asymptotic expansion converges if the applied forces are trigonometric polynomials and if the thickness of the plate is small enough. In a more general case, when the applied forces may be expanded in Fourier series :

$$\left\{ \begin{aligned} f(x) &= \sum_{k \in \mathbf{Z}^2} F_k(x_3)\, e^{2 i\pi k . x'} \\ g(x) &= \sum_{k \in \mathbf{Z}^2} G_k\, e^{2 i\pi k . x'}\,, \end{aligned} \right.$$

for a fixed value of the half thickness ε, there exists an order of frequency $K(\varepsilon)$ such that the expansion converges if we omit all the terms of frequency k with $|k| > K(\varepsilon)$.

We think that there is no convergence for the terms of frequency $|k| > K(\varepsilon)$. The reason is, although the integral operators

$$\mathfrak{I} : h \to \mathfrak{I}h \text{ such that : } (\mathfrak{I}h)\,(x_1, x_2, x_3) = \int_{-1}^{x_3} h(x_1, x_2, t)\, dt\,,$$

in (23) are of Volterra type with

$$\lim_{n \to \infty} \|\mathfrak{I}^n\| = 0\,,$$

the integral operator

$$\mathbf{J} : h \to \mathbf{J}h \text{ such that : } (\mathbf{J}h)\,(x_1, x_2, x_3) = \int_{-1}^{+1} h(x_1, x_2, t)\, dt ,$$

(see (24)) *is not quasinilpotent* and therefore the sequence $\{T^{n,k}\}_{n \in \mathbb{N}}$ *does not converge to zero* if the product $\varepsilon \,.\, |k|$ is not small enough. But, at present, we do not have a rigorous proof to this fact. We only have *a numerical test* where we can see the convergence *exactly when* $\varepsilon \,.\, |k|$ *is small enough.* (In fact, if $\varepsilon \,.\, |k|$ is not small enough, the divergence is very quick !) This phenomenon will be published elsewhere.

Finally, we point out that Theorems 1 and 2 hold in *the particular case of a plate with sliding edges* defined by the condition $n_i\, v_i = 0$ on Γ_0^ε where $v = (v_i)$ is the displacement vector field [9, Chapter V]. We omit the proof, but merely mention that one introduces the symmetry S_α in $\mathbf{R}^3$ with respect to the plane $x_\alpha = 0$, the group of symmetries $\mathcal{S} = \{I, S_1, S_2, S_1 S_2\}$ and the applied forces which are $\mathcal{S}$-invariant. (See [10, Section 6].)

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