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CURVES FROM VARIATIONAL PRINCIPLES

by Ch. A. MICCHELLI [1]

Abstract. — *Variational techniques have long been used as a guiding principle for the construction of curves and surfaces. In this spirit, we study a practical problem of curve design where the choice of the curve is governed by a variational principle. The problem is to find the shape of a deformed wire in the plane of maximum durability which is subject to continual wear due to external forces and the surrounding material. We use this problem as a prototype for a class of variational curve problems which we solve by using convex duality theory. Thus we characterize our optimal curve in terms of a finite dimensional minimum problem which can be solved numerically. Some numerical examples are given.*

Résumé. — Courbes déduites de principes variationnels. *Les techniques variationnelles ont depuis longtemps été utilisées comme principe directeur pour la construction de courbes et surfaces. Dans cet esprit nous étudions un problème pratique de conception de courbe où le choix de la courbe est piloté par un principe variationnel. Le problème à résoudre est de trouver la forme d'un câble déformé dans le plan de vie maximum sous l'usure continue provenant de forces extérieures et des matériaux environnants. Nous utilisons ce problème comme prototype pour une classe de problèmes variationnels de courbes que nous résolvons par la théorie de dualité convexe. Ainsi notre courbe optimale est caractérisée en termes d'un problème de minimisation de dimension finie qui peut se résoudre numériquement. Quelques exemples numériques sont donnés.*

1. INTRODUCTION

Variational techniques have long been used as a guiding principle for the construction of curves and surfaces. Most noteworthy are the fundamental notions of geodesic curve and minimal surface, *cf.* do Carmo [1].

Among the many interesting and useful advances in this direction we mention that recent variational methods have been successfully used to obtain nonparametric surfaces which interpolate prescribed data, *cf.* Franke [5]. For planar interpolatory curves, minimizing the integral of the square of its curvature is a criterion that has attracted some interest. This problem was

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investigated by Lee and Forsythe [10] and also by Jerome and Fisher [6] (with the additional constraint that the length of the curve is bounded). A linearized version of this problem leads to interpolatory (natural) spline curves, *cf.* Farin [2], and general results of this type are given in Sidhu and Weinert [15]. We also mention that variational principles have been also used to obtain GC^2 (twice geometrically continuous) interpolatory curves, Nielson [13].

In all these cases, a fixed parametrization of the curve is assumed. As this may dramatically affect the shape of the curve proposals have been made to even choose the parametrization of an optimal interpolatory curve by a variational principle, see Marin [11], and Scherer [14]. This important design issue of best parametrization is also discussed by Foley [3] and Foley and Nielson [13] from nonvariational perspectives.

In this spirit, we study a practical problem of curve design where the choice of the curve is governed by a variational principle. The problem is to find the shape a deformed wire in the plane of maximum durability which is subject to continual wear due to external forces and the surrounding material. Such a requirement could arise in various situations, for instance, for wires in matrix printers which connect actuators to print heads.

We use this problem as a prototype for a class of variational curve problems which we solve by using convex duality theory. Thus we reduce our infinite dimensional primal problem to a finite dimensional dual problem. We show that there is no duality gap and characterize our optimal curve in terms of the finite dimensional dual problem which can then be solved numerically. Some numerical examples are given.

2. THE WIRE PATH PROBLEM

The formulation of the problem we are about present is due to John Lew of IBM T. J. Watson Research Center. The description we give here is a physically idealized set up which leads to a satisfactory analytic solution to the problem.

Wire Path Problem : A thin elastic rod of constant circular cross-section, with moment of inertia I and Young's modulus E is subject to external forces which bends it from a straight line, its underformed state. Assume that the axial curve of the rod is parametrized as $\mathbf{r}(t) = (x(t), y(t))$, $t \in [0, 1]$ and that the external force is a vector-valued function $\mathbf{F}(t)$ which exerts no twist on the rod. Then small deformation theory, *cf.* Landau and Lifshitz, eq. (20.14) [9], yields the relation

$$\mathbf{r}^{(4)}(t) = \mathbf{f}(t), \quad 0 \leqslant t \leqslant 1, \tag{2.1}$$

where $\mathbf{f}(t) := \mathbf{F}(t)/IE$ and $\mathbf{r}(t)$, $\mathbf{r}^{(1)}(t)$, $\mathbf{r}^{(2)}(t)$, $\mathbf{r}^{(3)}(t)$ are continuous functions of $t \in [0, 1]$.

We suppose that the ends of the rod are constrained linearly (no curvature) which provides us with boundary conditions :

$$(\mathbf{r}(0), \mathbf{r}^{(1)}(0), \mathbf{r}^{(2)}(0), \mathbf{r}^{(3)}(0)) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{2.2}$$

and

$$(\mathbf{r}(1), \mathbf{r}^{(1)}(1), \mathbf{r}^{(2)}(1), \mathbf{r}^{(3)}(1)) = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{0}, \mathbf{0}) . \tag{2.3}$$

The material surrounding the rod determines a wear rate at each point t, assumed to be proportional to $|\mathbf{f}(t)|_2$ and given by $W(t)\,|\mathbf{f}(t)|_2$ for some known functions $W(t)$; here $|\mathbf{x}|_2 =$ the euclidean norm of $\mathbf{x} \in \mathbf{R}^2$.

To maximize the lifetime of the rod we want to find the curve $\mathbf{r}(t)$ which minimizes the maximum wear rate

$$\max \{W(t)\,|\mathbf{f}(t)|_2 : 0 \leqslant t \leqslant 1\} \tag{2.4}$$

over all curves $\mathbf{r}(t)$ subject to (2.1), (2.2) and (2.3).

3. ANALYSIS OF THE WIRE PATH PROBLEM

We write the competing curves in the form

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{r}^{(1)}(0)\, t + \mathbf{r}^{(2)}(0)\,\frac{t^2}{2!} + \mathbf{r}^{(3)}(0)\,\frac{t^3}{3!} + \frac{1}{3!}\int_0^t (t-\sigma)^3\,\mathbf{f}(\sigma)\,d\sigma .$$

Then the boundary conditions imply that

$$\mathbf{r}(t) = \frac{1}{3!}\int_0^t (t-\sigma)^3\,\mathbf{f}(\sigma)\,d\sigma \tag{3.1}$$

and

$$3!\,\mathbf{a}_0 = \int_0^1 (1-\sigma)^3\,\mathbf{f}(\sigma)\,d\sigma \tag{3.2}$$

$$2!\,\mathbf{a}_1 = \int_0^1 (1-\sigma)^2\,\mathbf{f}(\sigma)\,d\sigma \tag{3.3}$$

$$0 = \int_0^1 \mathbf{f}(\sigma)\,d\sigma = \int_0^1 (1-\sigma)\,\mathbf{f}(\sigma)\,d\sigma . \tag{3.4}$$

Thus there are eight linear constraints on $\mathbf{f}$ which we express as follows. We define $\mathbf{c} = (c_1, c_2, \ldots, c_8) = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4) \in \mathbf{R}^8$ and require that

$$\mathbf{c}_1 = \mathbf{a}_0, \quad \mathbf{c}_2 = \mathbf{a}_1, \quad \mathbf{c}_3 = \mathbf{c}_4 = 0 . \tag{3.5}$$

Also, we define the functions $\mathbf{x}_1, \ldots, \mathbf{x}_8 : [0, 1] \to \mathbf{R}^2$ by

$$\begin{aligned} \mathbf{x}_1(t) &= \frac{1}{3!}(1-t)^3(1,0), & \mathbf{x}_2(t) &= \frac{1}{3!}(1-t)^3(0,1) \\ \mathbf{x}_3(t) &= \frac{1}{2!}(1-t)^2(1,0), & \mathbf{x}_4(t) &= \frac{1}{2!}(1-t)^2(0,1) \\ \mathbf{x}_5(t) &= (1-t)(1,0), & \mathbf{x}_6(t) &= (1-t)(0,1) \\ \mathbf{x}_7(t) &= (1,0), & \mathbf{x}_8(t) &= (0,1). \end{aligned} \tag{3.6}$$

Then the constraints on $\mathbf{f}$ take the equivalent form

$$c_i = \int_0^1 (\mathbf{x}_i(t), \mathbf{f}(t))\, dt, \quad i = 1, \ldots, 8 \tag{3.7}$$

where $(\mathbf{x}, \mathbf{y})$ is the standard euclidean inner product for vectors $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^2$.

We now extend this problem to the following situation (further examples of the general case will be given later).

An External Problem : Given constants $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbf{R}^m$ and vector-valued functions $\mathbf{x}_1, \ldots, \mathbf{x}_m, : [0, 1] \to \mathbf{R}^n$ with $|\mathbf{x}_i(\,.\,)|_2 \in L^1[0, 1]$, $i = 1, 2, \ldots, m$. Find a curve $\mathbf{f}(\,.\,) : [0, 1] \to \mathbf{R}^n$, $|\mathbf{f}(\,.\,)|_2 \in L^\infty[0, 1]$ which minimizes

$$\max_{0 \leq t \leq 1} |\mathbf{f}(t)|_2 \tag{3.8}$$

among all curves $\mathbf{f}$ such that

$$c_i = \int_0^1 (\mathbf{x}_i(t), \mathbf{f}(t))\, dt, \quad i = 1, \ldots, m. \tag{3.9}$$

Note that this is a general version of the wire path problem when the wear rate is assumed to be constant.

For our first result we require the following hypothesis : given any $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbf{R}^m \setminus \{0\}$ the set

$$Z(\mathbf{d}\,; \mathbf{x}_1, \ldots, \mathbf{x}_m) = \left\{ t : 0 \leq t \leq 1, \sum_{j=1}^m d_j\, \mathbf{x}_j(t) = \mathbf{0} \right\} \tag{3.10}$$

has Lebesgue measure zero, which we express as

$$\operatorname{meas} Z(\mathbf{d}\,; \mathbf{x}_1, \ldots, \mathbf{x}_m) = 0. \tag{3.11}$$

Observe that in the case of the wire path problem this condition is valid since the set (3.11) is the common zero set of two cubics and hence it is either a finite set or the whole interval, [0, 1]. The latter case cannot occur, since obviously the curves $\mathbf{x}_1, \ldots, \mathbf{x}_8$ in (3.6) are linearly independent. Note that conditions (3.11) certainly implies, in general, that $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are linearly independent.

We will now proceed to solve the external problem under the condition (3.11).

THEOREM 3.1 : *There exists a unique curve* $\mathbf{f}^0 : [0, 1] \to \mathbf{R}^n$ *which minimizes* (3.8) *subject to* (3.9). *It is given by*

$$\mathbf{f}^0(t) = \lambda_0^{-1} \sum_{i=1}^{m} y_i \, \mathbf{x}_i(t) \Bigg/ \left| \sum_{i=1}^{m} y_i \, \mathbf{x}_i(t) \right|_2 , \; a.e., \, t \in [0, 1] , \qquad (3.12)$$

where $(y_1, \ldots, y_m) \in \mathbf{R}^m$ *is the unique minimum of the strictly convex function*

$$H(\mathbf{d}) = \int_0^1 \left| \sum_{i=1}^{m} d_i \, \mathbf{x}_i(t) \right|_2 dt, \quad \mathbf{d} = (d_1, \ldots, d_m) . \qquad (3.13)$$

subject to

$$\sum_{i=1}^{m} d_i \, c_i = 1 \qquad (3.14)$$

and

$$\lambda_0 := \int_0^1 \left| \sum_{i=1}^{m} y_i \, \mathbf{x}_i(t) \right|_2 dt . \qquad (3.15)$$

Note that the extremal curve is a.e. a motion on the sphere with radius λ_0 which is the minimum value of (3.8) that we seek.

Proof: Clearly, the function (3.13) has a unique minimum subject to (3.14) because it is a strictly convex function of $\mathbf{d}$. Let $\mathbf{y} \in \mathbf{R}^m \setminus \{0\}$ be the unique minimum. Then $\mathbf{y}$ satisfies the nonlinear (variational) equations

$$\int_0^1 \left(\left(\sum_{i=1}^{m} y_i \, \mathbf{x}_i(t), x_j(t) \right) \right) \Bigg/ \left| \sum_{i=1}^{m} y_i \, \mathbf{x}_i(t) \right|_2 dt = \lambda c_j, \quad j = 1, \ldots, m , \qquad (3.16)$$

for some Lagrange multiplier λ. Here we use the easily verified fact that for any $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbf{R}^m \setminus \{0\}$

$$\frac{\partial H(\mathbf{d})}{\partial d_j} = \int_0^1 \left(\sum_{i=1}^{m} d_i \, \mathbf{x}_i(t), \mathbf{x}_j(t) \right) \Bigg/ \left| \sum_{i=1}^{m} d_i \, \mathbf{x}_i(t) \right|_2 dt , \quad j = 1, \ldots, m . \qquad (3.17)$$

One should keep in mind that our hypothesis insures that the integral on the right hand side of (3.17) is finite.

Now, multiply both sides of equation (3.16) by y_j, and sum over j, $j = 1, \dots, m$. Since $\sum_{i=1}^{m} y_i c_i = 1$ we get

$$\lambda = \int_0^1 \left| \sum_{i=1}^{m} y_i \mathbf{x}_i(t) \right|_2 dt$$

which is what we call λ_0 in (3.15). This observation shows that $\mathbf{f}^0$, as defined by (3.12) satisfies our constraints (3.9). Furthermore, since $|\mathbf{f}^0(t)|_2 = \lambda_0^{-1}$, a.e. $t \in [0, 1]$ it only remains to show that any other curve $\mathbf{f}$ satisfying the linear constraints (3.9) has a larger maximum norm. To see this we now multiply both sides of equation (3.9) by y_i and sum over i, $1 \leqslant i \leqslant m$ to obtain

$$1 = \int_0^1 \left(\sum_{i=1}^{m} y_i \mathbf{x}_i(t), \mathbf{f}(t) \right) dt$$

which is, by the Cauchy-Schwarz inequality

$$\leqslant \int_0^1 \left| \sum_{i=1}^{m} y_i \mathbf{x}_i(t) \right|_2 \cdot |\mathbf{f}(t)|_2 \, dt$$
$$\leqslant \lambda_0 \max_{0 \leqslant t \leqslant 1} |\mathbf{f}(t)|_2 .$$

Therefore we get $\max_{0 \leqslant t \leqslant 1} |\mathbf{f}(t)|_2 \geqslant \lambda_0^{-1}$ which is the desired conclusion.

The above inequalities imply even more. They establish that $\mathbf{f}_0$ defined by (3.12) is indeed the unique solution to our extremal problem. To see this we let $\mathbf{f}$ be any other solution with $\max_{0 \leqslant t \leqslant 1} |\mathbf{f}(t)|_2 = \lambda_0^{-1}$. Therefore the above inequalities all become equalities, that is,

$$\int_0^1 \left(\sum_{i=1}^{m} y_i \mathbf{x}_i(t), \mathbf{f}(t) \right) dt = \int_0^1 \left| \sum_{i=1}^{m} y_i \mathbf{x}_i(t) \right|_2 |\mathbf{f}(t)|_2 \, dt$$
$$= \left(\int_0^1 \left| \sum_{i=1}^{m} y_i \mathbf{x}_i(t) \right|_2 dt \right) \max_{0 \leqslant t \leqslant 1} |\mathbf{f}(t)|_2 .$$

The last equation above, and our basic hypothesis (3.11) implies that $\mathbf{f}(t) = \sigma(t) \mathbf{f}^0(t)$ where σ is some function such that $|\sigma(t)| = 1$, a.e., $t \in [0, 1]$. To pin down $\sigma(t)$ we now use our linear constraints (3.9) which give

$$\int_0^1 \sigma(t)(\mathbf{x}_k(t), \mathbf{f}^0(t)) \, dt = \int_0^1 (\mathbf{x}_k(t), \mathbf{f}^0(t)) \, dt, \quad 1 \leqslant k \leqslant m . \quad (3.18)$$

First, we multiply equation (3.18) by y_k and sum over k, $1 \leqslant k \leqslant m$, and then from our formula (3.12) for $\mathbf{f}^0$ we get

$$\int_0^1 \sigma(t) \left| \sum_{k=1}^m y_k \mathbf{x}_k(t) \right|_2 dt = \int_0^1 \left| \sum_{k=1}^m y_k \mathbf{x}_k(t) \right|_2 dt .$$

But then our basic hypothesis (3.12) guarantees that $\sigma(t) = 1$, a.e., $t \in [0, 1]$.

Before we consider extensions/improvements of Theorem 3.1 let us return to our Wire Path Problem and see what conclusions we can draw from Theorem 3.1.

It is as convenient to consider a somewhat general situation. We suppose $u_1(t), \dots, u_r(t)$, $t \in [0, 1]$, are given linearly independant scalar valued functions and consider curves $\mathbf{f}$ constrained to satisfy

$$\mathbf{c}_j = \int_0^1 u_j(t)\, \mathbf{f}(t)\, dt , \quad j = 1, \dots, r . \tag{3.19}$$

Obviously the wire path problem is of this form, and generally (3.19) is a special case of the linear constraints (3.9) for the choice

$$\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_r) \in \mathbf{R}^m , \quad m = rn$$

and

$$\mathbf{x}_{(\ell - 1) n + j}(t) = u_\ell(t)\, \mathbf{e}_j , \quad \ell = 1, \dots, r , \quad j = 1, \dots, n \tag{3.20}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard coordinate basis in $\mathbf{R}^n$, $(\mathbf{e}_j)_k = 0$, $j \neq k$ and 1 otherwise.

Thus the dual minimum problem becomes, in this case,

$$\min \int_0^1 \left| \sum_{j=1}^r \mathbf{b}_j\, u_j(t) \right|^2 dt \tag{3.21}$$

subject to $\sum_{i=1}^r (\mathbf{b}_i, \mathbf{c}_i) = 1$, $\mathbf{b}_1, \dots, \mathbf{b}_r \in \mathbf{R}^n$. In particular to solve the wire path problem we are led to minimizing

$$\int_0^1 |\mathbf{p}(t)|_2\, dt$$

where $\mathbf{p}(t) = \mathbf{b}_0 + \mathbf{b}_1(1 - t) + \mathbf{b}_2(1 - t)^2/2! + \mathbf{b}_3(1 - t)^3/3!$ is a parametric cubic curve in the plane such that $(\mathbf{a}_0, \mathbf{b}_3) = (\mathbf{a}_1, \mathbf{b}_2)$. Although in this case $(\mathbf{r}^0)^{(4)}(t)$ is a motion on the circle of radius λ_0, the functional form of the actual curve $\mathbf{r}^0(t)$ is quite complicated. Nonetheless, the numerical calculation $\mathbf{r}^0(t)$ can be accomplished easily.

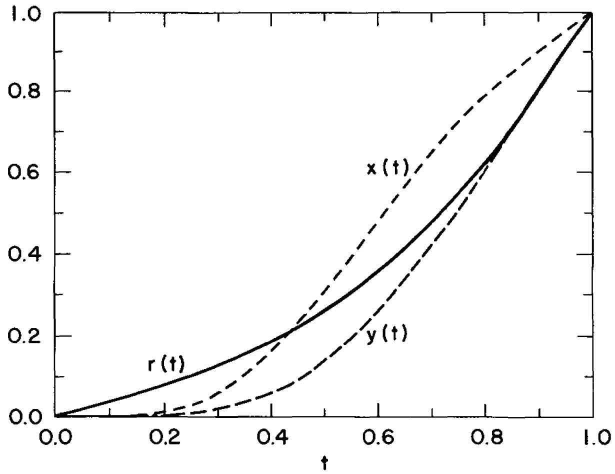

Figure 3.1. — $\mathbf{r}(1) = (1, 1)$, $\mathbf{r}^{(1)}(1) = (1, 2)$, $\mathbf{r}^{(2)}(1) = (0, 0)$, $\mathbf{r}^{(3)}(1) = (0, 0)$.

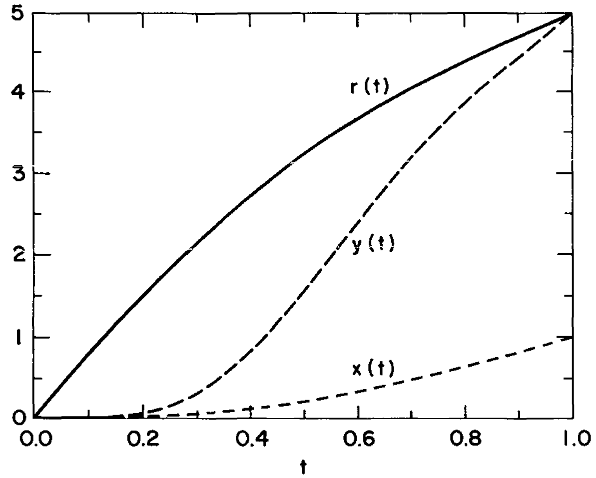

Figure 3.2. — $\mathbf{r}(1) = (1, 5)$, $\mathbf{r}^{(1)}(1) = (2, 6)$, $\mathbf{r}^{(2)}(1) = (3, 7)$, $\mathbf{r}^{(3)}(1) = (4, 8)$.

Software for minimizing a convex function of several (eight) variables, given a procedure to compute the partial derivative of the objective function, is required. The computation of the integrals can be accomplished by a quadrature which may take account of the zeros (integrable singularities) of the denominator. As a check it is advisable to employ a quadrature on the computed curve $\mathbf{r}^0(t)$ to check if it satisfies the boundary conditions and the variational equations. Above are the results of such computations using some Harwell Library subroutines. We have plotted the curve and each component as well.

4. EXTENSIONS

First let us note that the analysis extends easily to non constant continuous wear function $W(t)$ which are positive on $[0, 1]$. It will appear in the dual minimum problem as a reciprocal weight. Specifically (3.12) still holds where $(y_1, \ldots, y_m) \in \mathbf{R}^m$ is chosen to minimize

$$\int_0^1 \left| \sum_{i=1}^{m} d_i \, \mathbf{x}_i(t) \right|_2 W^{-1}(t) \, dt$$

subject to

$$\sum_{i=1}^{m} c_i \, d_i = 1$$

and λ_0 is likewise given by

$$\lambda_0 = \int_0^1 \left| \sum_{i=1}^{m} y_i \, \mathbf{x}_i(t) \right|_2 W(t) \, dt \, .$$

To remove the critical hypothesis (3.1) is another matter. A particularly interesting example when it doesn't hold is the problem of finding a curve $\mathbf{r} : [0, 1] \to \mathbf{R}^n$ which passes through prescribed points $\mathbf{r}_1, \ldots, \mathbf{r}_{q+k}$, at distinct points $t = t_1, \ldots, t_{q+k}$, $(t_1 = 0, t_{q+k} = 1)$ in [0, 1], respectively, that is,

$$\mathbf{r}(t_i) = \mathbf{r}_i, \quad i = 1, \ldots, q + k \tag{4.1}$$

and minimizes the quantity

$$\max_{0 \leq t \leq 1} |\mathbf{r}^{(q)}(t)|_2 \, . \tag{4.2}$$

As in the wire path problem we expand $\mathbf{r}(t)$ as

$$\mathbf{r}(t) = \sum_{j=0}^{q-1} \frac{\mathbf{r}^{(j)}(0)}{j!} + \frac{1}{(q-1)!} \int_0^1 (t - \sigma)_+^{q-1} \, \mathbf{r}^{(q)}(\sigma) \, d\sigma \, .$$

We evaluate both sides of this equation at $t = t_i, \ldots, t_{i+q}$ and take the divided difference of the resulting expressions. By doing so we obtain the equivalent problem of finding a curve $\mathbf{f}(t)(= r^{(q)}(t)) : [0, 1] \to \mathbf{R}^n$ which minimizes

$$\max_{0 \leq t \leq 1} |\mathbf{f}(t)|_2$$

subject to

$$[t_i, ..., t_{i+q}]\, \mathbf{r} = \int_0^1 M(t \,|\, t_i, ..., t_{i+q})\, \mathbf{f}(t)\, dt\,, \quad i = 1, ..., k\,. \tag{4.3}$$

Here

$$\mathbf{d}_i := [t_i, \ldots, t_{i+q}]\, \mathbf{r} = \sum_{j=0}^{q} \frac{\mathbf{r}_{i+j}}{\prod_{\ell \neq j} (t_{i+j} - t_{i+\ell})}\,,$$

is the q-th order divided difference vector, and

$$M(t \,|\, t_i, ..., t_{i+q}) = \frac{1}{(q-1)!} \sum_{j=0}^{q} \frac{(t_{i+j} - t)_+^{q-1}}{\prod_{\ell \neq j} (t_{i+j} - t_{i+\ell})}$$

is the $q-1$st degree B-spline with knots at $t_i, ..., t_{i+q}$. This problem is of the type (3.19), but now (3.11) fails to hold, since spline functions may vanish on intervals. Thus the dual objective function (3.13) or (3.21) is no longer differentiable.

We will only prove the following fact about minimizing (4.2) subject to (4.1).

THEOREM 4.1 : *There exists a curve* $\mathbf{r} : [0, 1] \to \mathbf{R}^n$ *which minimizes the maximum length of its* q-th *derivative,* (4.2) *subject to the interpolation conditions* (4.1) *such that*

(i) $|= r^{(q)}(t)|_2 =$ constant, a.e., $t \in [0, 1]$,

(ii) *For every* $\mathbf{a} \in \mathbf{R}^n$, *the function* $(\mathbf{a}, r^{(q)}(t))$ *has at most* $k-1$ *sign changes.*

In the scalar case $n = 1$, this result was proved by Karlin, [8].

Proof: Given $\varepsilon > 0$ we define

$$M_i^{\varepsilon}(t) = \int_0^1 G_{\varepsilon}(t, \sigma)\, M(\sigma \,|\, t_i, ..., t_{i+q})\, d\sigma\,, \quad i = 1, ..., k$$

where $G_{\varepsilon}(t, \sigma) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}((t-\sigma)/\varepsilon)^2\right)$.

Then

$$\lim_{\varepsilon \to 0^+} M_i^{\varepsilon}(t) = M(t \mid t_i, \dots, t_{i+q}), \; t \in (0, 1), \quad i = 1, \dots, k, \tag{4.4}$$

uniformly on compact subsets of (0, 1).

Moreover, from the variation diminishing property of B-spline series and the strict total positivity of the heat kernel $G_{\varepsilon}(x, y)$, *cf.* Karlin [7], it follows that $\{M_1^{\varepsilon}(t), \dots, M_k^{\varepsilon}(t)\}$ constitutes a Chebyshev system on (0, 1). Thus any nontrivial linear combination of these functions have a most $k - 1$ zeros on (0, 1). Hence, we may appeal to Theorem 3.1 and characterize the minimum of

$$\max_{0 \le t \le 1} |\mathbf{f}(t)|_2 \tag{4.5}$$

subject to

$$\int_0^1 M_i^{\varepsilon}(t)\, \mathbf{f}(t)\, dt = \mathbf{d}_i, \quad i = 1, \dots, k, \tag{4.6}$$

since the zero measure condition (3.11) is satisfied for the corresponding functions $\mathbf{x}_i$, given by (3.20), for $u_i = M_i^{\varepsilon}$, $i = 1, \dots, k$. We let λ_0^{ε} be the corresponding minimum to the dual problem and $\mathbf{f}_0^{\varepsilon}$ the unique solution to the primal problem. Then $|\mathbf{f}_0^{\varepsilon}(t)| = (\lambda_0^{\varepsilon})^{-1}$, a.e., $t \in [0, 1]$ and $(\mathbf{a}, \mathbf{f}_0^{\varepsilon}(t))$ has at most $k - 1$ sign changes for every $\mathbf{a} \in \mathbf{R}^n$. It is apparent that $\lim_{\varepsilon \to 0^+} \lambda_0^{\varepsilon} = \lambda_0$, where

$$\lambda_0 = \min \left\{ \int_0^1 \left| \sum_{i=1}^{k} \mathbf{b}_i\, M(t \mid t_i, \dots, t_{i+q}) \right|_2 dt : \sum_{i=1}^{k} (\mathbf{b}_i, \mathbf{d}_i) = 1 \right\}$$

which is positive since the B-splines are linearly independent. Thus $\mathbf{f}_0^{\varepsilon}$ converges weakly in L^1 (and hence a.e.), through some subsequence to an $\mathbf{f}_0$ which therefore minimizes (4.5) subject to (4.3) (which holds for $\mathbf{f} = \mathbf{f}_0$, because (4.6) holds for $\mathbf{f} = \mathbf{f}_0^{\varepsilon}$). Consequently, the curve $= \boldsymbol{r}^{(q)} = \mathbf{f}_0$ solves our minimum problem and inherits from $\mathbf{f}_0^{\varepsilon}$ the properties claimed in the Theorem.

It would be nice to have an efficient computational procedure to obtain an extremal curve for the problem above, even for the case that $q = 2$.

Various other extensions of our basic extremal problem are possible. Our previous analysis rests on being able to determine the norm conjugate to the primal norm

$$\|\mathbf{f}\|_{\infty, 2} := \max_{0 \le t \le 1} |\mathbf{f}(t)|_2 \,. \tag{4.7}$$

Specifically, we used in our proofs the fact that

$$\|\mathbf{f}\|_{\infty 2}^{*} = \max\left\{\int_0^1 (\mathbf{f}(t), \mathbf{g}(t))\, dt \quad \|\mathbf{g}\|_{\infty 2} \leq 1\right\}$$
$$= \int_0^1 |\mathbf{f}(t)|_2\, dt$$
$$= \|\mathbf{f}\|_{1 2}$$

That is, the norm conjugate to $\|\mathbf{f}\|_{\infty 2}$ is $\|\mathbf{f}\|_{1 2}$ This « mixed » norm, a combination of the ℓ_2 norm on $\mathbf{R}^n$, and sup norm on [0, 1], quite naturally arose in the wire path problem But, perhaps, one might have first thought of the simpler Hilbert space norm on curves in $\mathbf{R}^n$ given by

$$\|\mathbf{f}\|_{2 2}^2 = \int_0^1 |\mathbf{f}(t)|_2^2\, dt \tag{4 8}$$

This norm is self-conjugate

More complicated possibilities immediately come to mind, namely,

$$\|\mathbf{f}\|_{p r} = \left(\int_0^1 |\mathbf{f}(t)|_p^r\, dt\right)^{1/r} \tag{4 9}$$

where $|\mathbf{x}|_p^p = \sum_{i=1}^{n} |x_i|^p$, $\mathbf{x} = (x_1, \quad , x_n) \in \mathbf{R}^n$, is the ℓ^p norm on $\mathbf{R}^n$ The conjugate norm in this case is

$$\|\mathbf{f}\|_{p r}^{*} = \max\left\{\int_0^1 (\mathbf{f}(t), \mathbf{g}(t))\, dt \quad \|\mathbf{g}\|_{p r} \leq 1\right\} \tag{4 10}$$

and it easily follows from Holder's inequality that

$$\|\mathbf{f}\|_{p r}^{*} = \|\mathbf{f}\|_{q s} \tag{4 11}$$

where (p, q), (r, s) are conjugate pairs, i e

$$\frac{1}{p} + \frac{1}{q} = 1\,, \quad \frac{1}{r} + \frac{1}{s} = 1$$

For instance, as we have already pointed $\|f\|_{\infty 2}^{*} = \|f\|_{1 2}$

Using equation (4 11) a version of Theorem 3 1 will hold also for the mixed norms (4 9) Perhaps the special cases p, $r = 1$ or ∞ will lead to some further interesting variational problems for curves

Amongst all these norms the simplest is the (2, 2) norm because here the variational equations are linear. Thus, minimizing

$$\int_0^1 |\mathbf{f}(t)|_2^2\, dt \tag{4.12}$$

subject to

$$c_i = \int_0^1 (\mathbf{x}_i(t), \mathbf{f}(t))\, dt\,, \quad i = 1, \ldots, m \tag{4.13}$$

is solved by

$$\mathbf{f}_0(t) = \sum_{i=1}^{m} y_i\, \mathbf{x}_i(t) \tag{4.14}$$

where $y_1, \ldots, y_m$ are determined by the linear equations obtain by substituting $\mathbf{f} = \mathbf{f}_0$ in (4.13). In the special case (3.19), (3.20) the linear system has a block structure so that it reduces to an $r \times r$ (rather than an $m \times m$, $m = rk$) linear system. This means we can characterize the optimal curve by minimizing coordinate by coordinate. For instance, when (4.2) is replaced by the (2, 2) norm and the curve is constrained by the interpolation conditions (4.1) we see that the optimal curve is coordinatewise, natural spline interpolation. This fact was the departure point for the problem of optimal knot parametrization for spline interpolation mentioned earlier in the introduction.

Our final comments will focus on adding, in addition to our linear constraints, some global constraint on the curve. For instance, generally speaking we might want the curve to avoid certain obstacles. We can cavalierly describe this by saying $\mathbf{f}(t) \in K$, for all $t \in [0, 1]$, for some set $K \subseteq \mathbf{R}^n$. But, duality will only apply when K is convex, a strong hypothesis. We describe one possible result in the case that K is a closed convex cone. We restrict ourselves to the norm $\|\mathbf{f}\|_{\infty, 2}$ and use methods of [12]. Although that paper, is restricted to Hilbert spaces norms, i.e. the (2.2) case, the general methods used there suggest the following approach.

We suppose K is a closed convex subset of $\mathbf{R}^n$ and that $P_K : \mathbf{R}^n \to K$ is the orthogonal projection onto K, i.e.

$$\min_{y \in K} |\mathbf{x} - \mathbf{y}|_2 = |\mathbf{x} - P_K \mathbf{x}|_2,\ \mathbf{x}, \mathbf{y} \in \mathbf{R}^n\,.$$

In general, P_K is a nonlinear function of n variables, which, however can be determined for some simple sets. When K is a closed convex cone, the case we will restrict ourselves to it is known that

$$\frac{d}{dt}(|P_K(\mathbf{x}+t\mathbf{y})|_2^2)_{t=0} = (P_K\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^n, \tag{4.15}$$

$$(\mathbf{x}, \mathbf{y}) \leqslant (P_K(\mathbf{x}), \mathbf{y}), \quad \mathbf{x} \in \mathbf{R}^n, \quad \mathbf{y} \in K \tag{4.16}$$

and

$$(P_K(\mathbf{x}), P_K(\mathbf{x})) = (\mathbf{x}, P_K(\mathbf{x})), \quad \mathbf{x} \in \mathbf{R}^n, \tag{4.17}$$

cf. [12, Lemma 2.1]. Based on these facts we are lead to consider the (dual) minimum problem

$$H_K(\mathbf{d}) = \int_0^1 \left| P_K\left(\sum_{i=1}^m d_i\,\mathbf{x}_i(t)\right)\right|_2 dt \tag{4.18}$$

subject to $\sum_{j=1}^m d_j c_j = 1$. If $H_K(\mathbf{d})$ has a minimum at $\mathbf{d} = \mathbf{y}$ and $P_K\left(\sum_{i=1}^m y_i\,\mathbf{x}_i(t)\right) = 0$ only on a set of Lebesgue measure zero then remove using (4.15) there is a Lagrange multiplier λ_0 such that

$$\int_0^1 \left(P_K\left(\sum_{i=1}^m y_i\,\mathbf{x}_i(t)\right), \mathbf{x}_j(t)\right) \Bigg/ \left|P_K\left(\sum_{i=1}^m y_i\,\mathbf{x}_i(t)\right)\right|_2 dt \tag{4.19}$$
$$= \lambda_0 c_j, \quad j = 1, \dots, m.$$

As before,

$$\mathbf{f}_0(t) := \lambda_0^{-1} P_K\left(\sum_{i=1}^m y_i\,\mathbf{x}_i(t)\right) \Bigg/ \left|P_K\left(\sum_{i=1}^m y_i\,\mathbf{x}_i(t)\right)\right|_2 \tag{4.20}$$

satisfies

$$c_i = \int_0^1 (\mathbf{x}_i(t), \mathbf{f}_0(t))\,dt, \quad i = 1, \dots, m. \tag{4.21}$$

Multiplying both sides of (4.19) by y_j and summing over j, $1 \leqslant j \leqslant m$ and simplifying the resulting expression gives

$$\lambda_0 = \int_0^1 \left|P_K\left(\sum_{i=1}^m y_i\,\mathbf{x}_i(t)\right)\right| dt. \tag{4.22}$$

Now, let $\mathbf{f}$ be any curve such that $\mathbf{f}(t) \in K$, $t \in [0, 1]$, and (3.9) holds, that is,

$$c_i = \int_0^1 (\mathbf{x}_i(t), \mathbf{f}(t))\,dt, \quad i = 1, \dots, m.$$

Using (4.16), we have for all $t \in [0, 1]$

$$\left(\sum_{j=1}^{m} y_j \mathbf{x}_j(t), \mathbf{f}(t) \right) \leq \left(P_K \left(\sum_{j=1}^{m} y_j \mathbf{x}_j(t) \right), \mathbf{f}(t) \right)$$
$$\leq \left| P_K \left(\sum_{j=1}^{m} y_j \mathbf{x}_j(t) \right) \right|_2 \cdot |\mathbf{f}(t)|_2 . \qquad (4.23)$$

Integrating both sides over $t \in [0, 1]$ and using (4.21), and also (4.17), (4.22) gives

$$1 = \int_0^1 \left(\mathbf{f}_0(t), \sum_{j=1}^{m} y_j \mathbf{x}_j(t) \right) dt$$
$$\leq \left(\int_0^1 \left| P_K \left(\sum_{j=1}^{m} y_j \mathbf{x}_j(t) \right) \right|_2 dt \right) \cdot \left(\max_{0 \leq t \leq 1} |\mathbf{f}(t)|_2 \right) .$$

Hence we get

$$\lambda_0^{-1} \leq \max_{0 \leq t \leq 1} |\mathbf{f}(t)|_2 .$$

This proves most of the following theorem.

THEOREM 4.2 : *Let K be a closed convex cone in $\mathbf{R}^n$ and suppose $\mathbf{x}_1, \ldots, \mathbf{x}_m, y_1, \ldots, y_m$ are as in the extremal problem in section three. Suppose $P_K : \mathbf{R}^n \to K$ is the orthogonal projection of $\mathbf{R}^n$ onto K and it has the following property. For any $\mathbf{d} \in \mathbf{R}^m \setminus \{0\}$*

$$\text{meas} \left\{ t : t \in [0, 1], \ P_K \left(\sum_{j=1}^{m} d_j \mathbf{x}_j(t) \right) = \mathbf{0} \right\} . \qquad (4.24)$$

Then $\mathbf{f}_0$ defined by (4.20) is the unique curve which minimizes

$$\max_{0 \leq t \leq 1} |\mathbf{f}(t)|_2 \qquad (4.25)$$

over all curves in K satisfying

$$c_i = \int_0^1 (\mathbf{x}_i(t), \mathbf{f}(t)) \, dt, \quad i = 1, \ldots, m .$$

Proof: First, we observe that H_K indeed has a minimum. Let $\{\mathbf{d}^\ell : \ell = 1, 2, \ldots\}$ be any minimizing sequence i.e.

$$\lim_{\ell \to \infty} H_K(\mathbf{d}^\ell) = \inf \left\{ H_K(\mathbf{d}) : \sum_{i=1}^{m} d_i c_i = 1 \right\} .$$

If $\{\mathbf{d}^{\ell} : \ell = 1, 2, \ldots\}$ is bounded then we may choose a convergent subsequence and prove H_K has a minimum. If not, we consider the vectors $\tilde{\mathbf{d}}^{\ell} := \mathbf{d}^{\ell}/|\mathbf{d}^{\ell}|_2$ which surely have a convergent subsequence, say $\lim_{\ell' \to \infty} \tilde{\mathbf{d}}^{\ell'} = \mathbf{d}^0$. Then it easily follows, since $|\mathbf{d}^{\ell}|_2 \to \infty$, that

$$\int_0^1 \left| P_K\left(\sum_{j=1}^{m} d_j^0 \mathbf{x}_j(t) \right) \right|_2 dt = 0 .$$

But this contradicts our hypothesis (4.24) and so H_K indeed has a minimum. As for the uniqueness, the argument is similar to the one used in Theorem 3.1. Namely, if $\max_{0 \leq t \leq 1} |\mathbf{f}(t)| = \lambda_0^{-1}$ then the inequalities in (4.23) become equalities and so $\mathbf{f}(t) = \sigma(t)\, \mathbf{f}_0(t)$. But then the moment conditions (3.9) imply that

$$\int_0^1 \sigma(t) \left| P_K\left(\sum_{i=1}^{m} y_i\, \mathbf{x}_i(t) \right) \right|_2 dt = \int_0^1 \left| P_K\left(\sum_{i=1}^{m} y_i\, \mathbf{x}_i(t) \right) \right|_2 dt$$

which proves that $\sigma(t) = 1$, a.e., $t \in [0, 1]$ by using our hypothesis (4.24), once again.

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