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GEOMETRY PROCESSING : INTERSECTIONS, CONTOURS, AND CUBATURES

by R. E. BARNHILL [1]

Abstract. — *Geometry Processing is the calculation of geometric properties of curves, surfaces and volumes. In this paper we present windows to three aspects of Geometry Processing : intersections, contours and cubatures. The calculation of intersections between parametric surfaces is a fundamental problem in Geometric Modelling. Contours represent one means of visualization of higher dimensional « surfaces ». Cubatures, that is, numerical multiple integrations, provide numerical solutions to problems such as the calculation of volumes and moments, which is useful in Geometric Modelling and elsewhere. The mathematical software and algorithms are illustrated by pictures from interactive computer graphics sessions.*

Résumé. — Traitement géométrique : intersections, lignes de niveaux et intégrations. *Le Traitement Géométrique est le calcul des propriétés géométriques de courbes, surfaces et volumes. Dans le présent article nous présentons des « fenêtres » de trois aspects du Traitement Géométrique : intersections, iso-contours et intégrations (« cubatures »). Le calcul d'intersections entre surfaces à représentation paramétrique est un des problèmes fondamentaux en modélisation géométrique. Les iso-contours représentent un moyen de visualiser des « surfaces » de dimension supérieure à 2. L'intégration ou « cubatures », c'est-à-dire l'évaluation numérique d'intégrales multiples, apporte des solutions numériques à des problèmes tels que le calcul de volumes et moments, utile en Modélisation Géométrique entre autres. Les logiciels mathématiques et les algorithmes sont illustrés à l'aide d'images de sessions d'informatique graphique interactive.*

1. INTRODUCTION

Computer Aided Geometric Design is the design and representation of curves, surfaces, and volumes in an interactive computer graphics environment. Geometry Processing is the determination of geometric properties of already constructed curves, surfaces, and volume. We shall discuss three topics of current interest in Geometry Processing :

- Intersections of parametric surfaces.
- Contours of bivariate and trivariate surfaces.
- Numerical integration.

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Each of these topics has many applications : calculation of the intersections of surfaces is needed for geometric modeling systems. Contours are utilized to display surfaces more understandably. Surface areas, volumes, and moments are required in many applications : these are obtained by means of numerical integration.

2. SURFACE/SURFACE INTERSECTION

The most often used type of surface in CAGD is parametric surfaces. We consider the intersection of parametric surfaces which are usualy networks of patches. Our surface/surface intersection (SSI) algorithm includes the following high-level steps :

- Obtain starting points.
- March along the intersection curve.
- Sort and connect the disjoint intersection segments.

Each of these steps includes many substeps. Principal ingredients include obtaining bounding boxes to detect non-intersecting patches and performing geometric subdivisions according to a variety of criteria depending on surface curvatures and patch edge linearity. [Barnhill and Kersey '90].

Examples are shown below.

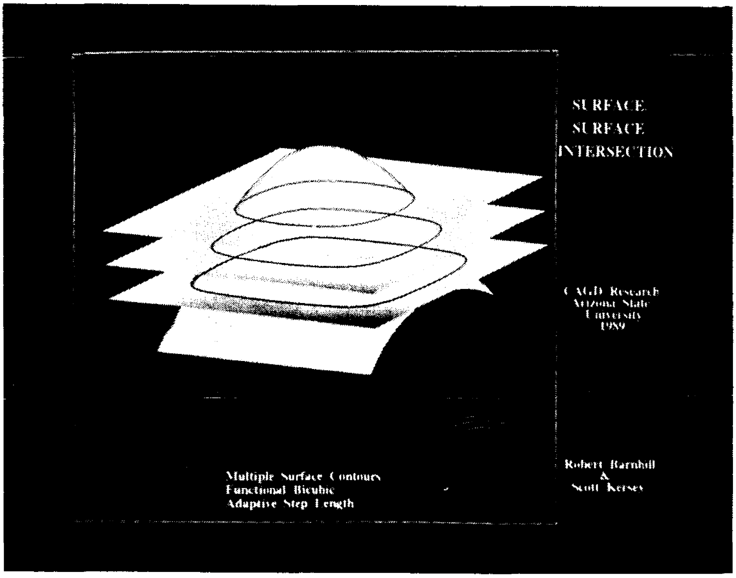


Figure 1. — Surface/surface intersection : a sculptured surface is intersected by four planes. The bottom diagrams show the trace of the intersection curves in the domains of the planes and of the sculptured surface.

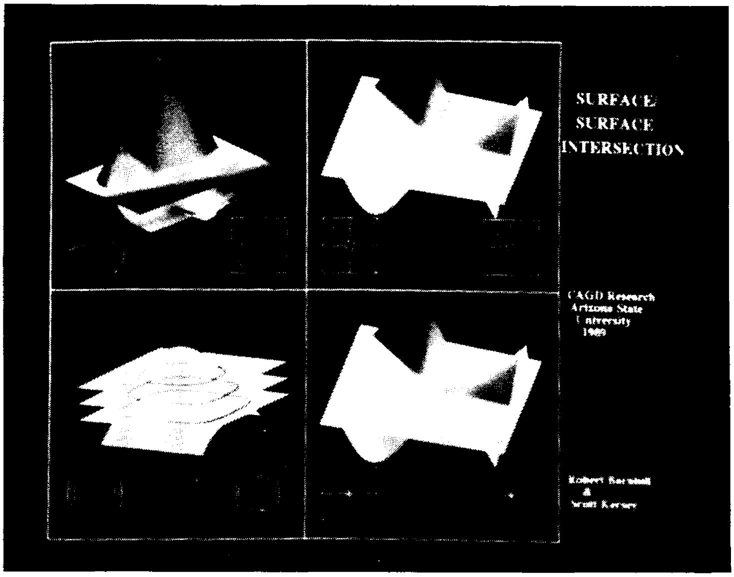


Figure 2. — Surface/surface intersection :
a variety of sculptured surface intersections is illustrated.

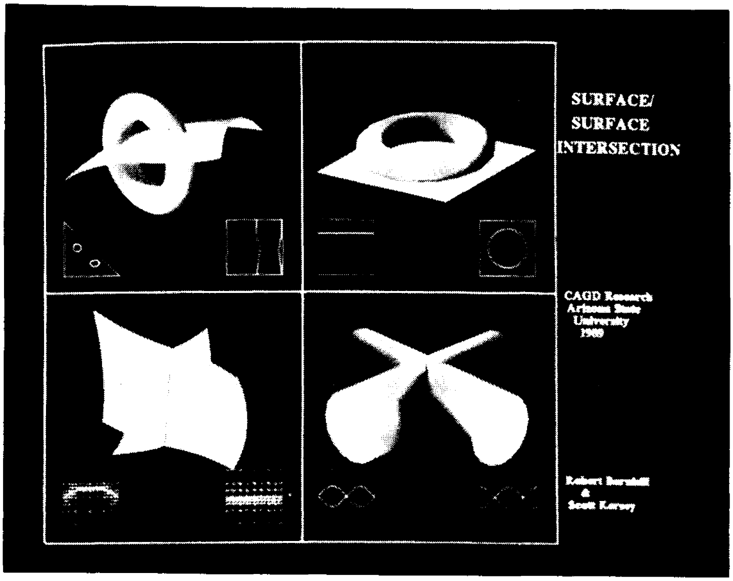


Figure 3. — Surface/surface intersection : examples include (counterclockwise from upper left) the intersection of a rectangular and triangular patch, of the selective subdivisions and of tangent track intersections.

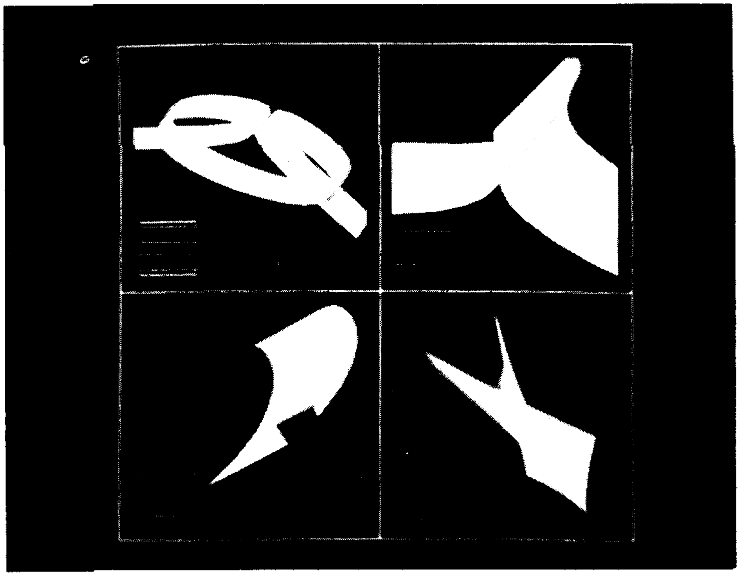


Figure 4. — Surface/surface intersection : tangent track intersections and three a self-intersecting surface.

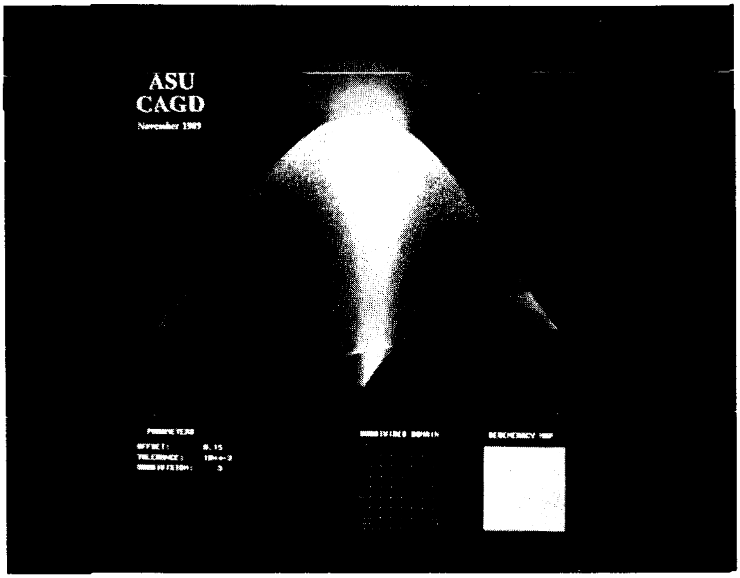


Figure 5. — Offset surfaces : a parametric surface and its offset are displayed. The lower left diagram indicates the amount of subdivision required for a given accuracy of approximation of the offset surface. The lower right diagram is a « degeneracy map » which indicates where self-intersections may occur.

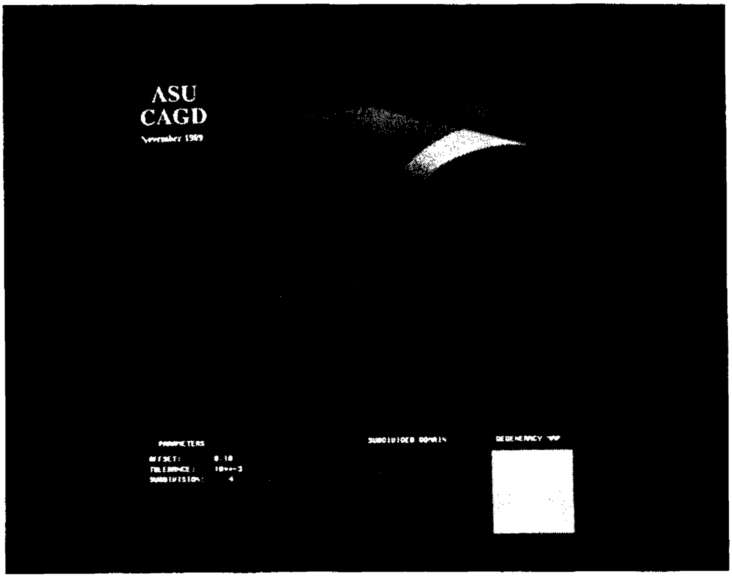


Figure 6. — Offset surfaces :
a second example of a parametric surfaces and its offset surface.

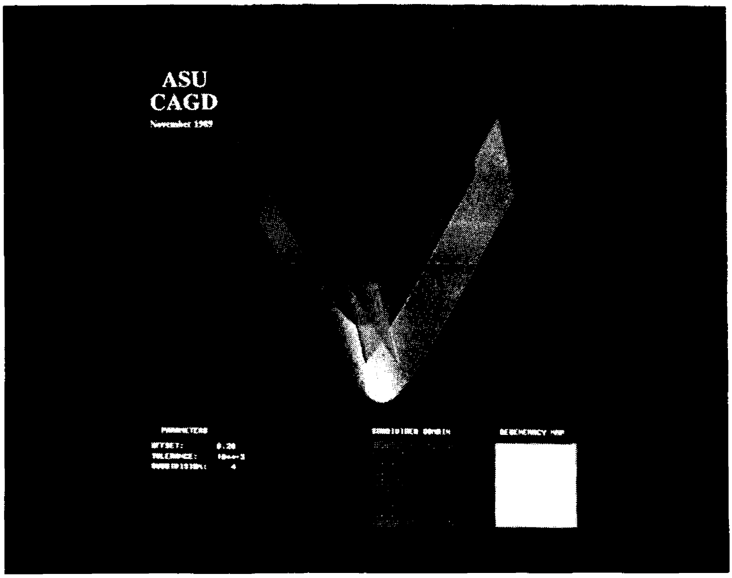


Figure 7. — Offset surfaces : a parametric surface and a self-intersecting offset surface.
The original figures are in color and some use transparency to see the parametric surface and its offset. The pictures are generated on interactive color graphics devices (Silicon Graphics IRIS workstations).

Self-intersecting surfaces arise in the subject of offset surfaces. If $\mathbf{r}(u, v)$ is a parametric surface, then its offset surface is given by $\mathbf{r}(u, v) + d\mathbf{n}$ where $\mathbf{n}$ is the surface normal of $\mathbf{r}$. If the offset distance d is too large compared to the minimum surface curvature, a self-intersection of the offset surface can occur.

For additional offset examples, see [Barnhill '89]. We are currently changing our method of approximation of the offset surface from Farouki's method [Farouki '86] based on tensor products to Piper's triangular interpolant method [Piper '87]. We shall report on these results in the future.

3. CONTOURS OF BIVARIATE AND TRIVARIATE SURFACES

I encouraged Leesa Brieger to compile a survey of contouring methods in 1980 [Brieger '80] as a first step in the subject of bivariate contouring. This was followed by Chip Petersen's thesis [Petersen '84] which developed new methods for trivariate contouring. Brett Bloomquist's just completed Masters thesis [Bloomquist '90] carries trivariate contouring further.

The basic idea of Petersen's and Bloomquist's applies to networks of polynomial patches as follows :

1. Find a patch that might contain the contour level.
2. If the polynomial patch is linear, contour it and go to 5. Else :
3. Reduce the degree and check that the lower degree polynomial is a suitable approximation. If so, go to 2. If not :
4. Subdivide the domain triangles while maintaining a triangulation. Go to 2.
5. Continue through the entire patch network.

The Bézier representation of polynomial interpolant helps as follows : bounding boxes to carry out step 1 are available from the Bézier polygon. Degree raising for Bézier representations is well known [Farin '90]. The inverse, degree reduction, is not uniquely defined of course, but a suitable least squares yields a good approximation. Finally, in generalizing the algorithm from bivariate interpolants over triangles to trivariate interpolants over tetrahedra, there are useful analogies.

There are at least two important differences between the bivariate and trivariate cases : 1) there are considerably fewer smooth tetrahedral interpolants, 2) there are more choices for splitting tetrahedra than for splitting triangles. Bloomquist concludes that splitting at the midpoint of the longest edge of the tetrahedra, with the concomitant changes to maintain a triangulation, is best. In addition to these mathematical considerations, there are interesting computer science implications in the sorting to maintain optimally large tetrahedra. By clever memory management Bloomquist reduced the execution time for substantial examples by 90 % !

We next display computer graphics illustrations of the trivariate contouring algorithm.

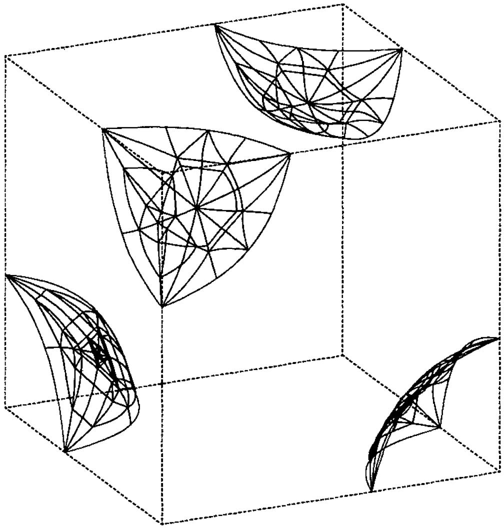


Figure 8. — Trivariate contouring : quadratic contour 0.03 of the function $F(x, y) = (x - 0.5)(y - 0.5)(z - 0.5)$. The test data are interpolated by a trivariate Clough-Tocher interpolant.

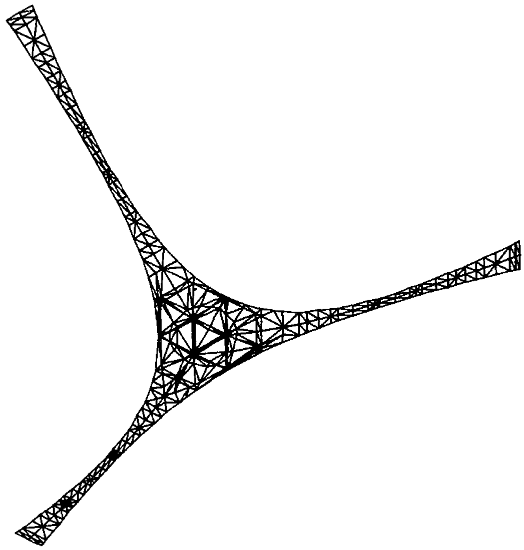


Figure 9. — Trivariate contouring : contour 0.45 of a quadratic. The contour level consists of 558 triangles.

There is a considerable future for contouring, both in research and application. One application of contouring is visualization : contouring reduces the dimension of the phenomenon visualized by one. This is particularly important for trivariate surfaces, since their contours, 3*D* surfaces, can be illustrated with interactive computer graphics.

4. CUBATURES

Numerical integration has a long and interesting history. The numerical integration of univariate functions, « quadrature », comes from the Greek *quadratos,* meaning to find a square with area equal to the area under a given curve. Numerical integration was the topic of my Ph. D. thesis [Barnhill '64], so I've had a long and continuing interest in it.

Several years ago we were asked by Lockheed to compute a singular integral for a potential theory problem. We used the current « off the shelf » cubature and obtained bad numbers. The algorithm was based upon integrating over a network of triangles. Frank Little had the ingenious idea of graphing the progressive subdivision of triangles on our (then — 1978 — new) Evans and Sutherland Picture System II. A snapshot of these triangulations is shown in figure 10.

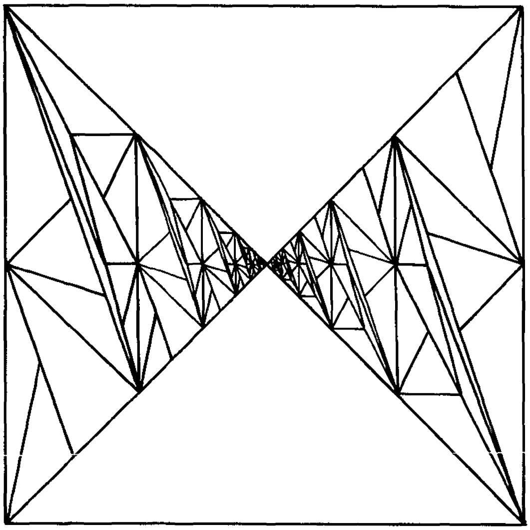
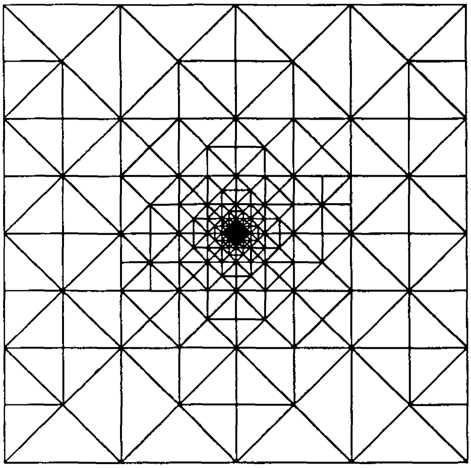


Figure 10. — Cubatures : An adaptive subdivision, based on the integrand only, applied to a singular integrand.

Given this motivation, we then devised an algorithm which considered both the integrand and the underlying geometry. We subdivided along the longest edge of the triangle with the largest error estimate [Barnhill and Little '84]. We used the « efficient » Radon 7 point precision 5 rule and the Cowper 13 point precision 7 rule as our basic bivariate rules.

Recently we enhanced this algorithm to solve additional types of bivariate problems as well as trivariate problems [Barnhill and Watson '89]. We used a 15 point precision 5 rule for the trivariate cubatures. More research is needed, particularly for adaptive trivariate cubatures.



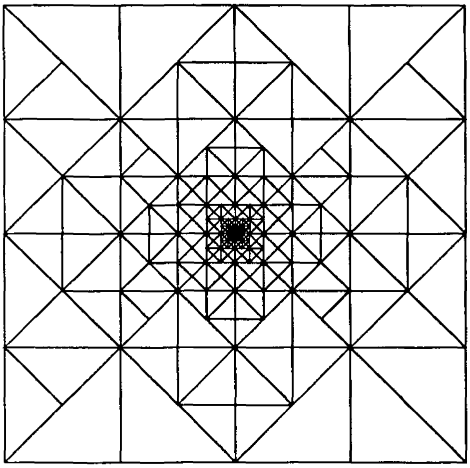
ADAPTIVE TRIANGULAR CUBATURE

Singular Problem e2=0.0001

7-5 Radon Rule

Longest Side Split
Cubature = -0.0000001192
Error Est = 0.00099842
Req Error = 0.00100000
Triangles = 428
Ftn Calls = 5978

Figure 11. — Cubatures : Our algorithm applied to the singular integrand. Basic rule : Radon.



ADAPTIVE TRIANGULAR CUBATURE

Singular Problem e2=0.0001

13-7 Cowper Rule

Longest Side Split
Cubature = -0.0104241100
Error Est = 0.00098963
Req Error = 0.00100000
Triangles = 264
Ftn Calls = 6838

Figure 12. — Cubatures : Our algorithm applied to the singular integrand. Basic rule : Cowper.

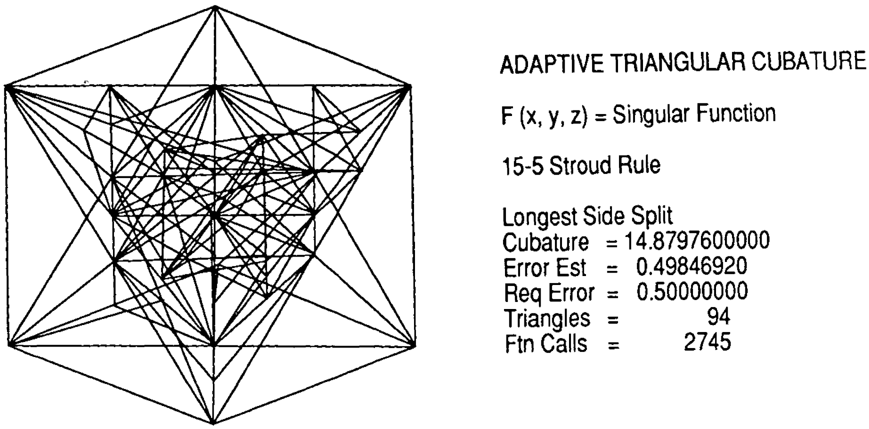


Figure 13. — Cubatures : Our trivariate algorithm applied to a singular integrand.

5. CONCLUSION

We have presented research, both old and new, on three topics of current interest in Computer Aided Geometric Design : intersections, contours and cubatures. All three subject areas have the common feature of providing both interesting theoretical questions and important applications.

ACKNOWLEDGMENTS

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