RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

N. DYN
D. LEVIN
I. YAD-SHALOM

Conditions for regular *B*-spline curves and surfaces

RAIRO – Modélisation mathématique et analyse numérique, tome 26, nº 1 (1992), p. 177-190.

<http://www.numdam.org/item?id=M2AN_1992__26_1_177_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(Vol. 26, n° 1, 1992, p. 177 à 190)

CONDITIONS FOR REGULAR B-SPLINE CURVES AND SURFACES

by N. DYN (1), D. LEVIN (1) and I. YAD-SHALOM (1)

Abstract. — *New sufficient conditions for the regularity of a B-spline curve are derived in terms of geometrical properties of the set of control points. These conditions exclude critical points and self-intersections in the curve, and are extendable to tensor-product B-spline surfaces.*

Résumé. — Conditions pour la régularité des courbes et surfaces B-spline. *De nouvelles conditions suffisantes pour la régularité d'une courbe B-spline sont obtenues en termes de propriétés géométriques de l'ensemble des points de contrôle. Ces conditions permettent d'exclure les points critiques et les points d'auto-intersections des courbes, et s'étendent à des surfaces produit tensoriels de B-spline.*

1. INTRODUCTION

A parametric B-spline curve in R^d, $d > 1$, where in practice $d = 2$ or $d = 3$, is defined by

$$S(t) = \sum_{i=1}^{n} P_i B_{i,k}(t), \quad k \leqslant t \leqslant n+1, \tag{1.1}$$

where $\{P_i\}_{i=1}^n \subset R^d$ and $B_{i,k}(t) = B_{0,k}(t-i)$ is the normalized uniform B-spline of order k (degree $k-1$) with $k+1$ knots : $i, i+1, \dots, i+k$.

Parametric B-spline curves are usually generated by binary subdivision schemes operating on the set of initial points $\{P_i\}_{i=1}^n$, thus $S(t)$ may be regarded as a limit curve of a subdivision scheme. For $k = 3$, $S(t)$ is obtained by the well known Chaikin's scheme, and for higher orders of k see Lane and Riesenfeld [3].

We define a curve with C^1 components as regular if it has a continuous unit tangent and if it has no self-intersections. Since $B_{i,k} \in C^{k-2}$, for $k \geqslant 3$ the components of $S(t)$ are C^1 functions but irregularities in $S(t)$ might occur.

(1) Beverly and Raymond Sackler Faculty of Exact Sciences, School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel.

M² AN Modélisation mathématique et Analyse numérique 0764-583X/92/01/177/14/$ 2.40

Clearly unit tangent discontinuities occur only in critical points, namely points where the tangent vector vanishes. It is of interest to note that if $S(t)$ is a non-degenerated polynomial curve in the neighbourhood of t, then the only possible unit tangent discontinuity is a cusp. This means that there are two tangents in opposite directions at t as is the case for the curve (t^3, t^2) at $t = 0$.

Each unit interval $[\ell, \ell + 1]$ with ℓ integer, $k \leqslant \ell \leqslant n$, is in the support of k B-splines

$$B_{\ell - k + 1, k}(t) \dots B_{\ell, k}(t)\,,$$

thus we have

$$S(t) = \sum_{i = \ell - k + 1}^{\ell} B_{i,k}(t)\, P_i\,, \quad t \in [\ell, \ell + 1]\,. \tag{1.2}$$

We show that if $\{P_i\}_{i = \ell - k + 1}^{\ell}$ satisfy the following regularity condition of order k then the curve segment in (1.2) has no self-intersections and has no critical points, thus it is a regular curve.

DEFINITION 1.1 : *A set of points* $Q := \{Q_j\}_{j=1}^{k}$ *satisfies the regularity condition of order* k, $k \geqslant 3$, *if*

$$\operatorname{conv}\left(\{Q_j\}_{j=1}^{r}\right) \cap \operatorname{conv}\left(\{Q_j\}_{j=r+1}^{k}\right) = \varnothing\,, \quad r \in W_k\,.$$

where

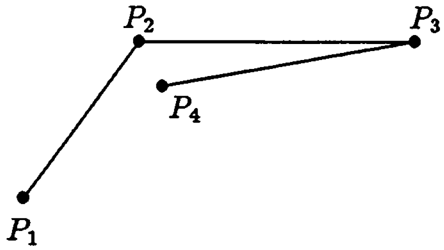
$$W_k = \begin{cases} \left\{\dfrac{k}{2}\right\}, & k \text{ even}\,, \\ \left\{\dfrac{k-1}{2}, \dfrac{k+1}{2}\right\}, & k \text{ odd}\,. \end{cases} \tag{1.3}$$

Here conv $(Q := \{Q_j\}_{j=1}^{\ell})$ *denotes the convex hull of* Q, *given by*

$$\operatorname{conv}(Q) = \left\{ \sum_{j=1}^{\ell} \alpha_j Q_j \;\middle|\; \sum_{j=1}^{\ell} \alpha_j = 1\,, \quad \alpha_j \geqslant 0\,, \quad j = 1, \dots, \ell \right\}. \tag{1.4}$$

In R^d, conv (Q) is a convex polytope and in particular it is a polygonal area in R^2 and a polyhedron in R^3. Since two convex polytopes in $R^3(R^2)$ have an empty intesection if and only if they are strictly separated by a plane (line), linear programming methods can be used to verify the regularity condition (see Edelsbruner [2]).

The regularity condition extends the results of Lau [4] where the control points are required to turn through an angle of at most π which is equivalent to the condition $0 \notin \operatorname{conv}\left(\{P_i - P_{i+1}\}_{i=1}^{k-1}\right)$. For example in the case $k = 4$ the following points are not accepted by Lau's condition, but accepted by the regularity condition since $\overline{P_1P_2}$ and $\overline{P_3P_4}$ are disjoint segments. Moreover, we show that any set of points satisfying Lau's condition also satisfies the regularity condition.



A full analysis of the case $k = 4$ can be found in Wang [6] and in Stone and DeRose [5]. Necessary and sufficient conditions for excluding self-intersections, critical points and inflection points are determined and efficient algorithms are presented.

Given a control polygon $\{P_i\}_{i=1}^{n}$, with each k successive points satisfying the regularity condition of order k, then the curve segments given in (1.2) are regular. However irregularities might occur if two curve segments intersect. While big loops are accepted in most applications, small loops in general have to be eliminated, and we show that if the regularity conditions of orders $k,\ k+1, \dots, k+\ell$ are imposed then $S(t_1) = S(t_2)$ implies $|t_1 - t_2| > \ell$. Notice that excluding $\{P_i\}_{i=1}^{k+\ell-1}$ and $\{P_i\}_{i=n-k-\ell+2}^{n}$ it suffices to verify only the regularity condition of order $k+\ell$.

In Section 3, the regularity of parametric tensor-product B-spline surfaces is discussed. Let $\{s_i(u, v)\}_{i=1}^{3}$ be a set of functions with continuous partial derivatives, then we define the surface $S(u, v) = (s_1(u, v), s_2(u, v), s_3(u, v))$ as regular if it has a non-vanishing normal and if it has no self-intersections. The normal may vanish only in case of a rank deficient Jacobian matrix and our sufficient conditions avoid this situation.

A tensor-product B-spline surface of order k is defined by

$$S(u, v) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} P_{ij}\, B_{i,k}(u)\, B_{j,k}(v)\,,$$

$$k \leq u \leq n_1 + 1\,, \quad k \leq v \leq n_2 + 1 \tag{1.5}$$

where $\{P_{ij}\}_{i=1\,j=1}^{n_1\,n_2} \subset R^3$.

On each unit square $[\ell_1, \ell_1 + 1] \times [\ell_2, \ell_2 + 1]$ with ℓ_1, ℓ_2 integers, $k \leq \ell_1 \leq n_1$, $k \leq \ell_2 \leq n_2$, $S(u, v)$ is given by

$$S(u, v) = \sum_{i=\ell_1-k+1}^{\ell_1} \sum_{j=\ell_2-k+1}^{\ell_2} P_{ij} B_{i,k}(u) B_{j,k}(v),$$

$$u \in [\ell_1, \ell_1 + 1], \quad v \in [\ell_2, \ell_2 + 1]. \tag{1.6}$$

We show that if $\{P_{ij}\}_{i=\ell_1-k+1}^{\ell_1} {}_{j=\ell_2-k+1}^{\ell_2}$ satisfy the following regularity condition of order (k, k) then the surface patch given by (1.6) is regular.

DEFINITION 1.2: *A set of points* $\{Q_{ij}\}_{i=1}^{k_1} {}_{j=1}^{k_2}$ *satisfies the surface regularity condition of order* (k_1, k_2) *if*

$$\text{conv}\,(\{P_{ij}\}_{i=1}^{r_1} {}_{j=1}^{k_2}) \cap \text{conv}\,(\{P_{ij}\}_{i=r_1+1}^{k_1} {}_{j=1}^{k_2}) = \varnothing, \quad r_1 \in W_{k_1} \tag{1.7}$$

and

$$\text{conv}\,(\{P_{ij}\}_{i=1}^{k_1} {}_{j=1}^{r_2}) \cap \text{conv}\,(\{P_{ij}\}_{i=1}^{k_1} {}_{j=r_2+1}^{k_2}) = \varnothing, \quad r_2 \in W_{k_2} \tag{1.8}$$

and

$$\text{cone}\,(\{P_{\ell j} - P_{ij}\}_{i=1}^{r_1} {}_{\ell=r_1+1}^{k_1} {}_{j=1}^{k_2}) \cap$$

$$\cap\ \text{cone}\,(\{P_{i\ell} - P_{ij}\}_{j=1}^{r_2} {}_{\ell=r_2+1}^{k_2} {}_{i=1}^{k_1}) = \{0\}, \tag{1.9}$$

$$\forall r_1, r_2, \qquad r_1 \in W_{k_1}, \qquad r_2 \in W_{k_2}.$$

where W_{k_1} *and* W_{k_2} *are defined by* (1.3)

The largest number of conditions occur when both k_1 and k_2 are odd, since (1.7) and (1.8) are two conditions each, and (1.9) includes four conditions. In case of even numbers k_1, k_2 then (1.7), (1.8) and (1.9) are one condition each. Here the cone of a set A is defined by

$$\text{cone}\,(A) = \{\alpha a \mid \alpha \in R, \quad a \in \text{conv}\,(A)\}. \tag{1.10}$$

Also, we define the chopped cone of a set A as $\{\alpha a \mid |\alpha| \leq 1, a \in \text{conv}\,(A)\}$. Note that if condition (1.9) is violated then the two cones intersect along a line through the origin. Thus it is sufficient to verify that the chopped cones corresponding to the cones in (1.9) have no intersection point except for the origin.

In analogy to the curve case, if all the regularity conditions of orders (i, j), $k \leq i \leq k + \ell_1$, $k \leq j \leq k + \ell_2$ are imposed then $S(u_1, v_1) = S(u_2, v_2)$ implies either $|u_1 - u_2| > \ell_1$ or $|v_1 - v_2| > \ell_2$.

Actually, condition (1.7) and (1.8) may be replaced by sharper ones (see Remark 3.6), but this improved regularity condition has no simple geometrical interpretation.

We conclude Section 3 by extending Lau's condition to tensor-product B-spline surfaces.

2. SELF-INTERSECTIONS AND CRITICAL POINTS OF B-SPLINE CURVES

Let $\{P_i\}_{i=1}^{k}$ be the control points then by (1.2):

$$S(t) = \sum_{i=1}^{k} B_{i,k}(t)\, P_i\,, \quad t \in [k, k+1]\,, \tag{2.1}$$

and the behavior of $B_{1,k}(t), \ldots, B_{k,k}(t)$ on $[k, k+1]$ is given by the following lemma.

LEMMA 2.1 : *If $k \geq 2$ is an even number then*

1. $B_{1,k}(t) \leq \cdots \leq B_{\frac{k}{2},k}(t)\,, \qquad t \in [k, k+1]$.
2. $B_{k,k}(t) \leq \cdots \leq B_{\frac{k}{2}+1,k}(t)\,, \qquad t \in [k, k+1]$.
3. $B_{1,k}(t), \ldots, B_{\frac{k}{2},k}(t)$ *are strictly decreasing on* $[k, k+1]$.
4. $B_{\frac{k}{2}+1,k}(t), \ldots, B_{k,k}(t)$ *are strictly increasing on* $[k, k+1]$.
5. $B_{\frac{k}{2},k}(t)$ *is a maximal B-spline on* $\left[k, k+\frac{1}{2}\right]$, $B_{\frac{k}{2}+1,k}$ *is maximal on* $\left[k+\frac{1}{2}, k+1\right]$.

If $k \geq 2$ is an odd number then

1. $B_{1,k}(t) \leq \cdots \leq B_{\frac{k+1}{2},k}(t), \qquad t \in [k, k+1]$.
2. $B_{k,k}(t) \leq \cdots \leq B_{\frac{k+1}{2},k}(t), \qquad t \in [k, k+1]$.
3. $B_{1,k}(t), \ldots, B_{\frac{k-1}{2},k}(t)$ *are strictly decreasing on* $[k, k+1]$.
4. $B_{\frac{k+3}{2},k}(t), \ldots, B_{k,k}(t)$ *are strictly increasing on* $[k, k+1]$.
5. $B_{\frac{k}{2},k}(t)$ *is a maximal B-spline on* $\left[k, k+\frac{1}{2}\right]$, $B_{\frac{k}{2}+1,k}$ *is maximal on* $\left[k+\frac{1}{2}, k+1\right]$.

Proof: Since $B_{i,k}(t) = B_{1,k}(t-i+1)$ and $B_{1,k}(t)$ is a strictly increasing function on the left half of its support and a strictly decreasing function on the right half of its support, the proof is completed. □

The following lemma provides a sufficient condition for a non-empty intersection of two polytopes. The proof of the lemma is trivial.

LEMMA 2.2 : *Given* $Q := \{Q_i\}_{i=1}^{r_1} \subset R^d$, $\tilde{Q} := \{\tilde{Q}_j\}_{j=1}^{r_2} \subset R^d$ *and non-negative coefficients* $\{\alpha_i\}_{i=1}^{r_1}$, $\{\tilde{\alpha}_j\}_{j=1}^{r_2}$ *with*

$$\sum_{i=1}^{r_1} \alpha_i Q_i = \sum_{j=1}^{r_2} \tilde{\alpha}_j \tilde{Q}_j \qquad \text{and} \qquad \sum_{i=1}^{r_1} \alpha_i = \sum_{j=1}^{r_2} \tilde{\alpha}_j \neq 0$$

then $\operatorname{conv}(Q) \cap \operatorname{conv}(\tilde{Q}) \neq \varnothing$.

THEOREM 2.3 : *Let* $\{P_i\}_{i=1}^{k}$, $k \geqslant 3$ *satisfy the regularity condition of Definition* 1.1, *then* $S(t)$ *as defined by* (2.1) *has no critical points.*

Proof: Using the differentiation formula for uniform B-splines ([1], p. 138)

$$B'_{i,k}(t) = B_{i,k-1}(t) - B_{i+1,k-1}(t)\,, \tag{2.2}$$

together with (2.1), we get

$$S'(t) = \sum_{i=1}^{k} \left(B_{i,k-1}(t) - B_{i+1,k-1}(t)\right) P_i\,, \quad k \leqslant t \leqslant k+1\,. \tag{2.3}$$

Let t_1 be a critical point then by (2.3)

$$\sum_{i=1}^{r} \left(B_{i+1,k-1}(t_1) - B_{i,k-1}(t_1)\right) P_i = $$
$$= \sum_{i=r+1}^{k} \left(B_{i,k-1}(t_1) - B_{i+1,k-1}(t_1)\right) P_i\,, \tag{2.4}$$

for any $1 \leqslant r \leqslant k$. Now, since $B_{1,k-1}(t)$ and $B_{k+1,k-1}(t)$ vanish on $[k, k+1]$, the coefficients on both sides sum up to $B_{r+1,k-1}(t_1)$. Let k be even, then in view of Lemma 2.1, $B_{\frac{k}{2}+1,k-1}(t_1) > 0$, is a maximal B-spline. The choice $r = \frac{k}{2}$ together with Lemma 2.1 implies that all the coefficients on both sides of (2.4) are non-negative, and sum up to $B_{\frac{k}{2}+1,k,k-1}(t_1) > 0$. Thus (2.4) satisfies the conditions of Lemma 2.2 and therefore

$$\operatorname{conv}\left(\{P_i\}_{i=1}^{k/2}\right) \cap \operatorname{conv}\left(\{P_i\}_{i=\frac{k}{2}+1}^{k}\right) \neq \varnothing\,,$$

which contradicts the regularity condition. The proof with k odd is analogous but we have to consider two choices of r, each contradicting the regularity condition. □

THEOREM 2.4 : *Let* $\{P_i\}_{i=1}^{k}$, $k \geqslant 3$ *satisfy the regularity condition, then* $S(t)$ *as defined by* (2.1) *has no self-intersections.*

Proof: Let t_1, $t_2 \in [k, k+1]$ with $t_1 < t_2$ satisfy $S(t_1) = S(t_2)$. In view of (2.1)

$$\sum_{i=1}^{k} P_i (B_{i,k}(t_2) - B_{i,k}(t_1)) = 0 , \tag{2.5}$$

which implies

$$\sum_{i=1}^{r} P_i (B_{i,k}(t_1) - B_{i,k}(t_2)) = \sum_{i=r+1}^{k} P_i (B_{i,k}(t_2) - B_{i,k}(t_1)) . \tag{2.6}$$

Since $\sum_{i=1}^{k} B_i(t_1) = \sum_{i=1}^{k} B_i(t_2) = 1$, the scalar coefficients in (2.5) sum up to zero and the sum of the coefficients on both sides of (2.6) is the same. Let k be even, then the choice $r = \frac{k}{2}$ together with Lemma 2.1 implies that all the coefficients on both sides of (2.6) are positive. Thus by Lemma 2.2

$$\operatorname{conv} (\{P_i\}_{i=1}^{k/2}) \cap \operatorname{conv} \left(\{P_i\}_{i=\frac{k}{2}+1}^{k} \right) \neq \varnothing ,$$

which contradicts the regularity condition.

The proof for k odd is analogous but two choices of r are required, each contradicting the regularity condition. □

COROLLARY 2.5 : *If the control points* $\{P_i\}_{i=1}^{k}$, $k \geqslant 3$ *satisfy the regularity condition, then the curve* $S(t)$ *defined by* (2.1) *is regular.*

In the following we discuss intersections of two curve segments of order k. Let $\{P_i\}_{i=1}^{k+\ell}$, $\ell \geqslant 1$ be the initial control points then by (1.1)

$$S(t) = \sum_{i=1}^{k+\ell} P_i \, B_{i,k}(t) , \qquad k \leqslant t \leqslant k + \ell + 1 , \tag{2.7}$$

and consider the two curve segments corresponding to $t \in [k, k+1]$ and $t \in [k+\ell, k+\ell+1]$, where $\ell > 0$ is an integer.

LEMMA 2.6 : *Let* $S(t)$ *be defined by* (2.7) *and let* t_1, t_2 *satisfy* $t_1 \in [k, k+1]$, $t_2 \in [k+\ell, k+\ell+1]$ *then there exists* $r \in W_{k+\ell}$ *such that*

$$B_{j,k}(t_1) \geqslant B_{j,k}(t_2) \qquad j = 1 \ldots r , \tag{2.8}$$

$$B_{j,k}(t_2) \geqslant B_{j,k}(t_1) \qquad j = r+1 \ldots k+\ell , \tag{2.9}$$

where W_k *is defined by* (1.3).

Proof: For each pair $t_1,\ t_2$ the existence of a positive integer r, for which (2.8), (2.9) hold, follows from the fact that the derivative of a B-spline function has one strong sign change. To computer r assume that $k+\ell$ is even. Then $B_{\frac{k+\ell}{2},k}(t)$ obtains its maximal value at $t = k + \frac{\ell}{2}$ and is symmetric about this point. Now since $\left|k + \frac{\ell}{2} - t_1\right| \leqslant \left|k + \frac{\ell}{2} - t_2\right|$ it follows that $B_{\frac{k+\ell}{2},k}(t_1) \geqslant B_{\frac{k+\ell}{2},k}(t_2)$ and by a similar argument $B_{\frac{k+\ell}{2}+1,k}(t_1) \leqslant B_{\frac{k+\ell}{2}+1,k}(t_2)$. The proof of the case where $k+\ell$ is odd is analogous. □

The following theorem is a direct consequence of Lemma 2.6. The proof is similar to the proof of Theorem 2.4.

THEOREM 2.7 : *Let $S(t)$ be defined by* (2.7), $3 \leqslant k$ *and let* $\{P_i\}_{i=1}^{k+\ell}$ *satisfy the regularity condition of order* $k+\ell$, *then the curve segments corresponding to* $\{P_i\}_{i=1}^{k}$ *and* $\{P_i\}_{i=\ell+1}^{k+\ell}$ *do not intersect.*

We conclude this section showing that the regularity condition is stronger than Lau's condition.

THEOREM 2.8 : *Let* $\{P_i\}_{i=1}^{k}$, $k \geqslant 3$ *satisfy Lau's condition then* $\{P_i\}_{i=1}^{k}$ *satisfies the regularity condition.*

Proof: For $k=3$ the conditions are equal, thus we consider $k>3$. Assume in contradiction that $\{P_i\}_{i=1}^{k}$ does not satisfy the regularity condition then there exists $1<r<k$ and non-negative coefficients $\{\alpha_i\}_{i=1}^{k}$ such that

$$\sum_{i=1}^{r} \alpha_i P_i = \sum_{i=r+1}^{k} \alpha_i P_i\,, \quad \sum_{i=1}^{r} \alpha_i = \sum_{i=r+1}^{k} \alpha_i = 1\,. \tag{2.10}$$

By (2.10) we get

$$\begin{aligned} \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)\,(P_2 - P_3) + \cdots + \\ + (\alpha_1 + \cdots + \alpha_{r-1})\,(P_{r-1} - P_r) + \\ + (\alpha_1 + \cdots + \alpha_r)\,P_r = (\alpha_{r+1} + \cdots + \alpha_k)\,P_{r+1} + \\ + (\alpha_{r+2} + \cdots + \alpha_k)\,(P_{r+2} - P_{r+1}) + \cdots + \\ + (\alpha_{k-1} + \alpha_k)\,(P_{k-1} - P_{k-2}) + \alpha_k(P_k - P_{k-1}) \end{aligned} \tag{2.11}$$

which implies

$$0 \in \operatorname{conv}\left(\{P_i - P_{i+1}\}_{i=1}^{k-1}\right) \tag{2.12}$$

contradicting Lau's condition. □

3. REGULAR TENSOR PRODUCT B-SPLINE SURFACES

In the following we discuss the regularity of a tensor-product B-spline patch given by

$$S(u, v) = \sum_{i=1}^{k} \sum_{j=1}^{k} P_{ij} B_{i,k}(u) B_{j,k}(v), \quad (u, v) \in [k, k+1]^2, \quad (3.1)$$

where $k \geq 3$, $\{P_{ij}\}_{i=1}^{k}{}_{j=1}^{k} \subset R^3$, and $\{P_{ij}\}_{i=1}^{k}{}_{j=1}^{k}$ satisfy the regularity condition given by Definition 1.2.

Given two sets X, Y, we define

$$X - Y = \{x - y \mid x \in X, y \in Y\} \quad (3.2)$$

and we make use of the following lemma.

LEMMA 3.1 : *Given two sets* $P = \{P_i\}_{i=1}^{n}$, $Q = \{Q_j\}_{j=1}^{m}$ then

$$\operatorname{conv}(P) - \operatorname{conv}(Q) = \operatorname{conv}(\{P_i - Q_j \mid i = 1, \dots, n, j = 1, \dots, m\}). \quad (3.3)$$

Proof: Obviously $\operatorname{conv}(P) - \operatorname{conv}(Q)$ is a convex set which contains $P_i - Q_j$, $i = 1, \dots, n$, $j = 1, \dots, m$ hence

$$\operatorname{conv}(\{P_i - Q_j \mid i = 1, \dots, n, j = 1, \dots, m\}) \subset \operatorname{conv}(P) - \operatorname{conv}(Q).$$

To see the inverse inclusion let $\{\alpha_i\}_{i=1}^{n}$, $\{\beta_j\}_{j=1}^{m}$ be non-negative numbers satisfying $\sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{m} \beta_j = 1$. Then

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i P_i - \sum_{j=1}^{m} \beta_j Q_j &= \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{m} \beta_j\right) P_i - \sum_{j=1}^{m} \beta_j \left(\sum_{i=1}^{n} \alpha_i\right) Q_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j (P_i - Q_j). \qquad \square\end{aligned}$$

THEOREM 3.2 : *Let* $\{P_{ij}\}_{i=1}^{k}{}_{j=1}^{k}$, $k \geq 3$ *satisfy the regularity condition of order* (k, k) *and let* $S(u, v)$ *be defined by* (3.1) *then the Jacobian matrix.*

$$\left(\frac{\partial S}{\partial u}, \frac{\partial S}{\partial v}\right) = \begin{pmatrix} \frac{\partial S_1}{\partial u} & \frac{\partial S_1}{\partial v} \\ \frac{\partial S_2}{\partial u} & \frac{\partial S_2}{\partial v} \\ \frac{\partial S_3}{\partial u} & \frac{\partial S_3}{\partial v} \end{pmatrix}, \quad (u, v) \in [k, k+1]^2 \quad (3.4)$$

is of rank 2.

Proof: In view of (2.2) we have

$$\frac{\partial S}{\partial u}(u, v) = \sum_{i=1}^{k} \sum_{j=1}^{k} P_{ij} B_{j,k}(v) \left(B_{i,k-1}(u) - B_{i+1,k-1}(u)\right)$$

$$\frac{\partial S}{\partial u}(u, v) = \sum_{i=1}^{k} \sum_{j=1}^{k} P_{ij} B_{i,k}(u) \left(B_{j,k-1}(v) - B_{j+1,k-1}(v)\right) \quad (3.5)$$

and by defining

$$F_i(v) = \sum_{j=1}^{k} P_{ij} B_{j,k}(v)$$

$$G_j(u) = \sum_{i=1}^{k} P_{ij} B_{i,k}(u), \quad (3.6)$$

we get

$$\frac{\partial S}{\partial u}(u, v) = \sum_{i=1}^{k} F_i(v) \left(B_{i,k-1}(u) - B_{i+1,k-1}(u)\right)$$

$$\frac{\partial S}{\partial v}(u, v) = \sum_{j=1}^{k} G_j(u) \left(B_{j,k-1}(v) - B_{j+1,k-1}(v)\right) \quad (3.7)$$

and in analogy to Theorem 2.3 it is clear that (1.7) implies $\frac{\partial S}{\partial u}(u, v) \neq 0$ and (1.8) implies $\frac{\partial S}{\partial v}(u, v) \neq 0$.

In order to establish the linear independence of $\frac{\partial S}{\partial u}$ and $\frac{\partial S}{\partial v}$ we rearrange (3.5) in the following form

$$\frac{\partial S}{\partial u}(u, v) = \sum_{j=1}^{k} B_{j,k}(v) \hat{U}_j(u), \quad (3.8)$$

where

$$\hat{U}_j(u) = \sum_{\ell=r_1+1}^{k} P_{\ell j}\left(B_{\ell,k-1}(u) - B_{\ell+1,k-1}(u)\right) -$$

$$- \sum_{i=1}^{r_1} P_{ij}\left(B_{i+1,k-1}(u) - B_{i,k-1}(u)\right). \quad (3.9)$$

Here, as in the curve case, there exists $r_1 \in W_k$ such that the expression (3.9) is of the form

$$\hat{U}_j(u) = \sum_{\ell=r_1+1}^{k} \alpha_\ell P_{\ell j} - \sum_{i=1}^{r_1} \beta_i P_{ij},$$

where

$$\alpha_\ell \geqslant 0\,, \quad \beta_i \geqslant 0\,, \qquad \sum_{\ell=r_1+1}^{k} \alpha_\ell = \sum_{i=1}^{r_1} \beta_i \neq 0 \tag{3.10}$$

and there exists a positive constant c independent of j with

$$c\,.\, \hat{U}_j(u) \in \operatorname{conv}\left(\{P_{\ell j}\}_{\ell=r_1+1}^{k}\right) - \operatorname{conv}\left(\{P_{ij}\}_{i=1}^{r_1}\right). \tag{3.11}$$

By Lemma 3.1 we get

$$c\,.\, \hat{U}_j(u) \in \operatorname{conv}\left(\{P_{\ell j} - P_{ij}\}_{i=1}^{r_1}\,{}_{\ell=r_1+1}^{k}\,{}_{j=1}^{k}\right)$$

and by (3.8)

$$\frac{\partial S}{\partial u}(u, v) \in \operatorname{cone}\left(\{P_{\ell j} - P_{ij}\}_{i=1}^{r_1}\,{}_{\ell=r_1+1}^{k}\,{}_{j=1}^{k}\right). \tag{3.12}$$

By a similar argument we get for $r_2 \in W_k$

$$\frac{\partial S}{\partial v}(u, v) \in \operatorname{cone}\left(\{P_{i\ell} - P_{ij}\}_{j=1}^{r_2}\,{}_{\ell=r_2+1}^{k}\,{}_{i=1}^{k}\right)$$

and (1.9) guarantees the linear independence. □

THEOREM 3.3 : *Let* $\{P_{ij}\}_{i=1}^{k}\,{}_{j=1}^{k}$, $k \geqslant 3$ *satisfy the regularity condition of order* (k, k) *and let* $S(u, v)$ *be defined by* (3.1) *then* $S(u, v)$ *has no self-intersections.*

Proof: Let $u_1 \neq u_2$, $v_1 \neq v_2$ *then*

$$S(u_1, v) - S(u_2, v) = \sum_{i=1}^{k} F_i(v)\,\left(B_{i,k}(u_1) - B_{i,k}(u_2)\right)$$

$$S(u, v_1) - S(u, v_2) = \sum_{j=1}^{k} G_j(u)\,\left(B_{j,k}(v_1) - B_{i,k}(v_2)\right), \tag{3.13}$$

where $F_i(v)$ and $G_j(u)$ are given by (3.6). In analogy to Theorem 2.4, (1.7) implies $S(u_1, v) \neq S(u_2, v)$, and (1.8) implies $S(u, v_1) \neq S(u, v_2)$.

Assume in contradiction

$$S(u_1, v_1) = S(u_2, v_2)\,, \tag{3.14}$$

then

$$S(u_1, v_1) - S(u_2, v_1) = S(u_2, v_2) - S(u_2, v_1) \tag{3.15}$$

which implies

$$\sum_{i=1}^{k} \sum_{j=1}^{k} P_{ij} B_{j,k}(v_1) (B_{i,k}(u_1) - B_{i,k}(u_2)) =$$
$$= \sum_{i=1}^{k} \sum_{j=1}^{k} P_{ij} B_{i,k}(u_2) (B_{j,k}(v_2) - B_{j,k}(v_1)). \quad (3.16)$$

The left side of (3.16) is an expression of the form

$$\sum_{j=1}^{k} B_{j,k}(v_1) \tilde{U}_j(u_1, u_2) \quad (3.17)$$

where

$$\tilde{U}_j(u_1, u_2) = \sum_{i=1}^{k} P_{ij}(B_{i,k}(u_1) - B_{i,k}(u_2)) \quad (3.18)$$

and in analogy to Theorem 3.2 there exists a constant $c \neq 0$ independent of j satisfying

$$c \cdot \tilde{U}_j(u_1, u_2) \in \operatorname{conv}(\{P_{\ell j}\}_{\ell=r_1+1}^{k}) - \operatorname{conv}(\{P_{ij}\}_{i=1}^{r_1}). \quad (3.19)$$

Hence

$$S(u_1, v_1) - S(u_2, v_1) \in \operatorname{cone}(\{P_{\ell j} - P_{ij}\}_{i=1}^{r_1}{}_{\ell=r_1+1}^{k}{}_{j=1}^{k}), \quad (3.20)$$

and by similar arguments

$$S(u_2, v_2) - S(u_2, v_1) \in \operatorname{cone}(\{P_{i\ell} - P_{ij}\}_{j=1}^{r_2}{}_{\ell=r_2+1}^{k}{}_{i=1}^{k}), \quad (3.21)$$

contradicting (1.9). □

COROLLARY 3.4 : *Let* $\{P_{ij}\}_{i=1}^{k}{}_{j=1}^{k}$, $k \geq 3$ *satisfy the regularity condition of Definition* 1.2, *then* $S(u, v)$ *as defined by* (3.1) *is a regular surface.*

The following Theorem is the analog of Theorems 2.7.

THEOREM 3.5 : *Let* $S(u, v)$ *be defined by*

$$S(u, v) = \sum_{i=1}^{k+\ell_1} \sum_{j=1}^{k+\ell_2} P_{ij} B_{i,k}(u) B_{j,k}(v),$$
$$k \leq u \leq k + \ell_1 + 1, k \leq v \leq k + \ell_2 + 1. \quad (3.22)$$

Let $\{P_{ij}\}_{i=1}^{k+\ell_1}{}_{j=1}^{k+\ell_2}$ *satisfy the regularity condition of order* $(k + \ell_1, k + \ell_2)$

then the patches corresponding to $\{P_{ij}\}_{i=1}^{k}{}_{j=1}^{k}$ *and* $\{P_{ij}\}_{i=\ell_1+1}^{k+\ell_1}{}_{j=\ell_2+1}^{k+\ell_2}$ *do not intersect.*

Proof: The proof of this theorem uses the same arguments as in Theorems 2.7 and 3.3. □

Note, that as in the curve case, given a set of points $\{P_{ij}\}_{i=1}^{n_1}{}_{j=1}^{n_2}$ with n_1, n_2 sufficiently large then except for a boundary layer of width consisting of $k+\ell-1$ points, it is sufficient to impose the regularity conditions of order $(k+\ell, k+\ell)$ on all control points, in order to guarantee that $S(u_1, v_1) = S(u_2, v_2)$ implies either $|u_1 - u_2| > \ell$ or $|v_1 - v_2| > \ell$.

Remark 3.6 : A close examination of Theorems 3.2, 3.3 and 3.5 shows that the analysis holds even if we improve the conditions (1.7) and (1.8). These new conditions allow a non-empty intersection of the convex-hulls but the matrix of the coefficients has to be of rank > 1. The improved (1.7) condition of order (k_1, k_2) is contradicted only if there exists a non-negative matrix $\{\alpha_{ij}\}_{i=1}^{k_1}{}_{j=1}^{k_2}$ of rank one and there exists $r_1 \in W_{k_1}$ such that $\sum_{i=1}^{r_1}\sum_{j=1}^{k_2}\alpha_{ij} = \sum_{i=r_1+1}^{k_1}\sum_{j=1}^{k_2}\alpha_{ij} = 1$, and $\sum_{i=1}^{r_1}\sum_{j=1}^{k_2}\alpha_{ij}P_{ij} = \sum_{i=r_1+1}^{k_1}\sum_{j=1}^{k_2}\alpha_{ij}P_{ij}$.

In the following we extend Lau's condition to the surface case. First we introduce the sets

$$\begin{aligned}\Delta_u P &= \operatorname{conv}\left(\{P_{ij} - P_{i-1,j} \mid i = 2 \dots k, j = 1 \dots k\}\right)\\ \Delta_v P &= \operatorname{conv}\left(\{P_{ij} - P_{i,j-1} \mid i = 1 \dots k, j = 2 \dots k\}\right).\end{aligned} \tag{3.23}$$

THEOREM 3.7 : *Let* $S(u, v)$ *be defined by* (3.1) *with* $k \geq 3$. *Assume*

$$0 \notin \Delta_u P, \quad 0 \notin \Delta_v P \tag{3.24}$$

$$\operatorname{cone}(\Delta_u P) \cap \operatorname{cone}(\Delta_v P) = \{0\} \tag{3.25}$$

then (3.1) *is a regular surface.*

Proof: Obviously

$$\begin{aligned}\operatorname{cone}\left(\{P_{\ell j} - P_{ij}\}_{i=1}^{r_1}{}_{\ell=r_1+1}^{k}{}_{j=1}^{k}\right) &\subset \operatorname{cone}(\Delta_u P)\\ \operatorname{cone}\left(\{P_{i\ell} - P_{ij}\}_{j=1}^{r_2}{}_{\ell=r_2+1}^{k}{}_{i=1}^{k}\right) &\subset \operatorname{cone}(\Delta_v P)\end{aligned} \tag{3.26}$$

thus (3.25) implies (1.9). By arguments similar to Theorem 2.8 it follows that (3.24) implies the improved conditions (1.7) and (1.8) stated in Remark 3.6. □

Remark 3.8 : The analysis in this section holds also for the case of mixed-orders tensor-product splines.

REFERENCES

[1] C. DE BOOR (1978), A Practical Guide to Splines, Springer-Verlag.

[2] H. EDELSBRUNER (1987), Algorithms in combinatorial geometry, Springer-Verlag.

[3] J. M. LANE and R. F. RIESENFELD (1980), A theoretical development for the computer generation and display of piecewise polynomial surfaces, *IEEE T. Pattern Anal.* 2, 35-46.

[4] K. H. LAU (1988), Conditions for avoiding loss of geometric continuity on spline curves, *Comput. Aided Geom. Design.* 5, 209-214.

[5] C. M. STONE and T. DEROSE (1989), A geometric characterization of parametric cubic curvès, *ACM Trans. Graph.* 8, 147-163.

[6] C. Y. WANG (1981), Shape classification of the parametric cubic curve and parametric *B*-spline cubic curve, *Comput. Aided Design.* 13, 199-206.