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APPROXIMATION AND/OR CONSTRUCTION OF CURVES BY MINIMIZATION METHODS WITH OR WITHOUT CONSTRAINTS (*)

by M BERCOVIER (1) and A JACOBI (1)

Abstract — *This paper presents a global method for approximation and/or construction of planar curves The method is based on the minimization of a functional which describes approximation and differential geometric characteristics The functional includes weighting factors which are used to control the approximation process It is also possible to combine constraints upon the approximation/construction of curves, in order to achieve desired geometrical or physical effects The numerical solution of the functional takes full advantage of the Finite-Elements Method with Bezier shape functions*

Keywords Bézier curve, Offset curve, Approximate Conversion, Geometric Continuity, Variational Problem Formulation, Finite-Element Method (FEM), Uzawa method

Résume — Approximation et/ou construction de courbes par des méthodes de minimisation avec ou sans contraintes *Cet article présente une méthode globale pour l'approximation et/ou la construction de courbes planes Cette méthode est basée sur la minimisation d'une fonctionnelle qui décrit les caractéristiques de l'approximation et de la géométrie différentielle de la courbe La fonctionnelle comprend des facteurs de pondération utilisés pour le contrôle de l'approximation Il est également possible d'inclure des contraintes dans l'approximation ou la construction des courbes pour obtenir des propriétés géométriques ou physiques La résolution numérique de la fonctionnelle utilise pleinement les principes de la méthode des Éléments Finis sur des fonctions de base de Bézier*

1. INTRODUCTION

Approximation of curves and construction of offset curves have a variety of applications in geometric modeling For example, industry standard geometric modeling systems for free form curves and surfaces require a proper exchange format of data, as they use different mathematical

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representations for curves and surfaces, and different polynomial bases. An exact conversion between representations is possible in the case of degree elevation, or if the degrees and the number of terms are the same in both representations. Otherwise, approximate conversion is needed. Another case is when a conversion is needed between non-polynomial representations (such as conic sections or rational curves) to polynomial ones. Another motivation for approximation is the ability of merging curves and surfaces in order to reduce information. Construction of offset curves is needed in tool paths planning for numerical control machines, or to describe a thick surface (outer surface of a car or aircraft model design).

There are several methods for approximation of curves and surfaces. One of the first for actual approximate conversion is due to Hölzle [7]. This method was extended by Dannenberg and Nowacki [4] to surfaces by interpreting a surface as a grid of curves. Their method is based on a combination of Hermitian interpolation and least square approximation. Another approach to the approximation problem (as well as the construction of offset curves) using Bézier curves, was presented by Hoschek [8, 9, 10, 11, 12]. It is a discrete method in which transformations of parametrization and geometric continuity conditions are considered. Another method which creates a polynomial approximation that uses constrained Chebyshev polynomials or the orthogonal polynomial method is described by Goult [6] and Lachance [13]. Nowacki *et al.* [16] present another approach for construction of Bézier curves from given interpolation conditions, boundary conditions and area constraints, using a second derivative fairness criterion.

Our strategy is a global and continuous method for the approximation and construction of parametric planar curves. The method is based on a variational formulation which includes geometrical relations between curves, and constraints upon the geometry and/or parametrization of the approximated curve. The variation is based on the squared integrals of the zero, first and second derivative (semi) norms of the approximation and approximated curves. A weighting factor is related to each of the derivative (semi) norm. These weighting factors allow one to control the approximation of the related norm. The solution of this variational problem is done by the Finite Element Method (FEM).

2. THE PROBLEM AND ITS SOLUTION

2.1. Problem Statement

We will first define the problem without constraints, working only in two dimensional space. Given a parametric curve $\mathbf{f}(u) = (f_1(u), f_2(u))$, $u \in [a, b]$, find the unknown vector function $\mathbf{x}(u) = (x_1(u), x_2(u))$, $u \in [a, b]$, which minimizes the functional $J^{\ell}(x)$ (defined below).

Let

$$E(x) = \alpha \int_a^b (\mathbf{x}(u) - \mathbf{f}(u))^2 \, du \tag{1}$$

$$\bar{E}(x) = \beta \int_a^b (\mathbf{x}'(u) - \mathbf{f}'(u))^2 \, du \tag{2}$$

$$\hat{E}(x) = \gamma \int_a^b (\mathbf{x}''(u) - \mathbf{f}''(u))^2 \, du \tag{3}$$

be the zero, first and second error (semi) norms respectively, and α, β and γ positive moduli which are used as weighting factors.

We write

$$\begin{cases} J^0(x) = E(x) \\ J^1(x) = E(x) + \bar{E}(x) \\ J^2(x) = E(x) + \bar{E}(x) + \hat{E}(x) . \end{cases}$$

2.2. Solution for the Problem Using the FEM Technique

The Finite Element Method for analysis is a popular technique for solving complex problems in engineering or physics. A FEM problem is given by a functional and an approximation space [17]. The solution of a FEM problem is a stepwise process and includes the following steps :

- Subdivision of the problem range into elements
- Approximation of the solution for each element using shape functions such as Lagrange or Hermite ones (i.e., definition of the approximation space V)
- Creation of the element stiffness matrix for the shape function
- Assemblage of the element matrices into matrices that correspond to the complete finite-element system
- General numerical solution of the system
- Error estimation of the solution.

The solution to the problem (4) will be described in three stages :

2.2.1. *The Approximation FEM Space*

Given the partition $\{u_i\}_{i=0}^{m-1}$ of order m of the interval $[a, b]$,

$$a = u_0 < u_1 < u_2 \cdots < u_m = b \tag{4}$$

such that each subinterval $[u_i, u_{i+1}]$ is an *element*.

We introduce an approximation n-dimensional space $V^{\ell,n}$, consisting of functions which are piecewise C^0 Bézier curves over the interval $[a, b]$.

A planar Bézier curve C of degree q is

$$\mathbf{P}_q(t) \equiv \sum_{j=0}^{q} \mathbf{b}_j \, B_j^q(t) \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \tag{5}$$

The $\mathbf{b}_0, \ldots, \mathbf{b}_q$ are the unknown 2-dimensional Bézier points,

$$B_j^q(t) = \binom{q}{j} (1-t)^{q-j} \, t^j \tag{6}$$

are the Bernstein polynomials.

Let

$$\begin{aligned} V^{\ell,n} = \{\mathbf{x}(u) : & \\ & \mathbf{x}(a) = \mathbf{f}(a), \mathbf{x}(b) = \mathbf{f}(b), && \text{such that there is a } q \leqslant n \text{ with} \\ & \mathbf{x}(u)\big|_{[u_i, u_{i+1}]} = \mathbf{P}_q(t(u)) && \text{and for all } 0 \leqslant i \leqslant m-1, \text{ where} \\ & u(t) = u_i(1-t) + u_{i+1}\, t && \text{for } 0 \leqslant t \leqslant 1 \\ & && \text{and } \ell = 0, 1, 2 \,\} \end{aligned} \tag{7}$$

where

$$\mathbf{x}(u) \in C^{\ell-1}([a, b]) \tag{8}$$

and C^{-1} is the space of functions with discontinuities at u_i only.

Set

$$V^{\ell} = \overline{\bigcup_{n=1}^{\infty} V^{\ell, nJ_\ell}} \tag{9}$$

where V^{ℓ} is called the *minimization space* and $V^{\ell,n}$ is a finite-dimensional subspace of V^{ℓ}, $\ell = 0, 1, 2$.

With the partition (4) we write

$$E_i(x) = \alpha \int_{u_i}^{u_{i+1}} (\mathbf{x}(u) - \mathbf{f}(u))^2 \, du \tag{10}$$

$$\bar{E}_i(x) = \beta \int_{u_i}^{u_{i+1}} (\mathbf{x}'(u) - \mathbf{f}'(u))^2 \, du \tag{11}$$

$$\hat{E}_i(x) = \gamma \int_{u_i}^{u_{i+1}} (\mathbf{x}''(u) - \mathbf{f}''(u))^2 \, du \tag{12}$$

to be the zero, first and second error (semi) norms for the element i respectively.

Let

$$J^{\ell}(x(u)) = \sum_{i=0}^{m-1} J_i^{\ell}(x(u)) \tag{13}$$

where

$$J_i^{\ell}(x(u)) = J^{\ell}(x(u))\big|_{u \in [u_i, u_{i+1}]} . \tag{14}$$

Our objective is to find for all n the function $\mathbf{x}(u)$ which is taken over the space $V^{\ell, n}$, in order to approximate $\mathbf{f}(u)$ in some sense to be defined later.

Remark : Element subdivision is done according to the curvature behavior of the given curve $\mathbf{f}(u)$. This is done by an algorithm developed by Hoschek [12], which creates the subdivision into generic curves according to the minimal absolute values of the spline curvature (vertices, inflection points). Separation points are determined midway between these absolute values according to the degree of the generic curve. For example,

- a generic cubic curve has at least one minimal absolute value
- a generic quintic curve has at least three minimal absolute values.

Therefore, in the cubic case the splitting will be between each absolute values, and in the quintic case between the third and fourth absolute values.

Hoscheks' algorithm for subdivision introduces some problems in cases such as the Cornu spiral, in which it will take the whole spiral as two cubic segments, and leads to a further segmentation during the approximation process.

After the partition, each element represents an approximation curve segment which is a Bézier curve of any given degree (elements may have different degrees as in the *p-method*, see [1]).

2.2.2. *Creation of the Element Stiffness Matrix and the Load Vector for the Bernstein-Bézier Basis*

For each element of degree n, there are $2(n+1)$ degrees of freedom ($n+1$ for each component), except for the first and the last elements, where they have $2n$ degrees of freedom, since the values of the end points are predetermined.

We have,

$$J_i^{\ell}(x) = \left\{ \frac{1}{2} \mathbf{b}_i^T M_i \mathbf{b}_i - \mathbf{m}_i^T \mathbf{b}_i \right\} \quad \text{for} \quad 0 \leqslant i \leqslant m-1 \tag{15}$$

where for the i-th element, $\mathbf{b}_i \equiv [\mathbf{b}_{i_0}, \ldots, \mathbf{b}_{i_n}]^T$ is the vector of the unknown Bézier points, $\mathbf{m}_i$ is the load vector and M_i is the element stiffness matrix. M_i and $\mathbf{m}_i$ are calculated as follows :

The r-th derivatives of a Bézier curve (5) is

$$\frac{d^r}{dt^r}\mathbf{P}(t) = \sum_{j=0}^{n-r} \Delta^r \mathbf{b}_j \, B_j^{n-r}(t) \quad \text{for} \quad 0 \leqslant t \leqslant 1 \tag{16}$$

where

$$\Delta^r \mathbf{b}_j = \Delta^{r-1}\mathbf{b}_{j+1} - \Delta^{r-1}\mathbf{b}_j \, . \tag{17}$$

We denote

$$B^n \equiv [B_0^n, \ldots, B_n^n] \tag{18}$$

and using

$$\int_0^1 B_i^n \, dt = \frac{1}{n+1} \tag{19}$$

with

$$B_i^m(t)\, B_j^n(t) = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}} B_{i+j}^{m+n}(t)\,, \quad \text{for} \quad i = 0, \ldots, n\,; \quad j = 0, \ldots, m \tag{20}$$

one yields for a single element

$$A_{i,j}^n \equiv \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}}\,, \quad \text{for} \quad i, j = 0, \ldots, n\,. \tag{21}$$

Therefore we have

$$B_n^0 \equiv \int_0^1 B^n B^{n^T} dt = \frac{1}{2n+1} A^n \, .$$

Using these results we can evaluate the square integrals in equations (10)-(12)

$$\alpha \int_{u_i}^{u_{i+1}} \mathbf{x}(u)^2 \, du = \alpha \, \Delta u_i \int_0^1 \mathbf{x}(t)^2 \, dt = \alpha \, \Delta u_i \, \mathbf{b}_i^T \int_0^1 B^n B^{n^T} \mathbf{b}_i \, dt \equiv \equiv \alpha \, \Delta u_i \, \mathbf{b}_i^T B_n^0 \, \mathbf{b}_i \tag{22}$$

$$\beta \int_{u_i}^{u_{i+1}} \mathbf{x}'(u)^2 \, du = \beta \, \Delta u_i \int_0^1 \mathbf{x}'(t)^2 \, dt = \beta \, \frac{n^2}{\Delta u_i} \Delta^1 \mathbf{b}_i^T B_{n-1} \, \Delta^1 \mathbf{b}_i \, dt \equiv \equiv \beta \, \frac{n^2}{\Delta u_i} \mathbf{b}_i^T B_n^1 \, \mathbf{b}_i \tag{23}$$

$$\gamma \int_{u_i}^{u_{i+1}} \mathbf{x}''(u)^2 \, du = \gamma \, \Delta u_i \int_0^2 \mathbf{x}''(t)^2 \, dt = \gamma \frac{n^2(n-1)^2}{(\Delta u_i)^3} \Delta^2 \mathbf{b}_i^T B_{n-2} \Delta^2 \mathbf{b}_i \, dt \equiv$$

$$\equiv \gamma \frac{n^2(n-1)^2}{(\Delta u_i)^3} \mathbf{b}_i^T B_n^2 \mathbf{b}_i \, . \quad (24)$$

From results (22)-(24) one yields the *stiffness matrix,*

$$M_i = B_n^0 + B_n^1 + B_n^2 \, . \quad (25)$$

Since M_i is a constant matrix for each n, a large part of the stiffness matrices calculation can be done in advance for the available degrees of the approximation curves. What remains is to calculate scalar-matrix multiplications with constants determined by the element subdivision and to perform matrix additions.

The element load vector is calculated as follows :

$$\mathbf{m}_i(x) = 2 \left\{ \alpha \int_{u_i}^{u_{i+1}} (\mathbf{x}(u) \, . \, \mathbf{f}(u)) \, du + \beta \int_{u_i}^{u_{i+1}} (\mathbf{x}'(u) \, . \, \mathbf{f}'(u)) \, du + \right.$$

$$\left. + \gamma \int_{u_i}^{u_{i+1}} (\mathbf{x}''(u) \, . \, \mathbf{f}''(u)) \, du \right\} \quad (26)$$

where the evaluation of the integrals in (26) is done by Gaussian quadrature, adapted to the degree of the element.

2.2.3. *Properties of the Element Stiffness Matrix in Relation to the Bernstein-Bézier Basis*

The Bézier element mass matrix B_n^0 is a symmetrical, positive definite matrix, with the elements in the main diagonal greater than zero, all the elements are positive and the largest element per row or column is in the main diagonal, the sum of all elements of a row (or column) is a constant for each n, and is influenced by the length of the elements range, Δu_i [15].

The Bézier stiffness element matrices B_n^1 and B_n^2, are symmetrical, positive semi-definite matrices, with the elements in the main diagonal greater than zero, the largest absolute value per row or column is in the main diagonal and the sum of all elements of a row (or column) is zero.

2.2.4. *General Solution of the Problem*

After integration element by element we obtain :

$$\mathcal{J}^\ell(x) = \sum_{i=0}^{m-1} \mathcal{J}_i^\ell(x_i) = \left\{ \frac{1}{2} \mathbf{b}^T M \mathbf{b} - \mathbf{m}^T \mathbf{b} \right\} \quad \ell = 0, 1, 2 \, . \quad (27)$$

The elements' stiffness matrices M_i, and the vector $\mathbf{m}_i$, $(i = 0, ..., m-1)$ are assembled into the global stiffness matrix M and the global load vector $\mathbf{m}$. Since we have divided the range into m elements, then for C^0 continuity between elements there are at most $2\,mn - 2$ degrees of freedom.

The global stiffness matrix is a square band and symmetric, and its order is at most $2\,mn - 2$. The matrix band width is at most $2\,n$.

The minimum of (27) is given by the approximation curve $\mathbf{x}(u)$ (5), (7), where $\mathbf{b}$ is the solution of the system

$$\nabla_b J^{\ell}(x) = M\mathbf{b} + \mathbf{m} = 0\,. \tag{28}$$

The system (28) is linear symmetric positive definite, and we use the LDL^T algorithm [17] to solve it. The coordinate components of $J^{\ell}(x)$ are decoupled, and the solution of the system refers to each coordinate component by itself.

2.2.5. *Error Estimation*

The zero, first and second derivative error (semi) norms (1)-(3) are used for error estimation. The zero derivative error norm (squared error integral) is the L^2-*norm*. The first and second derivative error (semi) norms are used to estimate the error in tangential and the second derivative displacements.

We also use the error (semi) norms $E(x)/L^2$, $\bar{E}(x)/L^2$ and $\hat{E}(x)/L^2$, which measure the error mean displacement per unit length, where L is an approximated length of $\mathbf{f}(u)$. For error in curvature, we use the curvature L^2 error E^{κ},

$$E^{\kappa} = \int_a^b (\tilde{\kappa} - \kappa)^2\,du \tag{29}$$

where $\tilde{\kappa}$ is the curvature of the approximated curve $\mathbf{x}(u)$, and the curvature mean deviation error E_m^{κ},

$$E_m^{\kappa} = \frac{\displaystyle\int_a^b (\tilde{\kappa} - \kappa)^2\,du}{\displaystyle\int_a^b (\kappa)^2\,du}\,.$$

Since the approximation depends on the parametrization of $\mathbf{x}(u)$ and $\mathbf{f}(u)$, it does not necessarily yield orthogonal error vectors between corresponding values of parameters. The absolute Euclidean minimum (or maximum) is at the point where two normals are collinear. Therefore a re-parametrization is needed, so that the correction of the parametrization will direct the error vectors to be as orthogonal as possible to the tangent of $\mathbf{x}(u)$. This will result in a better error estimation [9, 2].

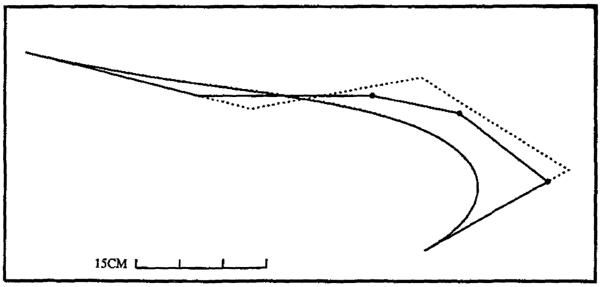

Figure 1.1. — Approximation of a Bézier curve of a degree 5 (solid line) by a Bézier curve of degree 4, with weighting factor values $\alpha = 1$, $\beta = 0$, $\gamma = 0$.

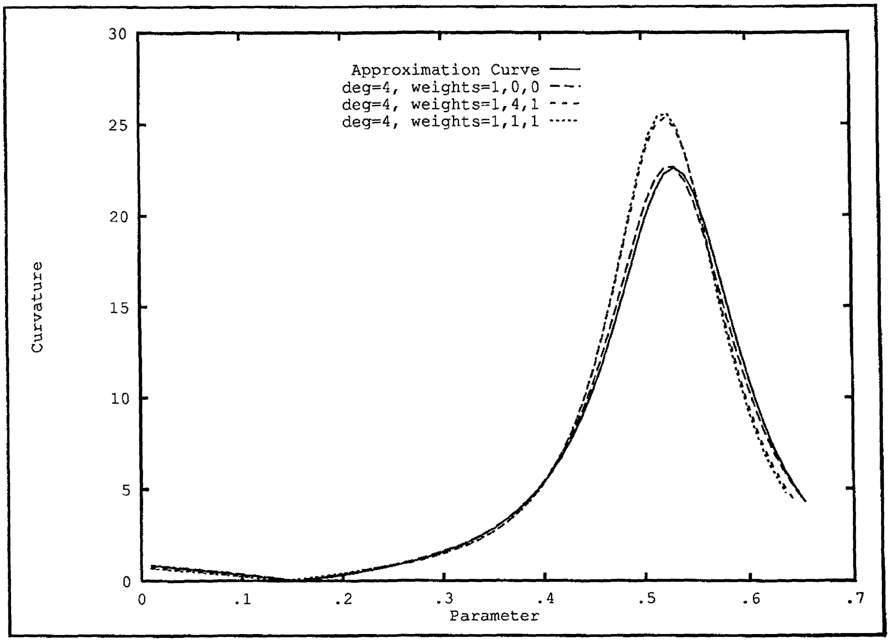

Figure 1.2. — Curvature profiles of approximation Bézier curves of degree 4 and with different weighting factor values (drawn in dashed lines), and of the given Bézier curve of degree 5. See figure 1.1.

A refinement to the partition $\{u_i^m\}_{i=0}^{m-1}$ is currently done by finding the positional error δ in the interval $[a, b]$ which is the maximum of the Euclidean norm :

$$\delta = \max \{ |\mathbf{x}(u) - \mathbf{f}(u)|, u \in [a, b] \} . \tag{30}$$

TABLE 1.1

Weighting factors relative to derivative error (semi) norm estin approximation of a Bézier curve of degree 5 by a Bézier curve of approximated length $L = 6.63297\ e - 01$ *Meter. See, figures* 1.1-1.3

α	β	γ	$E(x)$	$\bar{E}(x)$	$\hat{E}(x)$	$E(x)/L^2$
1	0	0	1.97933e-06	4.82373e-05	4.75987e-02	4.49885e-06
1	0	1	1.16714e-04	1.32565e-03	6.21261e-02	2.65282e-04
1	1	1	8.70318e-05	1.00876e-03	5.76158e-02	1.97816e-04
1	2	2	8.71392e-05	1.00990e-03	5.76312e-02	1.98060e-04
1	1	4	1.08845e-04	1.24180e-03	6.09395e-02	2.47396e-04
1	4	1	3.24070e-05	4.17382e-04	4.87316e-02	7.36586e-05
1	0	4	1.16946e-04	1.32811e-03	6.21585e-02	2.65808e-04
1	1	4	1.08845e-04	1.24180e-03	6.09395e-02	2.47396e-04
0	1	1	8.72467e-05	1.01105e-03	5.76465e-02	1.98305e-04
0	0	1	1.17023e-04	1.32893e-03	6.21693e-02	2.65983e-04
0	1	0	1.38500e-03	1.51124e-02	2.56872e-01	3.14799e-03

If δ is greater than a given ε, then the curve is split by the de Casteljau algorithm at the parameter value where δ occurs. This refinement procedure continues until δ is smaller than ε. For example, in figure 2.1 the curve was split at parameter value 0.25 for $\delta = 5.55925\,e-03$ (one can see the result in figure 2.2). In figure 3.1 the curve was split at parameter value $4.64286\,e-01$ for $\delta = 4.85932\,e-02$ (the resulting approximation is shown in figure 3.2). (Remark : δ is not an optimal criterion, since it is always greater or equal the maximum Euclidean norm after reparametrization.)

TABLE 1.2

Weighting factors relative to curvature error estimations for the approximation of a Bézier curve of degree 5 *by a Bézier curve of degree* 4, *with approximated length* $L = 6.63297\,e-01$ *Meter. See, figures* 1.1-1.3.

α	β	γ	$E^{\kappa}(x)$	$E^{\kappa}_{m}(x)$
1	0	0	3.91639e-02	1.37062e-03
1	0	1	7.28446e-01	2.54935e-02
1	1	1	4.24859e-01	1.48688e-02
1	2	2	4.25857e-01	1.49037e-02
1	1	4	6.44574e-01	2.25582e-02
1	4	1	1.88113e-02	6.58340e-04
1	0	4	7.30886e-01	2.55789e-02
1	1	4	6.44574e-01	2.25582e-02
0	1	1	4.26856e-01	1.49387e-02
0	0	1	7.31701e-01	2.56074e-02
0	1	0	2.33924e+01	8.18666e-01

2.2.6. *Properties of the Weighting Factors*

The weighting factors α, β and γ are positive moduli parameters. They allow one to augment a specific geometric or physical goal of approximation. Each of the error (semi) norms (1) to (3) has a different geometric or physical significance. Therefore, in many cases an approximation using only the zero, the first, or the second derivative error (semi) norm is required [14].

For example, given the Newton equation,

$$F = m\ddot{\mathbf{x}}(t)$$

where

$\mathbf{x}(t)$ is the path
$\dot{\mathbf{x}}(t)$ is the velocity (or momentum with the mass $m = 1$)
$\ddot{\mathbf{x}}(t)$ is the acceleration (or force with $m = 1$) .

• In a case where we would like to locate the position of a particle at every point in time, such as a bullet or an aircraft, then we only need to approximate $\mathbf{x}(t)$.

• If only the velocity or the momentum is needed in order to know for example the penetration of a bullet in a solid, then the only influencing factor is the momentum. Accordingly we need to obtain a good approximation for $\dot{\mathbf{x}}(t)$.

• A problem where a description of the forces which operate on different parts of a body in movement (such as a car in acceleration), can be solved by approximating $\ddot{\mathbf{x}}(t)$.

The weighting factors are attached to their error (semi) norms, meaning that a larger value of a weighting factor will increase the « influence » of its error (semi) norm upon the solution. On the other hand, we can decrease the effect of a certain error (semi) norm upon the solution by reducing the value of the weighting factor down to zero. For example, for angular momentum $m(\mathbf{x}(t) \times \dot{\mathbf{x}}(t))$ is linear in $\alpha\beta$.

The above example is a special case of the general form,

$$\mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)) \propto \mathbf{f}(\alpha, \beta, \gamma) \quad \text{with} \quad \alpha > 0 . \tag{31}$$

The relation between the weighting factors and the error (semi) norms can be seen in Tables 1.1, 1.2 and in figures 1.2, 1.3, where we can see the relation between curvature profiles of different weighting factors values, and the relation between the hodographs of those approximations. Tables 1.1, 1.2 and figures 1.2, 1.3 illustrate the following :

• When the values of the weighting factors β and γ are enlarged relative to α, then the effect of their relative error (semi) norms is augmented.

• The value of α *must be greater than zero*, since otherwise the order of approximation will decrease by 1 when $\alpha = 0$, $\beta > 0$, $\gamma = 0$, and by 2 when $\alpha = 0$, $\beta = 0$, $\gamma > 0$. This can be deduced from (23)-(24), for the construction of the stiffness matrices. In the separated bottom part of Tables 1.1, 1.2 and in figures 1.2, 1.3, it is evident that the results decrease in accuracy when this condition does not apply.

• With the correct weighting factors the approximation is improved up to a factor.

• The curvature of $\mathbf{x}(u)$, is dependent on both, first and second derivative error (semi) norms. Since $\bar{E}(x) \propto \beta$ and $\hat{E}(x) \propto \gamma$, then the curvature

$$|\kappa| = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3} \text{ is proportional to } \frac{\gamma}{\beta^2} . \tag{32}$$

• The effect of the weighting factors and of the different error (semi) norms upon the curvature is indicated in Tables 1-4. When the values of β

and/or γ become greater or smaller, then the values of the curvature L^2 error E^κ and the curvature mean deviation error E_m^κ behave proportionally. For the weighting factors $\alpha = 1$, $\beta = 4$, $\gamma = 1$ in Tables 1, 2 the curvature error (semi) norms have the best results, and for the weighting factors $\alpha = 1$, $\beta = 0$, $\gamma = 4$, the curvature error have relatively poor results. This example shows that despite $\gamma > 0$, κ is mainly dependent on β, as was expected from (32).

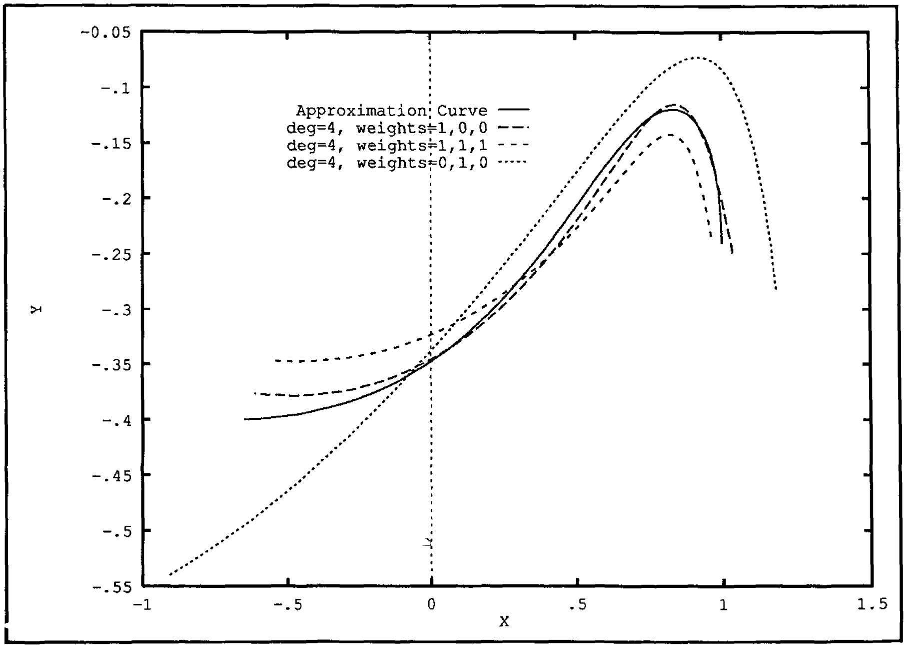

Figure 1.3. — Hodographs of approximation Bézier curves of degree 4 and with different weighting factor values (drawn in dashed lines), and of the given Bézier curve of degree 5. See figure 1.1.

2.2.7. *Integration of Higher Continuity Conditions*

The approximation curve $\mathbf{x}(u)$ is globally C^0 by construction. In order to create a higher order of global continuity approximation curves, one must include the proper continuity term in (28). For example, in the case of a two-element approximation curve which is globally C^1 over the interval $[u_0, u_2]$, [5]

TABLE 2.1

Derivative error (semi) norm estimations for approximation Bézier curves with different degrees, to a Bézier curve of degree 17 *with approximated length* $L = 1.39564\,e + 00$ *Meter, and weighting factors* ($\alpha = 1$, $\beta = 0$, $\gamma = 0$). *(The last tuple displays estimations for two elements approximation.) See figures* 2.1, 2.2.

Elm.	n	$E(x)$	$\bar{E}(x)$	$\hat{E}(x)$	$E(x)/L^2$	$\bar{E}(x)/L^2$	$\hat{E}(x)/L^2$
1	5	3.73829e-04	8.79422e-02	4.68648e+01	1.91922e-04	4.51491e-02	2.40602e+01
1	6	1.74646e-05	4.41127e-02	6.90335e+00	8.96622e-06	2.26472e-02	3.54415e+00
1	7	9.98117e-06	8.28747e-03	1.58263e+01	5.12429e-06	4.25475e-03	8.12516e+00
1	8	5.41058e-06	5.11381e-03	9.63218e+00	2.77777e-06	2.62541e-03	4.94511e+00
2	6	1.86566e-06	8.71906e-04	5.81642e-01	9.56063e-07	4.46811e-04	2.98064e-01

TABLE 2.2

Curvature error estimations and max Euclidean norm for approximation Bézier curves with different degrees, to a Bézier curve of degree 17 *with approximated length* $L = 1.39564\,e + 00$ *Meter, and weighting factors* ($\alpha = 1$, $\beta = 0$, $\gamma = 0$). *See figures* 2.1, 2.2.

Elm.	n	$E^{\kappa}(x)$	$E^{\kappa}_{m}(x)$	δ
1	5	1.45890e+00	1.22916e-01	4.80794e-02
1	6	5.77705e-01	5.29460e-02	1.42843e-02
1	7	4.42768e-01	2.54834e-02	7.83667e-03
1	8	1.99949e-01	1.31151e-02	5.55925e-03
2	6	7.14427e-02	4.96162e-03	1.28239e-04

$$\mathbf{x}(u) = \begin{cases} x_0(t_0) = \sum_{i=0}^{m} \mathbf{b}_i\, B_i^m(t_0)\,, & u = u_0(1 - t_0) + u_1\, t_0 \\ x_1(t_1) = \sum_{j=0}^{n} \mathbf{b}_{m+j}\, B_j^n(t_1)\,, & u = u_1(1 - t_1) + u_2\, t_1 \end{cases} \tag{33}$$

the C^1 continuity term is

$$\mathbf{b}_{m-1} = \frac{\Delta u_0 + \Delta u_1}{\Delta u_1}\,\mathbf{b}_m - \frac{\Delta u_0}{\Delta u_1}\,\mathbf{b}_{m+1}\,. \tag{34}$$

The integration of the term (34) in the system (28) is done by eliminating the row and column where $\mathbf{b}_{m-1}$ appears in the stiffness matrix and the load vector, and inserting (34) for $\mathbf{b}_{m-1}$. We obtain a new functional representation by the following transformation T [15],

$$\begin{aligned} \bar{M} &= T^T M_i\, T\,, \\ \bar{\mathbf{m}} &= T^T \mathbf{m}_i \end{aligned} \tag{35}$$

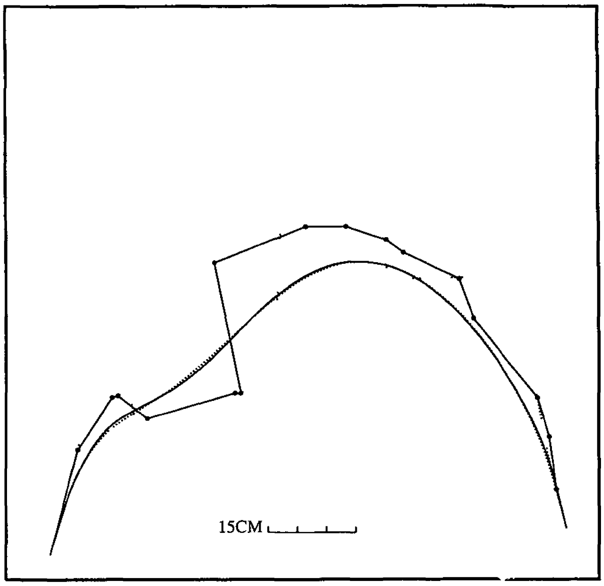

Figure 2.1. — Reduction of a Bézier curve of degree 17 (solid line) to one Bézier segment of degree 8, with weighting factors $\alpha = 1$, $\beta = 0$, $\gamma = 0$.

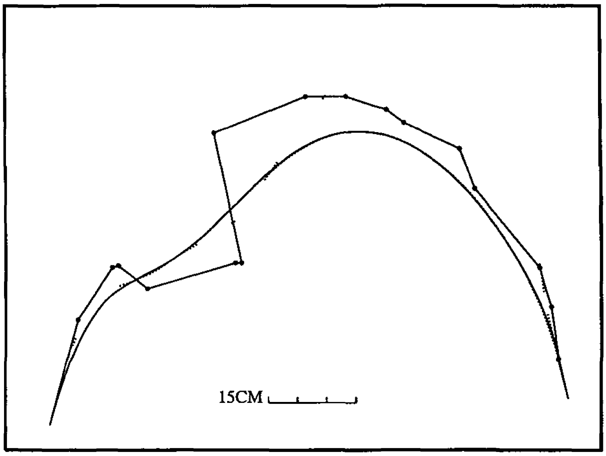

Figure 2.2. — Reduction of a Bézier curve of degree 17 (solid line) to two Bézier segments of degree 6, with weighting factors $\alpha = 1$, $\beta = 0$, $\gamma = 0$.

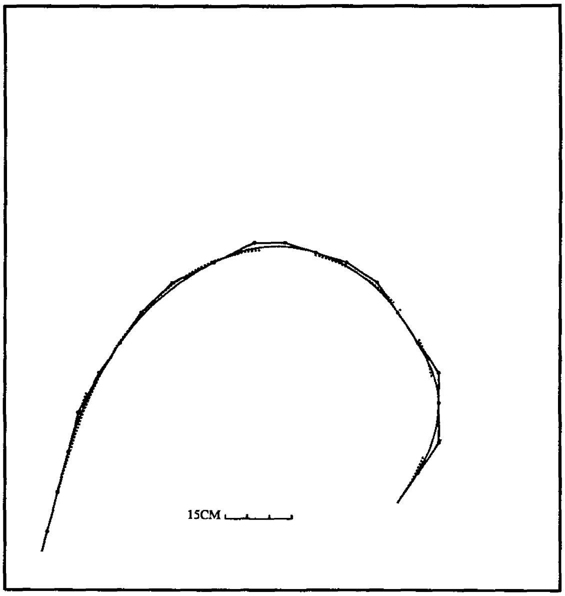

Figure 3.1. — Merging of seven C^0-Bezier segments of degree 3 to one Bezier curve degree 6.

where

$$T_{i,j} = \begin{cases} \delta_{i\,j} & \text{for} \quad 0 \geqslant i \leqslant m-2,\, 0 \geqslant j \leqslant m-2 \\ \dfrac{\Delta u_0 + \Delta u_1}{\Delta u_1} & \text{for} \quad i = m-1, j = m-1 \\ \dfrac{-\Delta u_0}{\Delta u_1} & \text{for} \quad i = m-1, j = m \\ \delta_{i,j} & \text{for} \quad m \leqslant i \leqslant m+n,\, m-1 \leqslant j \leqslant m+n-1 \end{cases} \tag{36}$$

The new functional representation with the global C^1 approach is

$$\bar{J}_i^q(x_i) = \left\{ \frac{1}{2}\, \bar{\mathbf{b}}_i^T \bar{M}_i\, \bar{\mathbf{b}}_i - \bar{\mathbf{m}}_i^T \mathbf{b}_i \right\} \tag{37}$$

The condition for C^k continuity between elements i and $i+1$ is (See [5])

$$\left(\frac{1}{\Delta u_i} \right)^j \Delta^j \mathbf{b}_{m-1} = \left(\frac{1}{\Delta u_{i+1}} \right)^j \Delta^j \mathbf{b}_m \quad j = 0, \quad , k \tag{38}$$

If continuity order of k is implemented then there will be at most $2\,mn - 2 - 2\,k\,(n-1)$ degrees of freedom, since there are $n-1$ connection points between elements and the $2k$ degrees of freedom per connection point are reduced.

3. EXAMPLES

The approximation method was used in the following applications,

• *Reduction of degree* of high order polynomial curves to polynomials of a lower order. The degree reduction can be done by reducing or increasing the curve segmentation. Figure 2.1 shows an approximation of a Bézier curve with degree 17, by Bézier curve of degree 8. Figure 2.2 presents an approximation of two elements of degree 6, to the same degree 17 Bézier curve. Here there is an improvement in the accuracy of the approximation. Tables 2.1, 2.2 present error estimations for the approximation with single element and two elements. There is a significant improvement in the accuracy in the case of the two elements approximation. Table 2.2 also provides the max Euclidean norm δ.

• Curves can be merged through degree reduction or by degree elevation. Figures 3.1, 3.2 present a merge of a C^0 Bézier spline curve with 7 cubic segments, has been done using a single Bézier curve of degree 6, and a Bézier-spline-curve with two segments of degrees 3 and 5.

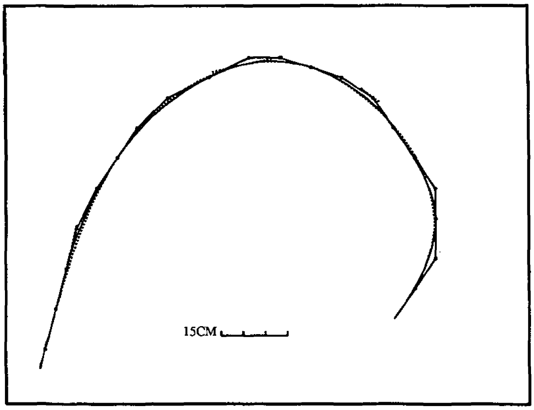

Figure 3.2. — Merging of seven C^0-Bézier segments of degree 3 to two Bézier segments of degrees 3 and 5.

• *Construction of a parametric offset curve* is done by approximating the offset curve

$$\mathbf{x}_d(u) = \mathbf{f}(u) + d\mathbf{n}(u), \tag{39}$$

where $\mathbf{n}(u)$ is the principal normal vector, and d is the distance along $\mathbf{n}(u)$.

In figure 4 there are eight approximation offset Bézier curves of degree 8, to a given (intermediate) Bézier curve of degree 5.

The offset curve construction given here does not solve problems such as loops or cusps in offset curves. For the solution of such problems, see [8].

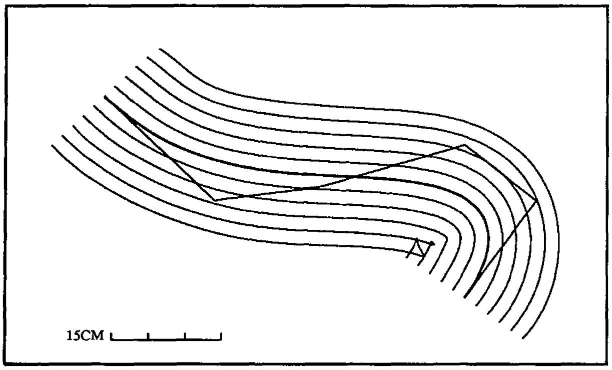

Figure 4. — Approximate offset Bézier curves of a degree 8 to a Bézier curve of degree 5.

4. INTEGRATION OF CONSTRAINTS

We introduce a Lagrangian multiplier formulation for the constrained functional, and solve the resulting optimization problem by the Uzawa method.

Given the inequality convex constraints functions,

$$\Phi(x) \equiv \{\varphi_0(x) \dots \varphi_\ell(x)\} \tag{40}$$

with $\varphi_i(x) \leq 0$, find u such that

$$\begin{cases} u \in U \equiv \{v \in V : \varphi_i(v) \leq 0, 1 \leq i \leq m\} \\ J(u) = \inf_{v \in U} J(v) . \end{cases}$$

Uzawa's method, is an iterative method which allows one to solve an inequality constrained minimization problem by replacing it with a sequence of unconstrained minimization problems [3].

The iteration starts with an arbitrary value for the element $\lambda^0 \in R_+^m$, a sequence of pairs $(\lambda^k, u^k) \in R_+^m \times V$, $k \geq 0$, is defined by means of the following recursion formula :

$$u^k : J(u_k) + \sum_{i=1}^{m} \lambda_i^k \varphi_i(u^k) = \inf_{v \in V} \left\{ J(v) + \sum_{i=1}^{m} \lambda_i^k \varphi_i(v) \right\} \tag{41}$$

where

$$\lambda_i^{k+1} : \lambda_i^{k+1} = \max \left\{ \lambda_i^k + \rho \varphi_i(u^k), 0 \right\}, 1 \leq i \leq m \tag{42}$$

and $u^k \equiv u_{\lambda^k}$.

The parameter ρ is chosen « as best may be » (for a criterion for choosing ρ and about the convergence of the method, see [3]).

The new constrained functional representation is,

$$J^{\Phi}(x) = J^{\ell}(x) + \sum_{i=1}^{m} \lambda_i \varphi_i(x) . \tag{43}$$

Minimization of (43) with regard to the free variables in $\mathbf{b}$ yields the following system of equations,

$$M\mathbf{b} - \mathbf{m} + \frac{\partial \sum_{i=1}^{m} \lambda_i \varphi_i(x)}{\partial b} = 0 \tag{44}$$

which is solved in every iteration k until convergence is reached or the number of iterations exceeds a given limit.

4.1. An Example with Constraint

The goal of this constraint is to eliminate loops in the construction process of an approximation cubic Bézier curve. When a self intersection in a cubic Bézier polygon is detected, the polygon is constrained to open itself thus eliminating the loop.

Remarks : This condition for loop detection in a cubic Bézier curve, is sufficient but not necessary. It is important to see that the approximation of the curve is still preserved while opening the loop (as in *fig.* 5, the upper part of the loop is merged with the open curve), this is due to functional (1).

Given a cubic Bézier polygon $\mathbf{Q}$, formed by $\mathbf{b}_0, \ldots, \mathbf{b}_3$. A *self intersection* in $\mathbf{Q}$, constrains the two vectors $\mathbf{u} = (\overrightarrow{\mathbf{b}_3\mathbf{b}_0} \times \overrightarrow{\mathbf{b}_0\mathbf{b}_1})$ and $\mathbf{v} = (\overrightarrow{\mathbf{b}_1\mathbf{b}_2} \times \overrightarrow{\mathbf{b}_2\mathbf{b}_3})$ to have opposite directions. Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to the same plane, the first two coordinates of $(\mathbf{u} \times \mathbf{v})$ vanish. Therefore, the constraint upon the polygon is $\varphi(\mathbf{b}) = u_3 v_3 > 0$. The constraint $\varphi(\mathbf{b})$ causes $\mathbf{u}$ and $\mathbf{v}$ to have the same direction. This constraint also *couples the coordinates components* in $J^{\Phi}(x)$.

It is difficult to determine the initial value of the parameter ρ for this constraint since (44) is non-linear. Several solutions may exist (different local minimum values), and different solutions can be obtained by different values of ρ. In this case the optimal ρ is determined by trial and error.

Figure 5 shows an example of the loop constraint, where the loop was opened with the initial value of the parameter $\rho = 20$ and with $n = 100$ iterations.

Since here the system (44) is nonlinear, it is solved using the Newton-Raphson method.

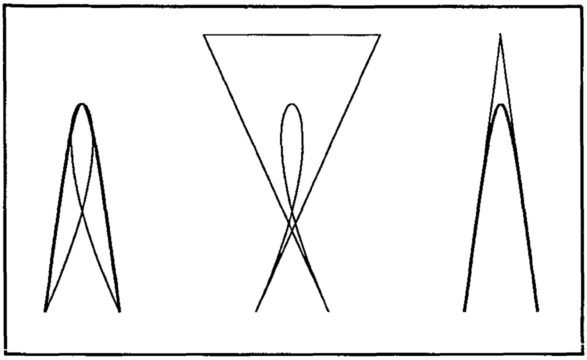

Figure 5. — Example of the loop opening constraint, with $\rho = 20$ and $n = 100$.

5. CONCLUSIONS

In this paper we introduced a global and continuous method for approximation and/or construction of planar curves. It is based on a minimization of a *functional* which describes approximation and differential geometric characteristics. Weighting factors were integrated in the functional to allow one to control the approximation. It is also possible to integrate constraints upon the approximation/construction process in order to achieve desired geometrical or physical effects. The numerical solution of the functional uses FEM.

The method introduces some advantages ;

• Functional which make use of a global and continuous criteria (the squared integrals of the displacements between the zero, first and second derivatives of the approximation and approximated curves) for approximation and construction.

• The weighting factors in the functional allow one to augment a certain error norm which is included in the functional (or to emphasize the appropriate physical task), by the determination of the weighting factor values. This is a vital flexibility in the control of the approximation process.

• The use of FEM with the Bernstein-Bézier representation for the shape functions has some cardinal advantages :

— Every element is treated separately, and its « influence » is added to the general stiffness matrix such that there is no limitation on the form of the general range combined from a collection of elements.

— It is possible to approximate different elements with different degrees of Bézier curves.

— System of equations (28) is linear for any order of continuity C^n between elements.

— Segmentation of the original curve is natural to FEM because of the subdivision of a FEM range problem into elements.

— The use of Bézier-Bernstein representation, saves much of the approximation calculation, which can be prepared in advance and be used regardless of the subdivision into elements.

• Integration of geometrical and physical constraints is modular and relatively easy. The constraints can be linear or nonlinear, with or without inequalities.

• The solution of a constrained minimization problem (43) uses the Uzawa iterative method, which allows integration of constraints that other methods, such as the gradient method, fail to solve [3].

The present approximation method is currently extended for approximation and construction of surfaces.

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